Consider the function $w = f(x, y, z)$ on the set $D = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$.

Note: The output values, $w$, are not represented in this picture.

\[ \sum_{k=1}^{n} f(x^k, y^k, z^k) \Delta V \]

If $\lim_{b \to 0} \sum_{k=1}^{n} f(x^k, y^k, z^k) \Delta V_k$ exists for all partitions and all choices of $(x^k, y^k, z^k)$, then $f$ is integrable on $D$ and the value is the triple integral of $f$ over $D$:

\[ \int_D f(x, y, z) \, dV = \lim_{b \to 0} \sum_{k=1}^{n} f(x^k, y^k, z^k) \, dV_k \]

Note: Volume of $D = \iiint_D 1 \, dV$ [why?]

Theorem:
If $f$ is continuous on the region $D = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$, then $f$ is integrable over $D$ and

\[ \iiint_D f(x, y, z) \, dV = \int_a^b \int_c^d \int_r^s f(x, y, z) \, dz \, dy \, dx \]
Note: There are a total of 6 orders of integration.

\[ \iiint_D f(x,y,z) \, dV = \iiint_c b a r f(x,y,z) \, dz \, dy \, dx \quad (\text{since } D \text{ is a rectangular box}) \]

Also, you try

Ex: Evaluate \( \iiint_D 3x^2y^2 \, dV \) where \( D = \{(x,y,z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2, 1 \leq z \leq 3\} \)

\[
\begin{align*}
&= \int_0^1 \int_0^2 \int_1^3 3x^2y^2 \, dz \, dy \, dx \\
&= \int_0^1 \int_0^2 \left[ 3x^2y^2 \right]_{z=1}^{z=3} \, dy \, dx \\
&= \int_0^1 \int_0^2 [9x^2y^2 - 3x^2y^2] \, dy \, dx \\
&= \int_0^1 \left[ 6x^2y^2 \right]_{y=0}^{y=2} \, dx \\
&= \int_0^1 [24x^2] \, dx \\
&= 8x^3 \bigg|_0^1 = 8
\end{align*}
\]

OR

\[
\begin{align*}
&= \int_1^3 \int_0^2 \int_0^1 3x^2y^2 \, dx \, dy \, dz \\
&= \int_1^3 \left[ \frac{3x^2y^2}{2} \right]_{x=0}^{x=1} \, dy \, dz \\
&= \int_1^3 \left[ \frac{3}{2}y^2 \right] \, dy \, dz \\
&= \left[ \frac{3}{2} \frac{y^3}{3} \right]_{y=0}^{y=2} \, dz \\
&= \frac{1}{2} \left[ 2x^3 \right]_{z=1}^{z=3} = \frac{1}{2} \left[ 8 \right] = 4
\end{align*}
\]

More general regions in \( \mathbb{R}^3 \)

\[ D = \{(x,y,z) \mid (x,y) \in R, G(x,y) \leq z \leq H(x,y)\} \]

Recall: There are 2 ways to describe \( R \) (possibly) using constant bounds for one of the variables.
\[ R = \{ (x,y) \mid \text{a} \leq x \leq \text{b}, \ g(x) \leq y \leq h(x) \} \]

Then \( D = \{ (x,y,z) \mid \text{a} \leq x \leq \text{b}, \ g(x) \leq y \leq h(x) \}, \ G(x,y) \leq z \leq H(x,y) \} \)

(\( (x,y) \text{ is in } R \))

\[ R = \{ (x,y) \mid \text{c} \leq y \leq \text{d}, \ g(y) \leq x \leq h(y) \} \]

Then \( D = \{ (x,y,z) \mid \text{c} \leq y \leq \text{d}, \ g(y) \leq x \leq h(y) \}, \ G(x,y) \leq z \leq H(x,y) \} \)

Other possibilities...

\[ y = G(x,z) \]

\[ y = H(x,z) \]

\[ D = \{ (x,y,z) \mid (x,z) \text{ in } R, \ G(x,z) \leq y \leq H(x,z) \} \]

Possibly 2 ways to describe

Practice?

Ex/ Let \( D \) be the solid bounded by \( z = 2y - 2x, \ x = 0, \ y = 0, \) and \( y = 1. \)

Describe \( D \) in the 6 possible ways.
**Projection of \( D \) onto xy-plane**

\[ D = \{(x, y, z) \mid 0 \leq x \leq 1, \; x \leq y \leq 1, \; 0 \leq z \leq 2y - 2x \} \]

**Projection of \( D \) onto xz-plane**

\[ D = \{(x, y, z) \mid 0 \leq y \leq 1, \; 0 \leq x \leq y, \; 0 \leq z \leq 2y - 2x \} \]

**Projection of \( D \) onto yz-plane**

\[ D = \{(x, y, z) \mid 0 \leq y \leq 1, \; 0 \leq z \leq 2y, \; 0 \leq x \leq y - \frac{1}{2}z \} \]

**Projection of \( D \) onto xz-plane**

\[ D = \{(x, y, z) \mid 0 \leq y \leq 1, \; 0 \leq z \leq 2y, \; 0 \leq x \leq y - \frac{1}{2}z \} \]
Theorem - If \( f \) is continuous over the region

\[
D = \{ (x, y) \mid a \leq x \leq b, \ g(x) \leq y \leq h(x), \ G(x, y) \leq z \leq H(x, y) \}\]

where \( g, h, G, \) and \( H \) are continuous,

then \( f \) is integrable over \( D \) and

\[
\iiint_D f(x, y, z) \, dV = \int_a^b \int_{g(x)}^{h(x)} \left( \int_{G(x, y)}^{H(x, y)} f(x, y, z) \, dz \right) \, dy \, dx
\]

Similar statements hold for the other 5 orders of description/integration.

Note:

\[
\int_a^b \int_{g(x)}^{h(x)} \left( \int_{G(x, y)}^{H(x, y)} f(x, y, z) \, dz \right) \, dy \, dx
\]

cannot depend on \( z \)

\( a \) is constant

\( b \) is constant

\( \mathcal{D} \) is constant

Note:

\[
\iiint_D f(x, y, z) \, dV = \iiint_{\mathcal{R}} \left( \int_{G(x, y)}^{H(x, y)} f(x, y, z) \, dz \right) \, dA
\]

where \( \mathcal{R} = \{ (x, y) \mid a \leq x \leq b, \ g(x) \leq y \leq h(x) \} \)

Ex/ Find the volume of the solid \( D \) bounded by \( z = 2y - 2x, \ z = 0, \ x = 0, \) and \( y = 1 \) in 6 different ways.

\[
V = \iiint_D 1 \, dV
\]
Using $D = \{(x, y, z)| 0 \leq x \leq 1, x^2 + y^2 \leq 1, 0 \leq z \leq 2y - 2x\}$

$$SSS \, dV = \int_0^1 \int_x^{2y-2x} \int_0^1 dz \, dy \, dx$$

$$= \int_0^1 \int_x^{2y-2x} [z]_0^1 \, dy \, dx = \int_0^1 (2y-2x) \, dy \, dx$$

$$= \int_0^1 [y^2 - 2xy]_x^{2y-2x} \, dx = \int_0^1 (1 - 2x) - (x^2 - 2x^2) \, dx$$

$$= \int_0^1 (x^2 - 2x + 1) \, dx = \left(\frac{1}{3}x^3 - x^2 + x\right)_0^1 = \frac{1}{3}$$

You try the other $S$ ways...

**Ex/** Find the volume of the solid $D$ bounded above by $z = f(x, y)$ and below by $z = g(x, y)$ over the region $R$ in the $xy$-plane.

$$V = SSS \, dV = SS\left[ \int_R f(x, y) \, dz \right] \, dA = SSS (f(x, y) - g(x, y)) \, dA$$

Look familiar?

**Ex/** Let $D$ be the solid region bounded by $z = 4, z = x^2 + y^2$ that lies in the 1st octant.

Let $\rho(x, y, z)$ be the density function for the solid $D$ in kg/m$^3$.

Find the mass of the solid $D$.

$$\text{mass} \approx \sum_{k=1}^n \rho(x_k^*, y_k^*, z_k^*) \, DV$$

Then $DV$ in m$^3$

Let $\rho(x, y, z) = x$
$D = \{(x,y,z) \mid 0 \leq y \leq 2, \ 0 \leq x \leq \sqrt{4-y^2}, \ x^2+y^2 \leq z \leq 4 \}$

\[
\text{mass of } D = \iiint_D \rho(x,y,z) \, dV = \iiint_D \rho(x,y,z) \, dx \, dy \, dz
\]

\[
= \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-y^2}} [x^2 \cdot \sqrt{4-y^2} - x^2] \, dx \, dy
\]

\[
= \int_0^2 \left[ \frac{1}{2} x^2 \sqrt{4-y^2} - \frac{1}{3} x^3 \right]_0^{\sqrt{4-y^2}} \, dy
\]

\[
= \int_0^2 \left[ \frac{1}{2} \sqrt{4-y^2} \cdot \sqrt{4-y^2} - \frac{1}{3} (4-y^2) \right] \, dy
\]

\[
= \left. \left[ 4y - \frac{3}{4} y^3 + \frac{1}{2} y^4 \right] \right|_0^2 = \frac{64}{15} \text{ kg}
\]

Try other orders of integration, such as using

\[
D = \{(x,y,z) \mid 0 \leq z \leq 2, \ x^2 + y^2 \leq 4, \ 0 \leq y \leq \sqrt{2-x^2} \}
\]

\[
\text{mass} = \iiint_D \rho(x,y,z) \, dx \, dy \, dz
\]

\[
\text{Evaluate } \iiint_D (x^2 + z^2)^{3/2} \, dV \text{ where } D \text{ is the solid bounded}
\]

by $x = y^2 + z^2$ and $x = 8 - y^2 - z^2$
\( D = \{ (x, y, z) | (y, z) \text{ in } \mathbb{R} \text{ and } y^2 + z^2 = x \leq 8 - y^2 - z^2 \} \)

\[
\iiint_D (y^2 + z^2)^{3/2} \, dV = \iiint_R \left[ \int_{y^2+z^2}^{8-y^2-z^2} (y^2+z^2)^{3/2} \, dx \right] \, dA
\]

\[
= \iiint_R \left[ (y^2+z^2)^{3/2} \left. x \right|_{y^2+z^2}^{8-y^2-z^2} \right] \, dA
\]

\[
= \iiint_R (y^2+z^2)^{3/2} (8-2y^2-2z^2) \, dA
\]

Try a polar representation of the \( y^2 = \text{plane} \) \( \frac{y^2 + z^2 = r^2}{y = r \cos \theta, z = r \sin \theta} \)

\[
\begin{aligned}
&= \int_0^{2\pi} \int_0^2 (r^3)(8-2r^2) \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^2 (8r^4 - 2r^6) \, dr \, d\theta = \ldots \text{you finish}
\end{aligned}
\]

\[
\text{Definition - Average Value of a Function of Three Variables}
\]

If \( f \) is continuous on a region \( D \) of \( \mathbb{R}^3 \), then the average value of \( f \) over \( D \) is

\[
\bar{f} = \frac{1}{\text{volume}(D)} \iiint_D f(x, y, z) \, dV
\]