Lines and Curves in Space

Vector-Valued Function

\[ \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \]

One variable input, \( t \).
Output is a vector, \( \mathbf{r}(t) \).

Ex/ \( \mathbf{r}(t) = \langle t, t+2 \rangle \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \mathbf{r}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>0</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>1</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>2</td>
<td>(2, 4)</td>
</tr>
</tbody>
</table>

If we use the position vector for \( \mathbf{r}(t) \), we would have

Moreover, we can relate these position vectors to the points at the head of the vectors

This yields a parametric curve

\[ (t, t+2) \]

\[ x = t, \ y = t+2 \]
Lines in Space

The line \( l \) through the 2 points
\[ P(x_0, y_0, z_0) \]
\[ Q(x_1, y_1, z_1) \]
\[ \vec{PQ} = (x_1 - x_0, y_1 - y_0, z_1 - z_0) \]
\[ \text{a direction vector for} \ l \]

Let \( a = x_1 - x_0 \)
\[ b = y_1 - y_0 \]
\[ c = z_1 - z_0 \]
Then \( \vec{v} = (a, b, c) \) is a direction vector for \( l \).

Any point on \( l \) can be represented by
\[ \vec{r} = \vec{r}_0 + t\vec{v} \]
where \( t \) is some scalar.

Equation of a Line

passing through the point \( P(x_0, y_0, z_0) \) in the direction of vector \( \vec{v} = (a, b, c) \) is \( \vec{r} = \vec{r}_0 + t\vec{v} \), or
\[ \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle, \quad -\infty < t < \infty \]

Equivalently, the parametric equations of the line are
\[ x = x_0 + at \]
\[ y = y_0 + bt \]
\[ z = z_0 + ct \]
\[ -\infty < t < \infty \]
Ex/ Find an equation of the line passing through the point \( P(-3, 2, 0) \) that is perpendicular to the vectors \( \vec{u} = <-1, 1, 0> \) and \( \vec{w} = <2, 0, 1> \).

Need a point \( \vec{P}_0 = <-3, 2, 0> \).

2) the direction \( \vec{v}? \)

We can use \( \vec{u} \times \vec{w} \) (why?)

\[
\begin{vmatrix}
\vec{u} & \vec{w} & k \\
2 & 1 & 0 \\
-1 & 1 & 0 \\
2 & 0 & 1
\end{vmatrix}
\]

\[
\vec{u} \times \vec{w} = (1-0)i - (-1-0)j + (0-2)k
\]

\[
= <1, 1, -2>
\]

Thus,

\[
\vec{v} = <-3, 2, 0> + t <1, 1, -2>
\]

or \[
\begin{align*}
x &= -3 + t \\
y &= 2 + t \\
z &= -2t
\end{align*}
\]

Ex/ Find an equation of the line passing through the point \( P(5, 2, -1) \) that is parallel to the line \( \vec{r}(t) = <6-t, 3+2t, 7t> \).

1) a point \( \vec{P}_0 = <5, 2, -1> \).

2) direction \( \vec{v}? \) rewrite

\[
\vec{v} = <6, 3, 0> + t <1, 2, 7>
\]

Thus

\[
\vec{r}(t) = <5, 2, -1> + t <-1, 2, 7>
\]
Equation of a line segment between \( P(x_0, y_0, z_0) \) and \( Q(x_1, y_1, z_1) \):

A point \( P(x_0, y_0, z_0) \)

direction \( \vec{v} = \overrightarrow{PQ} = \langle x_1-x_0, y_1-y_0, z_1-z_0 \rangle \)

\[
\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle x_1-x_0, y_1-y_0, z_1-z_0 \rangle
\]

Segment between \( P \) and \( Q \) \( \Rightarrow \quad 0 \leq t \leq 1 \)

Why? check

Ex/ Does the line \( \vec{r}(t) = \langle 3+2t, 1-2t, 6+2t \rangle \) pass through the point \( (5,5,5) \)?

\[
\begin{align*}
5 &= 3+2t \\
5 &= 1-2t \\
5 &= 6+2t
\end{align*}
\]

Solve for \( t \)

\( t = 2 \)

Sub into 2 \( \Rightarrow \)

\( 5 = 1 - 2(2) \)

\( 5 = -1 \) \( \text{NO} \)

What about the point \( (4, -1, 8) \)?

\[
\begin{align*}
4 &= 3+2t \\
1 &= 1-2t \\
8 &= 6+2t
\end{align*}
\]

\( t = 1 \)

Yes, when \( t = 1 \)

Ex/ Do the following lines intersect?

\[
\begin{align*}
\vec{r}_1(t) &= \langle 2t, 1-t, 4+t \rangle \\
\vec{r}_2(s) &= \langle 3-5s, 7-2s, 1+s \rangle
\end{align*}
\]
Solve \( 2t = 3 - s \) \[ \Rightarrow s = 3 - 2t \]
\( 1-t = 7 - 2s \)
\( 4 + t = 1 + s \)

Solve into (2)

\( 1-t = 7 - 2(3-2t) \)
\( 1-t = 7 - 6 + 4t \)
\( 5t = 0 \)
\( t = 0 \) then \( s = 3 - 2(0) = 3 \)
check in (3)
\( 4 + (0) = 1 + (3) \)

\[ \begin{align*}
\vec{v}(0) &= \langle 0, 1, 4 \rangle \\
\vec{v}(3) &= \langle 0, 1, 4 \rangle
\end{align*} \]

Yes, at the point \((0,1,4)\)

Note: Nonparallel lines might NOT intersect in IR^3 (these lines are said to be skew.)

Example: Project the line segment \( \vec{v}(t) = \langle 2 + t, 4-t, 3 + 2t \rangle, 0 \leq t \leq 1 \)
onto the xy-plane.

In the xy-plane, \( z = 0 \).

The projection is then \( x = 2 + t, y = 4 - t, z = 0, 0 \leq t \leq 1 \)

We can eliminate the parameter to get
\[ t = 4 - y \rightarrow x = 2 + (4 - y) \]
\( 0 \leq 4 - y \leq 1 \rightarrow -1 \leq y - 4 \leq 0 \rightarrow 3 \leq y \leq 4 \)
\[ y = -x + 6 \]
\[ 3 \leq y \leq 4 \]
\[ (6y, 2 \leq x \leq 3) \]

\[ (2,3,0) \]
\[ (2,4,0) \]
\[ (2,4,3) \]
\[ (3,3,5) \]
Curves in Space

\( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \)

or \( x(t)^2 + y(t)^2 + z(t)^2 \)

on an interval \( a \leq t \leq b \)

The Domain of \( \mathbf{r}(t) \) is the largest set of values of \( t \) for which all of \( x(t), y(t), \) and \( z(t) \) are defined.

The orientation of the curve, \( C \), is the positive direction of \( C \). The positive (or forward) direction is the direction in which the curve is generated as the parameter increases.

\[ \text{Ex: } \mathbf{r}(t) = \langle \cos t, \sin t, 3 \rangle, \ 0 \leq t \leq 2\pi \]

\[ \text{Ex: } \mathbf{r}(t) = \langle \cos t, 3, \sin t \rangle, \ 0 \leq t \leq 2\pi \]
Ex/ \ \overrightarrow{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle, \ 0 \leq t < \infty

Projection on xy-plane
\langle 2 \cos t, 2 \sin t, 0 \rangle

Projection on xz-plane
\langle 2 \cos t, 0, t \rangle

Projection on yz-plane
\langle 0, 2 \sin t, t \rangle

Ex/ Find the domain of \overrightarrow{r}(t) = \frac{3}{t+2} \overrightarrow{i} - \sqrt{t+6} \overrightarrow{j} + e^{t} \overrightarrow{k}

\frac{3}{t+2} \rightarrow \text{Domain} \rightarrow -2

-\sqrt{t+6} \rightarrow \text{Domain} \rightarrow -6

e^{t} \rightarrow \text{Domain} \rightarrow -

Thus, Domain of \overrightarrow{r}(t)
\left[ -6, -2 \right) \cup \left( -2, \infty \right)
Definition - Limit of a Vector-Valued Function

A vector-valued function \( \vec{r} \) approaches the limit \( \vec{L} \) as \( t \) approaches \( a \), written

\[
\lim_{t \to a} \vec{r}(t) = \vec{L}
\]

provided

\[
\lim_{t \to a} |\vec{r}(t) - \vec{L}| = 0
\]

Suppose that \( \vec{L} = \langle L_1, L_2, L_3 \rangle \), \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) and

\[
\begin{align*}
\lim_{t \to a} x(t) &= L_1 \\
\lim_{t \to a} y(t) &= L_2 \\
\lim_{t \to a} z(t) &= L_3
\end{align*}
\]

Then

\[
\lim_{t \to a} \vec{r}(t) = \left( \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right) = \langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \rangle = \langle L_1, L_2, L_3 \rangle
\]

Prove that

\[
\lim_{t \to a} |\vec{r}(t) - \vec{L}| = 0 \quad \text{if and only if} \quad \ast
\]

Note: This is a (scalar) real-valued function of \( t \).

Note:

\[
\lim_{t \to a} (\vec{r}(t) + \vec{s}(t)) = \lim_{t \to a} \vec{r}(t) + \lim_{t \to a} \vec{s}(t)
\]

\[
\lim_{t \to a} c \vec{r}(t) = c \lim_{t \to a} \vec{r}(t)
\]

provided \( \lim_{t \to a} \vec{s}(t) \) and \( \lim_{t \to a} \vec{r}(t) \) exist.
Continuity

\[ \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \]

is continuous at \( t = a \) provided

1) \( \lim_{t \to a} \vec{r}(t) \) exists

2) \( \vec{r}(a) \) exists

3) \( \lim_{t \to a} \vec{r}(t) = \vec{r}(a) \)

\( \vec{r}(t) \) is continuous on an interval \( I \) if it is continuous for all \( t \) in \( I \).

Ex:

\[ \vec{r}(t) = \frac{\sin t}{t} \hat{i} + \ln(3+t) \hat{j} + e^t \hat{k} \]

Then

\[ \lim_{t \to 0} \vec{r}(t) = \left( \lim_{t \to 0} \frac{\sin t}{t} \right) \hat{i} + \left( \lim_{t \to 0} \ln(3+t) \right) \hat{j} + \left( \lim_{t \to 0} e^t \right) \hat{k} \]

\[ = \hat{i} + \ln 3 \hat{j} + \hat{k} \] or \( <1, \ln 3, 1> \)

Also, \( \vec{r}(0) \) is undefined (why?)

So, \( \vec{r}(t) \) is not continuous at \( t = 0 \)

[at \( <1, \ln 3, 1> \)]

\( \vec{r}(t) \) is continuous on its domain

which is \( (-3, 0) \cup (0, \infty) \)
Ex/ \ \vec{r}(t) = \cos\left(\frac{t\pi}{t^2+1}\right) \hat{i} + \sin\left(\frac{t\pi}{t^2+1}\right) \hat{j} + \frac{\pi}{t^2+1} \hat{k}, \ t \geq 0

\lim_{t \to \infty} \vec{r}(t) = \left(\lim_{t \to \infty} \cos\left(\frac{t\pi}{t^2+1}\right)\right) \hat{i} + \left(\lim_{t \to \infty} \sin\left(\frac{t\pi}{t^2+1}\right)\right) \hat{j} + \left(\lim_{t \to \infty} \frac{\pi}{t^2+1}\right) \hat{k}

= \cos 0 \hat{i} + \sin 0 \hat{j} + 0 \hat{k} = <1, 0, 0>

\vec{r}(t) \text{ is continuous for all } t > 0 \text{ (why?)}

Ex/ Find the points at which the curve \( \vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle \) intersects the plane \( z = \frac{1}{2} \).

Solve for \( t \) \quad \begin{align*}
\frac{1}{2} &= \sin(2t) \\
2t &= \frac{\pi}{2} + 2k\pi \text{ or } 2t = \frac{3\pi}{2} + 2k\pi \\
t &= \frac{\pi}{4} + k\pi \text{ or } t = \frac{3\pi}{4} + k\pi
\end{align*}

when...

\( t = \frac{\pi}{12} \rightarrow (x = \cos \frac{\pi}{12}, y = \sin \frac{\pi}{12}, z = \frac{1}{2}) \quad \text{(cos} \frac{\pi}{12}, \sin \frac{\pi}{12}, \frac{1}{2})

\( t = \frac{5\pi}{12} \rightarrow (x = \cos \frac{5\pi}{12}, y = \sin \frac{5\pi}{12}, z = \frac{1}{2}) \quad \text{(cos} \frac{5\pi}{12}, \sin \frac{5\pi}{12}, \frac{1}{2})

\( t = \frac{13\pi}{12} \rightarrow (\cos \frac{13\pi}{12}, \sin \frac{13\pi}{12}, \frac{1}{2}) \quad \text{[4 points in total]}

\( t = \frac{19\pi}{12} \rightarrow (\cos \frac{19\pi}{12}, \sin \frac{19\pi}{12}, \frac{1}{2}) \)