Conservative Vector Fields

Definition -
A vector field \( F \) is said to be conservative on a region (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)) if there exists a scalar function \( \phi \) such that \( F = \nabla \phi \) on that region.

Note: In such a case, \( \phi \) is called the potential function for \( F \).

Example: Let \( \phi(x,y) = x^3y - x \) defined on all of \( \mathbb{R}^2 \)

Then \( \nabla \phi(x,y) = \langle 3x^2y - 1, x^3 \rangle \)

Let \( F(x,y) = \langle 3x^2y - 1, x^3 \rangle \)

Then \( F \) is a conservative vector field on \( \mathbb{R}^2 \)
since \( F = \nabla \phi \). And \( \phi \) is the potential function for \( F \).

Note: Given \( \phi(x,y,z) \), and letting \( F(x,y,z) = \nabla \phi(x,y,z) = \langle \phi_x, \phi_y, \phi_z \rangle \)

then \( F \) is a conservative vector field, and \( \phi \) is the potential function for \( F \).

1. However, given a vector field \( F \), is there a function \( \phi \) such that \( F = \nabla \phi \)? That is, is \( F \) conservative? Is there a potential function for \( F \)?
2. If \( F \) is conservative, how can we find the potential function \( \phi \)?
3. What is special about conservative vector fields?

Some Topology

Definition -
Suppose \( C \) is a curve (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)) described by \( \vec{r}(t) \) for \( a \leq t \leq b \).

1. \( C \) is a simple curve if \( \vec{r}(t_1) \neq \vec{r}(t_2) \) for all \( t_1 \) and \( t_2 \) where \( a < t_1, t_2 < b \),
   that is, \( \vec{r}(t) \) is one-to-one except possibly at the endpoints.
   In other words, \( C \) does not intersect itself between its endpoints.
2. \( C \) is a closed curve if \( \vec{r}(a) = \vec{r}(b) \).

\[ \text{Not closed} \quad \text{Not closed} \quad \text{Closed} \quad \text{Closed} \]
\[ \text{Simple} \quad \text{Not simple} \quad \text{Simple} \quad \text{Not simple} \]
Definition:
Suppose \( R \) is a region in \( \mathbb{R}^2 \) (or \( D \) is a region in \( \mathbb{R}^3 \)).

1. \( R \) (or \( D \)) is path-connected if any two points in \( R \) (or \( D \)) can be connected by a continuous curve lying entirely in \( R \) (or \( D \)).

2. \( R \) (or \( D \)) is simply connected if it is path-connected AND every closed simple curve in \( R \) (or \( D \)) can be continuously deformed and contracted to a point in \( R \) (or \( D \)) without leaving \( R \) (or \( D \)).

[Equivalently, given any two points in \( R \) (or \( D \)) there is a path and every path can be continuously transformed into another without leaving \( R \) (or \( D \)).]

Informally, \( R \) (or \( D \)) consists of one piece and does not have any holes that pass all the way through the region.

Path connected  Not path connected  Path connected
Simply connected  Not simply connected  Not simply connected

Theorem - Test for Conservative Vector Fields
Let \( \mathbf{F}(x, y, z) = M(x, y, z) \mathbf{i} + N(x, y, z) \mathbf{j} + P(x, y, z) \mathbf{k} \) be a vector field defined on an open simply-connected region \( D \) of \( \mathbb{R}^3 \).

Suppose that \( M, N, \) and \( P \) have continuous first partial derivatives on \( D \).

Then \( \mathbf{F} \) is a conservative vector field if and only if

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}
\]

(For \( \mathbf{F} = M \mathbf{i} + N \mathbf{j} \) on \( \mathbb{R}^2 \), we have a similar theorem with the single condition \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \).)
Ex/ Let $F(x,y) = <-y, x>$

Is $F$ conservative?

$M(x,y) = -y$, $N(x,y) = x$

\[
\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 1 \quad \text{so} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}
\]

$F$ is NOT conservative.

Ex/ Let $F(x,y,z) = <y, x^2 y, x^2>$

$M(x,y,z) = y$, $N(x,y,z) = x^2 y$, $P(x,y,z) = x^2$

\[
\frac{\partial M}{\partial y} = 1 \quad \frac{\partial M}{\partial z} = 0 \quad \frac{\partial N}{\partial x} = 2xy \quad \frac{\partial N}{\partial z} = 0 \quad \frac{\partial P}{\partial x} = 2x \quad \frac{\partial P}{\partial y} = 0
\]

So, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} \neq \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$.

$F$ is NOT conservative.

Ex/ Let $F(x,y) = \frac{-y^2}{x^2 + y^2} \frac{T}{M} + \frac{x}{x^2 + y^2} \frac{J}{N}$ where $R_1 = \{(x,y) | 0 < x^2 + y^2 < 1\}$

Is $F$ conservative on $R_1$?

\[
\frac{\partial M}{\partial y} = \frac{-(x^2 + y^2) - (2y)(-y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

\[
\frac{\partial N}{\partial x} = \frac{1(x^2 + y^2) - (2x)x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

R is Not simply connected

So, we cannot conclude that $F$ is conservative.
(Infact, it is NOT on $R_1$)

However, consider $R_2 = \{(x,y) | x,y > 0\}$

Then $R_2$ is simply connected and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$ so, $F$ is conservative on $R_2$. 

If we know \( \mathbf{F} \) is conservative, how do we find a potential function?

Ex: Let \( \mathbf{F}(x,y) = \langle 5 + 3x^2y, x^3 - 2y \rangle \)

Is \( \mathbf{F} \) conservative on \( \mathbb{R}^2 \)?

\[
\frac{\partial M}{\partial y} = 3x^2, \quad \frac{\partial N}{\partial x} = 3x^2 \quad \text{So} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

\( \therefore \mathbf{F} \) is conservative.

We now want to find \( \phi(x,y) \) such that \( \nabla \phi = \mathbf{F} \).

Thus, we want
\[
\begin{align*}
\phi_x &= M \\
\phi_y &= N
\end{align*}
\]

1. \( \phi_x(x,y) = 5 + 3x^2y \)
2. \( \phi_y(x,y) = x^3 - 2y \)

Then, \( \phi(x,y) = \int \phi_x \, dx \) \( \text{(treat } y \text{ a constant?)} \)

\[
= \int (5 + 3x^2y) \, dx = 5x + 3\frac{x^3}{3}y + g(y)
\]

\[
= 5x + x^3y + g(y) \quad \text{Constant of integration may depend on } y \quad \text{?}
\]

Thus, \( \phi(x,y) = 5x + x^3y + g(y) \)

We still need to find \( g(y) \).

\[
\phi_y = \frac{\partial}{\partial y} \phi = \frac{\partial}{\partial y} [5x + x^3y + g(y)]
\]

\[
= 0 + x^3 + g'(y)
\]

By 2, we have
\[
\int \phi_y \, dy = x^3 + g'(y)
\]

So, \( g'(y) = -2y \)
\[ g(y) = \int g'(y) \, dy = \int -2y \, dy = -y^2 + C \]

Thus
\[ \Phi(x, y) = 5x + x^3y - y^2 + C \]

\[ \Phi(x, y) = 5x + x^3y - y^2 + C \] Check if \( \nabla \Phi = F \)

Try this again by starting with \( \int \Phi \, dy \) ...

Ex/ Let \( F(x, y, z) = (e^{y^2} + e^x - ye^{x^2}) \, \hat{x} + (zxe^{y^2} - e^{x^2}) \, \hat{y} + (xye^{y^2} - xye^{x^2}) \, \hat{z} \)

Conservative?

\[
\begin{align*}
\frac{\partial M}{\partial y} &= 2e^{y^2} - xe^{x^2} \\
\frac{\partial N}{\partial x} &= 2e^{y^2} - xe^{x^2} \\
\frac{\partial M}{\partial x} &= 2e^{y^2} - xe^{x^2} \\
\frac{\partial N}{\partial y} &= ye^{y^2} - ye^{x^2} - xye^{x^2} \\
\frac{\partial P}{\partial z} &= ye^{y^2} - ye^{x^2} - xye^{x^2} \\
\frac{\partial M}{\partial z} &= xe^{y^2} + xye^{y^2} - xe^{x^2} \\
\frac{\partial P}{\partial x} &= xe^{y^2} + xye^{y^2} - xe^{x^2} \\
\frac{\partial N}{\partial z} &= \frac{\partial P}{\partial y}
\end{align*}
\]

\( F \) is conservative

Next, find \( \Phi \) such that \( \nabla \Phi = F \)

So, 1. \( \Phi_x = e^{y^2} + e^x - y^2 \, e^{x^2} \)
2. \( \Phi_y = zxe^{y^2} - e^{x^2} \)
3. \( \Phi_z = xye^{y^2} - xye^{x^2} \)

By 1
\[ \Phi = \int (e^{y^2} + e^x - y^2 \, e^{x^2}) \, dx \quad (y, z: \text{constant}) \]

\[ = xe^{y^2} + e^x - ye^{x^2} + g(y, z) \]

constant of integration

\( \Phi = xe^{y^2} + e^x - ye^{x^2} + g(y, z) \)
Then
\[ \phi_y = \frac{\partial}{\partial y} [xe^{y^2} + e^x - ye^{x^2} + g(y, z)] \]
\[ = xze^{y^2} + 0 - e^{x^2} + \frac{\partial g}{\partial y} \]

Thus, comparing with (2) we get
\[ xze^{y^2} - e^{x^2} = xze^{y^2} - e^{x^2} + \frac{\partial g}{\partial y} \]
\[ \text{So, } \frac{\partial g}{\partial y} = 0 \]
\[ \text{then } g(y, z) = \int \frac{\partial g}{\partial y} dy = \int 0 dy = k(z) \text{ constant} \]

So (3) becomes
\[ \phi(x, y, z) = xe^{y^2} + e^x - ye^{x^2} + k(z) \]

Then
\[ \phi_z = \frac{\partial}{\partial z} [xe^{y^2} + e^x - ye^{x^2} + k(z)] \]
\[ = xye^{y^2} + 0 - xye^{x^2} + k'(z) \]

Comparing with (3) yields
\[ xye^{y^2} - xye^{x^2} = xye^{y^2} - xye^{x^2} + k'(z) \]
\[ \text{So, } k'(z) = 0 \]
\[ k(z) = \int 0 dz = c \]

Thus (3) becomes
\[ \phi(x, y, z) = xe^{y^2} + e^x - ye^{x^2} + c \]

Check if \[ \nabla \phi = \vec{F} \]

[Try again using a different order...]
\[ \mathbf{F} = \langle 3x^2y, x^3-2yz, 4z^3-y^2 \rangle \]

Conservative? (yes, show it!) 

\[ \nabla \phi = \mathbf{F}, \quad \phi? \]

1. \[ \phi_x = 3x^2y \]
2. \[ \phi_y = x^3 - 2yz \]
3. \[ \phi_z = 4z^3 - y^2 \]

By 1:

\[ \phi = \int (3x^2y) \, dx = x^3y + g(y, z) \]

\[ \phi = x^3y + g(y, z) \]

\[ \phi_y = \frac{\partial}{\partial y} [x^3y + g(y, z)] = x^3 + \frac{\partial g}{\partial y} \]

Compare with 2:

\[ x^3 - 2yz = x^3 + \frac{\partial g}{\partial y} \quad \text{so} \quad \frac{\partial g}{\partial y} = -2yz \]

\[ g(y, z) = \int (-2yz) \, dy = -y^2z + h(z) \]

So, 1 becomes

\[ \phi = x^3y - y^2z + h(z) \]

\[ \phi_z = -y^2 + h'(z) \]

Compare with 3:

\[ 4z^3 - y^2 = -y^2 + h'(z) \quad \text{so} \quad h'(z) = 4z^3 \]

\[ h(z) = \int 4z^3 \, dz = z^4 + C \]

Thus, 1 becomes

\[ \phi = x^3y - y^2z + z^4 + C \]

\[ \nabla \phi = \mathbf{F} \]

(Try again, different order...)
Theorem - The Fundamental Theorem for Line Integrals

Let \( C \) be a smooth curve given by \( \mathbf{r}(t) \) for \( a \leq t \leq b \). Let \( \mathbf{F} \) be a differentiable function of two or three variables whose gradient vector field is continuous on \( C \).

Then

\[
\int_C \nabla \mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(\mathbf{r}(b)) - \mathbf{F}(\mathbf{r}(a))
\]

Note: Given \( \mathbf{F}(x,y,z) \) and \( \mathbf{r}(t) = <x(t), y(t), z(t)> \), then

If we let \( \omega = \mathbf{F}(x(t), y(t), z(t)) \) we have

\[
\frac{d\omega}{dt} = \frac{dx}{dt} \frac{d\omega}{dx} + \frac{dy}{dt} \frac{d\omega}{dy} + \frac{dz}{dt} \frac{d\omega}{dz},
\]

and so

\[
\frac{d\omega}{dt} = \frac{dx}{dt} \frac{d\mathbf{F}}{dx} + \frac{dy}{dt} \frac{d\mathbf{F}}{dy} + \frac{dz}{dt} \frac{d\mathbf{F}}{dz}.
\]

That is

\[
\frac{d\mathbf{F}}{dt} \cdot \mathbf{r}(t) = \frac{dx}{dt} \frac{d\mathbf{F}}{dx} + \frac{dy}{dt} \frac{d\mathbf{F}}{dy} + \frac{dz}{dt} \frac{d\mathbf{F}}{dz}
\]

\[
= \left< \frac{d\mathbf{F}}{dx}, \frac{d\mathbf{F}}{dy}, \frac{d\mathbf{F}}{dz} \right> \cdot \left< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right>
\]

\[
= \nabla \mathbf{F} \cdot \mathbf{r}'(t)
\]

Thus

\[
\int_C \nabla \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_a^b \frac{d\mathbf{F}}{dt} \cdot \mathbf{r}(t) dt
\]

\[
= \mathbf{F}(\mathbf{r}(b)) - \mathbf{F}(\mathbf{r}(a))
\]

Definition -

If \( \mathbf{F} \) is a continuous vector field on a domain \( D \) in \( \mathbb{R}^2 \) (or \( D \) in \( \mathbb{R}^3 \)) we say that the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path if

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

for any two paths \( C_1 \) and \( C_2 \) in \( D \) that have the same initial and terminal points.

Path means a piecewise smooth curve.
Note: If \( f \) is differentiable and \( \nabla f \) is continuous on an open connected region \( R \), then \( \int_{C} \nabla f \cdot dr \) is independent of path.

\[
\int_{C_{1}} \nabla f \cdot dr = f(B) - f(A)
\]

\[
\int_{C_{2}} \nabla f \cdot dr = f(C) - f(A)
\]

**Theorem**

Let \( F \) be a continuous vector field on an open connected region \( R \). Then, there exists a potential function \( \phi \) with \( F = \nabla \phi \) (\( F \) is conservative) if and only if \( \int_{C} F \cdot dr \) is independent of path in \( R \).

Consider a closed path in an open connected region \( R \), (closed piecewise smooth curve)

Assume \( \int_{C} F \cdot dr \) is independent of path in \( R \).

Then, choosing two points on \( C \)

we get

\[
\int_{C} F \cdot dr = \int_{C_{1}} F \cdot dr + \int_{C_{2}} F \cdot dr = \int_{C_{1}} F \cdot dr - \int_{C_{1}} F \cdot dr = 0.
\]

Thus,

\[
\int_{C} F \cdot dr = \int_{C_{1}} F \cdot dr + \int_{C_{2}} F \cdot dr = \int_{C_{1}} F \cdot dr - \int_{C_{1}} F \cdot dr = 0.
\]
So, \( \oint_C \vec{F} \cdot d\vec{r} \) is independent of path in \( \mathbb{R} \)

\( \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \) for every closed path in \( \mathbb{R} \).

Conversely:

Assume \( \oint_C \vec{F} \cdot d\vec{r} = 0 \) for every closed path in \( \mathbb{R} \).

Then, choose any two paths from \( A \) to \( B \) in \( \mathbb{R} \).

Then, let \( C = c_1 \cup -c_2 \) (a closed path)

\( 0 = \oint_C \vec{F} \cdot d\vec{r} = \oint_{c_1} \vec{F} \cdot d\vec{r} + \oint_{-c_2} \vec{F} \cdot d\vec{r} = \oint_{c_1} \vec{F} \cdot d\vec{r} - \oint_{c_2} \vec{F} \cdot d\vec{r} \)

Thus \( \oint_{c_1} \vec{F} \cdot d\vec{r} = \oint_{c_2} \vec{F} \cdot d\vec{r} \) (independence of path ?)

So,

\( \oint_C \vec{F} \cdot d\vec{r} = 0 \) for every closed path in \( \mathbb{R} \)

\( \Rightarrow \oint_C \vec{F} \cdot d\vec{r} \) is independent of path in \( \mathbb{R} \).

Theorem:

Let \( R \) be an open region.

\( \oint_C \vec{F} \cdot d\vec{r} \) is independent of path in \( R \)

if and only if

\( \oint_C \vec{F} \cdot d\vec{r} = 0 \) for every closed path \( C \) in \( R \)
Theorem - Let $\mathbf{F}$ be a vector field that is continuous on an open connected region $R$. Then the following conditions are equivalent:

1) $\mathbf{F}$ is conservative
2) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $R$
3) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path in $R$

Ex/ Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = x^2y^3 + 4x$ and $C$ is the piecewise smooth curve (oriented) below.

Parameterize, yuck!

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(4,3) - \mathbf{F}(-1,-1)
\]

\[
= [(4)^2(3)^3 + 4(4)] - [(-1)^2(-1)^3 + 4(-1)] = 353
\]

Note: $\mathbf{F} = \langle 2xy^3 + 4, 3x^2y^3 \rangle$

So, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = \cdots = 353$

Also, $\int_C (2xy^3 + 4) dx + 3x^2y^3 dy = \int_C \mathbf{F} \cdot d\mathbf{r} = \cdots = 353$

Ex/ Evaluate $\int_C (5 + 3x^2y) dx + (x^2 - 2y) dy$ where $C$ is the piecewise smooth curve (oriented) below.

Note: $\mathbf{F} = \langle 5 + 3x^2y, x^2 - 2y \rangle$

Hopefully $\mathbf{F}$ is conservative?
It is (see earlier problem) and we found the potential function

$$\phi(x,y) = 5x + x^2y - y^2 + C$$

So,

$$\oint_C \left(5 + 3x^2y\right)dx + \left(x^3 - 2y\right)dy = \oint_C \left(5 + 3x^2y, x^3 - 2y\right) \cdot d\vec{r}$$

$$= \int_C \nabla \phi \cdot d\vec{r} = \phi(5, 2) - \phi(0, 0)$$

$$= \left[5(5) + (5)^3(2) - (2)^2 + C\right] - \left[5(0) + (0)^3(0) - (0)^2 + C\right]$$

$$= 271 + C - C = 271$$

Ex/ Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = <\frac{1}{y}, -xy^2>$

and $C$ is any curve from $(1, 1)$ to $(3, 1)$ that does not intersect the $x$-axis.

- Hopefully $\vec{F}$ is conservative?

- $M = \frac{1}{y}$, $N = -xy^2$

- $\frac{\partial N}{\partial x} = -y^2$, $\frac{\partial M}{\partial y} = -y^2$

- Equal for $y \neq 0$

So, $\vec{F}$ is conservative on $R = \{(x, y) \mid y > 0\}$

Find $\phi$ such that $\nabla \phi = \vec{F}$.

$$\phi = \int \frac{1}{y} dy = \frac{x}{y} + g(y)$$

$$\phi_y = -xy^2 + g'(y)$$

So, $-xy^2 = -xy^2 + g'(y)$

$$g'(y) = 0$$

$$g(y) = C$$

Thus, $\phi(x, y) = \frac{x}{y} + C$

Choose $C = 0$, $\phi(x, y) = \frac{x}{y}$
So, \( \int_C \vec{F} \cdot d\vec{r} = \phi(3,1) - \phi(1,1) = \frac{3}{1} - \frac{1}{1} = 2 \)

Evaluate \( \oint_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = \langle y, x \rangle \) and \( C \) is the unit circle oriented counterclockwise.

\[ M = y, \quad N = x \]
\[ \frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1 \]
\( \vec{F} \) is conservative on \( \mathbb{R}^2 \).

Find \( \phi \) such that \( \nabla \phi = \vec{F} \).

\[ \phi_x = y \implies \phi = \int y \, dx = xy + g(y) \]

\( g \) is arbitrary...

\( \phi(x,y) = xy \)

\( \oint_C \vec{F} \cdot d\vec{r} = 0 \)

Evaluate \( \oint_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \) and \( C \) is the unit circle oriented counterclockwise.

From an earlier problem, we know \( \vec{F} \) is conservative on any region that does not encircle the origin.

Cannot find an open simply connected region containing \( C \) on which \( \vec{F} \) is conservative.

Parameterize \( C \)

\[
\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \leq t \leq 2\pi
\]

\[
\vec{r}'(t) = \langle -\sin t, \cos t \rangle
\]
Then
\[ \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left( -\sin t \ \cos t \right) \cdot \left( -\sin t, \ \cos t \right) dt \]
\[ = \int_0^{2\pi} \left( \sin^2 t + \cos^2 t \right) dt = \int_0^{2\pi} 1 dt = 2\pi \]
(not 0, why?)

Ex/ Evaluate \( \oint_C \vec{F} \cdot d\vec{r} \) with \( \vec{F} \) as above, but \( C \) is the closed curve below.

Ex/ Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = \left< -y^2 \sin x, 2y \cos x \right> \)
and \( C \) is the curve below.

\[ M = -y^2 \sin x \]
\[ N = 2y \cos x \]
\[ \frac{\partial M}{\partial y} = -2y \sin x \]
\[ \frac{\partial N}{\partial x} = -2y \sin x \]
\( \vec{F} \) is conservative on \( \mathbb{R}^2 \).

Know your options! We could find \( \phi \) such that \( \nabla \phi = \vec{F} \)
then compute \( \phi(\pi, 5) - \phi(\pi, 0) \)

OR ...

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left< -2t^2 \sin \pi t, 10t \cos \pi t >, <0, 5> \cdot <0, 5> \right> dt \]
\[
= \int_{0}^{1} -50t \, dt = -25t^2 \bigg|_{0}^{1} = -25
\]

Thus (by path independence)

\[\int_{C} F \cdot dr = -25\]

Ex: The Law of Conservation of Energy

Kinetic Energy, \( k = \frac{1}{2}mv^2 \)

Potential Energy, \( p(x, y, z) = -f(x, y, z) \) where \( f \) is the potential function for the conservative vector "force" field \( \vec{F} \) (\( \nabla f = \vec{F} \))

Work done by \( \vec{F} \) along an oriented smooth curve from \( A \) to \( B \)

is \( W = \int_{C} \vec{F} \cdot d\vec{r} = \int_{C} \nabla p \cdot d\vec{r} = p(x, y, z) \bigg|_{A}^{B} = p(B) - p(A) \)

On the other hand... parameterizing \( C \) as \( \vec{r}(t) \), \( a \leq t \leq b \)

\( W = \int_{a}^{b} \vec{F} \cdot d\vec{r} = \int_{a}^{b} \vec{F} \cdot \dot{\vec{r}}(t) \, dt \)

\( = \int_{a}^{b} \vec{F} \cdot \vec{v}(t) \, dt = \int_{a}^{b} [m\ddot{v}(t) \cdot \vec{v}(t)] \, dt \)

\( = \int_{a}^{b} m(v'(t) \cdot v(t)) \, dt = \frac{m}{2} \int_{a}^{b} \dot{v}^2(t) \, dt = \frac{m}{2} \int_{a}^{b} \dot{v}^2(t) \, dt \)

\( = \frac{m}{2} \left[ v(t) \right]_{a}^{b} = \frac{m}{2} (v(b) - v(a))^2 = k(B) - k(A) \)

Hence, \( k(B) - k(A) = p(A) - p(B) \)

So, \( p(A) + k(A) = p(B) + k(B) \)