Analysis Algorithms for Large-Scale Networks

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Algorithm Analysis Overview

Asymptotic Complexity

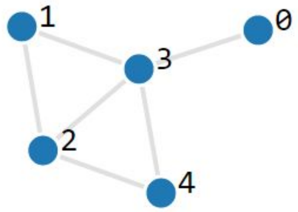
- Algorithms measured on time complexity and space complexity
  - Time complexity - how long an algorithm takes to complete
  - Space complexity - how much memory is needed for computation

O(n) Run time grows at least as fast as n	Ω(n) Run time grows at most as fast as n	Θ(n) Run time grows exactly as fast as n

Graph Representations

- Adjacency matrix
  - Space: $\Theta(V^2)$
  - Element query: $\Theta(1)$
- Adjacency list
  - Space: $\Theta(V + E)$
  - Element query: $\Theta(\text{degree}(V))$

0	0	0	1	0
0	0	1	1	0
0	1	0	1	1
1	1	1	0	1
0	0	1	1	0

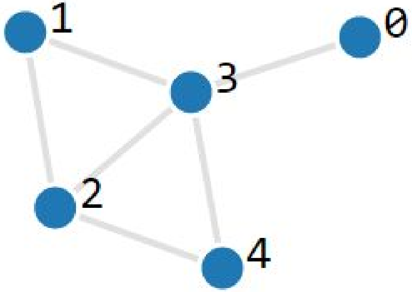


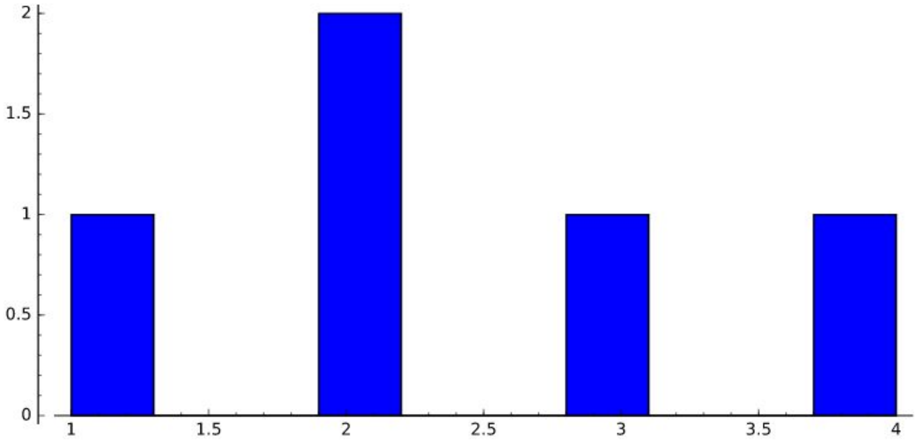
0	3			
1	2	3		
2	1	3	4	
3	0	1	2	4
4	2	3		

Degree Distribution

Degree Distribution

- Set of all degrees in a network [5]
    - Mean degree used as a measure of density of the network





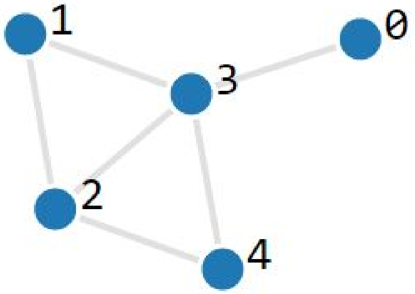
Degree Distribution

- Intuitively, will need to visit each vertex v and count edges incident on v
  - Can be performed in O(V + E) with an adjacency list
    - Loop through adjacency list
    - At each vertex, count the number of edges

- Alternatively, use a variant of breadth-first or depth-first search, both of which are O(V + E)

Degree Distribution

- Intuitive approach

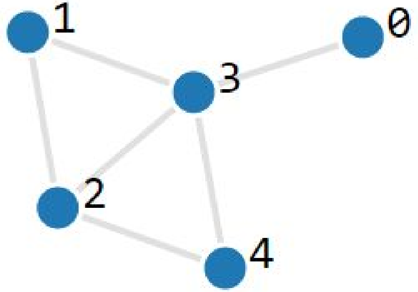


0	3				1
1	2	3			2
2	1	3	4		3
3	0	1	2	4	4
4	2	3			2

Characteristic Path Length

Characteristic Path Length

- Average length of shortest paths between all pairs of vertices in a graph [7]
- Can also look at diameter - longest shortest path

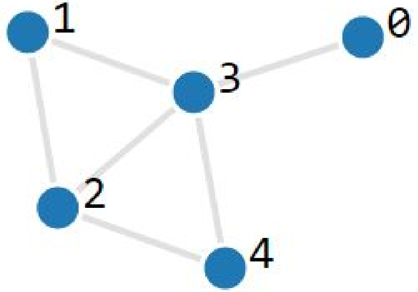


V	0	1	2	3	4
0	0	2	2	1	2
1	2	0	1	1	2
2	2	1	0	1	1
3	1	1	1	0	1
4	2	2	1	1	0

$L = 1.12$

Characteristic Path Length

- Need to solve all-pairs-shortest-path problem with an unweighted graph
    - Given a graph G, find the minimum distance $d_G(s, t)$ for all s, t $\in$ V



0 → 3 → 1	1 → 3 → 0	2 → 3 → 0	3 → 0	4 → 3 → 0
0 → 3 → 2	1 → 2	2 → 1	3 → 1	4 → 2 → 1
0 → 3	1 → 3	2 → 3	3 → 2	4 → 3 → 1
0 → 3 → 4	1 → 2 → 4	2 → 4	3 → 4	4 → 2
	1 → 3 → 4			4 → 3

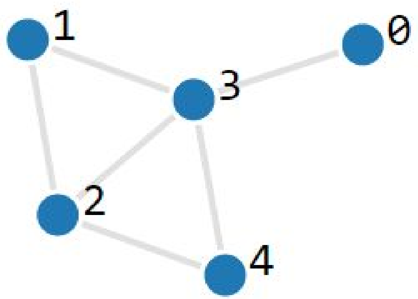
Characteristic Path Length

- Naive approach
    - Breadth-first search repeated for each vertex
    - BFS runs in $O(E + V)$ for one source node, so overall runtime is $O(EV + V^2)$
    - If graph is dense, this approaches $O(V^3)$
- Faster method
    - Reduce to matrix multiplication [6]
    - Runtime: $O(V^{2.376}\log V)$
- Even better method
    - dynamic programming - iteratively optimize a V*V matrix of shortest path lengths [3]
    - Runtime: $O(V^2\log V)$
    - Space: $O(V^2)$

Betweenness Centrality

Betweenness Centrality

- Probability that a given vertex falls on a randomly-selected shortest path between two other vertices in the network [2]



0 → 3 → 1	1 → 3 → 4
0 → 3 → 2	1 → 2 → 4
0 → 3 → 4	2 → 4
1 → 2	

$$C_B(3) = 4 / 7$$

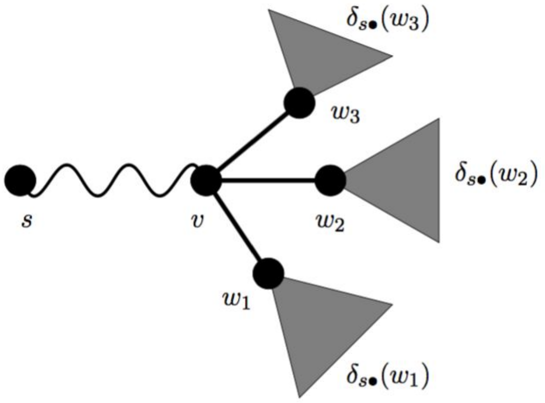
Betweenness Centrality - exact

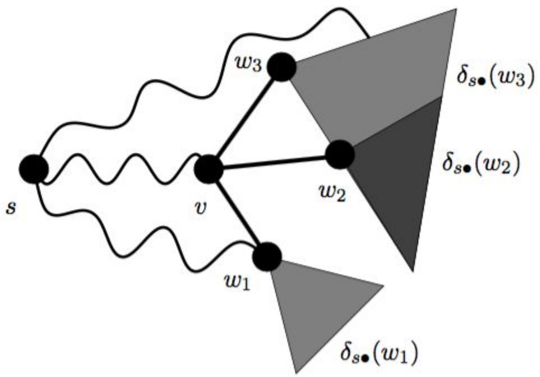
- Basic approach [1]
    1. Compute length and number of shortest paths between all pairs
        - Variation of all-pairs-shortest-path problem
    2. Sum all pair-dependencies
        - Pair-dependency - ratio of shortest paths between s and t containing v

- Takes $O(V^3)$ time to sum all pair-dependencies, and $O(V^2)$ space to store shortest paths

Betweenness Centrality - exact

- Faster method [1]
  - Runtime: O(VE) on unweighted graphs
    O(VE + V^2logV) on weighted graphs
  - Space: O(V + E)
  - Based on BFS for unweighted graphs or Djikstra's algorithm for weighted graphs
  - Use the fact that v is a predecessor of w to calculate a partial sum for dependency of s on v
  - Adding these partial sums together over all predecessors of w yields the pair-dependencies needed to calculate betweenness centrality





Betweenness Centrality - exact

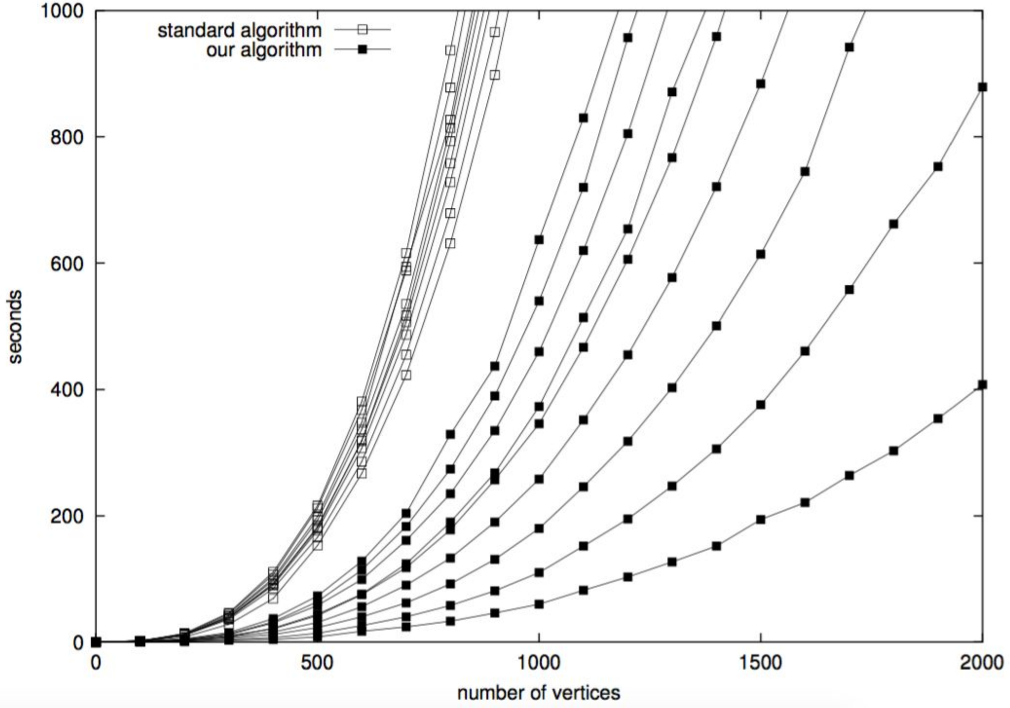
Algorithm 1: Betweenness centrality in unweighted graphs

```
C_B[v] ← 0, v ∈ V;
for s ∈ V do
    S ← empty stack;
    P[w] ← empty list, w ∈ V;
    σ[t] ← 0, t ∈ V;   σ[s] ← 1;
    d[t] ← −1, t ∈ V;   d[s] ← 0;
    Q ← empty queue;
    enqueue s → Q;
    while Q not empty do
        dequeue v ← Q;
        push v → S;
        foreach neighbor w of v do
            // w found for the first time?
            if d[w] < 0 then
                enqueue w → Q;
                d[w] ← d[v] + 1;
            end
            // shortest path to w via v?
            if d[w] = d[v] + 1 then
                σ[w] ← σ[w] + σ[v];
                append v → P[w];
            end
        end
    end
    δ[v] ← 0, v ∈ V;
    // S returns vertices in order of non-increasing distance from s
    while S not empty do
        pop w ← S;
        for v ∈ P[w] do δ[v] ← δ[v] + σ[v]/σ[w] · (1 + δ[w]);
        if w ≠ s then C_B[w] ← C_B[w] + δ[w];
    end
end
```

- Run modified BFS from source s
    1. Compute shortest path lengths and predecessor lists from s to v ∈ V
    2. Update betweenness centrality values for all v ∈ V based on dependency of s on v

- Repeat for all s ∈ V

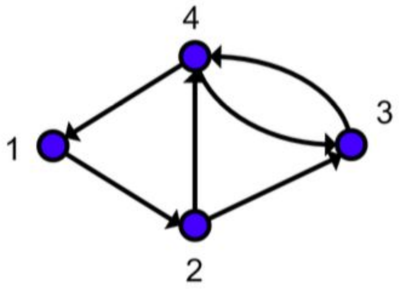
Betweenness Centrality - exact



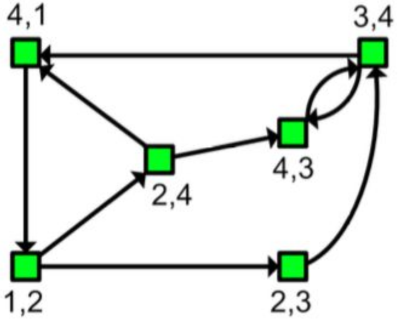
- Results on random, undirected, unweighted graphs for size 100-2000 vertices and density 10%-90% of all possible edges

Betweenness Centrality - approximate

- LINERANK algorithm [4]
  - Measure the importance of a node by summing the importance score of its incident edges
  - Importance score of an edge is the probability that a random walker traversing edges via nodes (with random restarts) will stay at the edge
    - Defined using a directed line graph



Original graph



Directed line graph

Betweenness Centrality - approximate

- LINERANK runtime: O(kE)
  - Run for k iterations
  - Each iteration improves the accuracy of the estimate, but reasonable accuracy can be achieved after only a few iterations
- LINERANK space: O(E)
  - Algorithm uses two incidence matrices, which hold only non-zero elements of the directed line graph, of which there are E elements

Questions?

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