Brenier’s polar factorization theorem and McCann’s generalization

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Theorem (Brenier's factorization theorem)

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $s: \Omega \to \mathbb{R}^n$ be a Borel map which does not map positive volume into zero volume. Then $s$ uniquely decomposes into the form $s = t \circ u$, where $u: \Omega \to \Omega$ is a volume preserving map and $t = \nabla \psi: \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a convex function $\psi: \mathbb{R}^n \to \mathbb{R}$.

McCann generalizes this result to Riemannian manifolds.

Question: What is the relation between this and optimal transport?

Answer: Proof depends on the solution of Monge-Kantorovich problem.
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Monge problem

- Let $M$ be a (topological) space and $\mu, \nu$ be (Borel) measures on $M$. Let $c : M \times M \rightarrow [0, \infty]$ be a function (it is called the cost function.)
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  $S(\mu, \nu) := \{ G : M \to M : G_*(\mu) = \nu \}$.

- Monge problem is finding the cost minimizing transport map $G$.
- Existence of the solution depends on properties $c$. In this presentation we assume that $M$ is a metric space and $c = d^2 / 2$. 

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Kantorovich problem

Let $p, q : M \times M \to M$ denote the projection onto the first coordinate and second coordinate respectively. The set of all transport plans from $\mu$ to $\nu$ is defined by

$$\Gamma(\mu, \nu) := \{\gamma \text{ a Borel measure on } M \times M : p_*(\gamma) = \mu, q_*(\gamma) = \nu\}.$$
Kantorovich problem

- Let $\Gamma(\mu, \nu) := \{\gamma \text{ a Borel measure on } M \times M : p_*(\gamma) = \mu, q_*(\gamma) = \nu\}$. The mathematical term for a transport plan is a coupling.
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$$C(\gamma) = \int_{M \times M} c(x, y) d(\gamma(x, y)).$$
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Kantorovich problem is a relaxation of the Monge problem in the following sense:
The map $S(\mu, \nu) \to \Gamma(\mu, \nu)$ given by $G \mapsto (id_M \times G)_*(\mu)$ is a cost preserving embedding.
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The image of the map above is the set of measures in \( \Gamma(\mu, \nu) \) whose support is a graph.
Relation between Monge and Kantarovich Problem

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- $\Gamma(u, v)$ is a convex subset of a Banach space (i.e. dual space of the continuous functions $(C(M \times M), l_\infty)$). This is helpful for showing the existence and uniqueness of solutions.
Existence of Monge solutions, uniqueness of Kantorovich solutions

Let $M$ be an $n$-dimensional connected compact Riemannian manifold, and $\mu, \nu$ be Borel measures on $M$. Then there is a convex potential function $\psi : M \to \mathbb{R}$ such that

- $G(x) := \exp_x(\nabla \psi)$ is a transport map.
- $G$ is the only transport map arising this way. It solves Monge’s problem.
- Kantorovich problem has a unique solution.
- Kantorovich problem is obtained from $G$. 
McCann’s Factorization Theorem

Let $M$ be a connected compact Riemannian manifold. Let $s : M \to M$ be a Borel map which never maps positive volume into zero volume. Then $s$ factors uniquely into the form $s = t \circ u$, where

$u : M \to M$ is a volume preserving map and

$t = \exp(\nabla \psi) : M \to M$

where $\psi$ is a convex function $\psi : M \to \mathbb{R}$. 
Idea of Proof

- Let $\mu$ be the Riemannian volume measure on $M$ and let $\nu = s_*(\mu)$. 

- Let $t$ be the solution of the Monge problem $S(\mu, \nu)$ arising from the potential $\psi : M \to \mathbb{R}$.

- Let $t^*$ be the solution of the Monge problem $S(\nu, \mu)$. Show that $t, t^*$ are inverses almost everywhere.

- Let $u = t^* \circ s$.

- Then $t \circ u = s$, $\mu$ almost everywhere. Furthermore $u^*(\mu) = t^* \circ s^*(\mu) = t^* \circ \nu = \mu$, hence $u$ is measure preserving.
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- Then $t \circ u = s$, $\mu$ almost everywhere. Furthermore $u_*(\mu) = t^*_s(\mu) = t^*(\nu) = \mu$, hence $u$ is measure preserving.