Some Definition and Example of Markov Chain

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Introduction

- Definition and Notation
- Simple example of Markov Chain

Aim

Have some taste of Markov Chain and how it relate to some applications

Definition

A sequence of random variables $(X_0, X_1, \ldots)$ is a **Markov Chain** with state space Ω and transition matrix P if for all $x, y \in \Omega$, all $t \geq 1$, and all events $H_{t-1} = \cap_{s=0}^{t-1}\{X_s = x_s\}$ satisfying $P(H_{t-1} \cap \{X_t = x\}) > 0$, we have:

$$P\{X_{t+1} = y | H_{t-1} \cap \{X_t = x\}\} = P\{X_{t+1} = y | X_t = x\} = P(x, y).$$

We store distribution information in a row vector μ_t, we have:

$$\mu_t = \mu_{t-1} P \text{ for all } t \geq 1.$$

μ_t has a limit π (whose value depend on p and 1), as $t \rightarrow 0$, satisfying:

$$\pi = \pi P$$

Definition

if we multiply a column vector f by P on the left and f is a function on the state space Ω:

$$Pf(x) = \sum_y P(x,y)f(y) = \sum_y f(y)P_x\{X_1 = y\} = E_x(f(X_1))$$

That is, the $x - th$ entry of Pf tells us the expected value of the function f at tomorrow's state, given that we are at state x today. Multiplying a column vector by P on the left takes us from a function on the state space to the expected value of that function tomorrow.

Definition

A **random mapping representation** of a transition matrix P on state space Ω is a function $f : \Omega \times \Lambda \Rightarrow \Omega$, along with a Λ-valued random variable Z, satisfying:

$$P\{f(x, Z) = y\} = P(x, y).$$

Irreducibility and Aperiodicity

A chain P is called **irreducible** if for any two states $x, y \in \Omega$ there exists an integer t (possibly depending on x and y) such that $P^t(x, y) > 0$.

let $\Gamma(x) := \{t \geq 1 | P^t(x, x) > 0\}$ be the set of times when it is possible for the chain to return to starting position x. The **period** of state x is define to be the greatest common divisor of $\Gamma(x)$.

LEMMA

If P is irreducible, then $gcd\ \Gamma(x) = gcd\ \Gamma(y)$ for all $x, y \in \Omega$.

Irreducibility and Aperiodicity

The chain will be called **aperiodic** if all states have period 1. If a chain is not aperiodic, we call it **periodic**.

Given an arbitrary transition matrix P, let $Q = \frac{I+P}{2}$ (I is the $|\Omega| \times |\Omega|$ identity matrix), we call Q a **lazy version** of P

Random Walks on Graph

Given a graph $G = (V, E)$, we can define **simple random walk** on G to be the Markov chain with state space V and transition matrix $P(x, y) = \frac{1}{deg(x)}$ if x y, 0 otherwise.

Stationary Distribution

Recall that a distribution π on Ω satisfying

$$\pi = \pi P$$

We cal π satisfying a **stationary distribution** of the Markov Chain.
In the simple random walk example:

$$\pi(y) = \sum_{x \in \Omega} \pi(x) P(x, y) = \frac{deg(y)}{2|E|}$$

Stationary Distribution

We define a **hitting time** for $x \in \Omega$ to be

$$\Gamma_x := \min\{t \geq 0 : X_t = x\},$$

and **first return time**

$$\Gamma_x^+ := \min\{t \geq 1 : X_t = x\} \text{ when } X_0 = x$$

LEMMA

For any x, y of an irreducible chain, $E_x(\Gamma_y^+) < \infty$

Classifying States

Given $x, y \in \Omega$, we say that y is **accessible** from x and write $x \to y$ if there exists an $r > 0$ such that $P^r(x, y) > 0$.

A state $x \in \Omega$ is called **essential** if for all y such that $x \to y$ it is also true that $y \to x$.

We say that x communicates with y and write $x \leftrightarrow y$ if and only if $x \to y$ and $y \to x$. The equivalence classes under $\leftrightarrow$ are called communicating classes. For $x \in \Omega$, the communicating class of x is denoted by [x].

If [x]=$\{$x$\}$, such state is called **absorbing**.

LEMMA

If x is an essential state and $x \to y$, then y is essential

Examples

Gambler

Assume that a gambler making fair unit bets on coin flips will abandon the game when her fortune falls to 0 or rises to n. Let X_t be gambler's fortune at time t and let τ be the time required to be absorbed at one of 0 or n. Assume that $X_0 = k$, where $0 \le k \le n$. Then

$$P_k\{X_\tau = n\} = k/n$$

and

$$E_k(\tau) = k(n-k)$$

Examples

Coupon Collecting

Consider a collector attempting to collect a complete set of coupons. Assume that each new coupon is chosen uniformly and independently from the set of n possible types, and let τ be the (random) number of coupons collected when the set first contains every type. Then

$$E(\tau) = n \sum_{k=1}^{n} \frac{1}{k}$$

Examples

Random walk on Group

Given a probability distribution μ on a group (G, Δ), we define the random walk on G with increment distribution μ as follows: it is a Markov chain with state space G and which moves by multiplying the current state on the left by a random element of G selected according to μ. Equivalently, the transition matrix P of this chain has entries

$$P(g, hg) = \mu(h) \text{ for all } g, h \in G$$