1) (20 points) Let $R$ be the region bounded by the curves $y = x^2$ and $y = 2 - x^2$.

(a) Sketch $R$ and find its area.

\[
\begin{align*}
2 - x^2 &= x^2 \\
2x^2 - 1 &= 0 \\
2(x-1)(x+1) &= 0 \\
x &= -1, 1
\end{align*}
\]

\[
A = \int_{-1}^{1} \left| (2 - x^2) - x^2 \right| \, dx
\]

\[
= \int_{-1}^{1} (2 - 2x^2) \, dx
\]

\[
= 2x - \frac{2}{3}x^3 \bigg|_{-1}^{1}
\]

\[
= (2 - \frac{2}{3}) - (-2 + \frac{2}{3}) = 4 - \frac{4}{3} = \frac{8}{3}
\]

(b) Set up (but do not evaluate) an integral to find the volume of the solid obtained by rotating $R$ about the line $x = 2$. You do not need to simplify the integrand.

\[
V = \int_{-1}^{1} 2\pi x h \, dx = \int_{-1}^{1} 2\pi (2-x)(2-2x^2) \, dx
\]

(c) Set up (but do not evaluate) an integral or a pair of integrals to calculate the length of the perimeter of $R$. You do not need to simplify the integrand.

\[
P = \text{length of "top" curve } + \text{length of "bottom" curve}
\]

\[
= \text{length of curve } y = 2 - x^2, -1 \leq x \leq 1 + \text{length of curve } y = x^2, -1 \leq x \leq 1
\]

\[
= \int_{-1}^{1} \sqrt{1 + \left( \frac{d}{dx}[2 - x^2] \right)^2} \, dx + \int_{-1}^{1} \sqrt{1 + \left( \frac{d}{dx}[x^2] \right)^2} \, dx
\]

\[
= \int_{-1}^{1} \sqrt{1 + 4x^2} \, dx + \int_{-1}^{1} \sqrt{1 + 4x^2} \, dx = 2 \int_{-1}^{1} \sqrt{1 + 4x^2} \, dx
\]
2) (20 points) The region bounded by the curves $y = x^3$ and $y = -x^3$ and the line $y = 8$ forms the back of a large wall-mounted aquarium. Cross sections parallel to the floor (i.e. $x$-axis) are half-circles.

(a) Set up (but do not evaluate) an integral to find the total volume of the aquarium. You do not need to simplify the integrand.

$$V = \int_0^8 \frac{1}{2} \pi r^2 \, dy$$

$$= \int_0^8 \frac{1}{2} \pi \left( \frac{2}{3} \sqrt[3]{y} \right)^2 \, dy$$

$$= \int_0^8 \frac{1}{2} \pi \frac{4}{9} y^{2/3} \, dy$$

(b) Set up (but do not evaluate) an integral to calculate the work needed to pump all of the water out of the aquarium if it has been filled to a depth of 7 m. You do not need to simplify the integrand.

(The density of water is 1000 kg/m³, and the acceleration due to gravity is 9.8 m/s².)

$$dW = \text{Weight}_{\text{water}} \cdot \text{distance}_{\text{shovt}}$$

$$= (\rho_{\text{water}} \cdot \delta \cdot y) \cdot (8 - y)$$

$$= \frac{1}{2} \pi y^{2/3} \, dy \cdot 1000 \cdot 9.8 \cdot (8 - y)$$

$$W = \int_0^7 \left[ \frac{1}{2} \pi y^{2/3} \cdot 1000 \cdot 9.8 \cdot (8 - y) \right] \, dy$$
3) (20 points) Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 40 cm. (Don’t forget to include units in your final answers.)

(a) How much work is needed to stretch the spring from 35 cm to 40 cm?

\[ 2 = \int_0^{10 \text{ cm}} kx \, dx = \int_0^{10 \text{ cm}} kx \, dx = k \frac{x^2}{2} \bigg|_0^{1/10} = \frac{k}{2} \cdot \left(\frac{1}{10}\right)^2 = \frac{k}{200} \]

so \( k = 2 \cdot 200 = 400 \)

\[ W = \int_{5 \text{ cm}}^{10 \text{ cm}} kx \, dx = \int_{1/20}^{1/10} 400x \, dx = 200 \left( \frac{x^2}{2} \right) \bigg|_{1/20}^{1/10} = 200 \left( \frac{1}{100} - \frac{1}{400} \right) = 200 \left( \frac{3}{400} \right) = \frac{3}{2} \text{ J} \]

(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

\[ 30 = k \cdot x_o = 400 \cdot x_o \]

\[ x_o = \frac{30}{400} \text{ m} = \frac{15}{100} \text{ m} = 15 \text{ cm} \]
4) (20 points) Evaluate the following limits.

(a) \[ \lim_{x \to 0} \frac{\sin 4x}{\tan 5x} = \lim_{x \to 0} \frac{4 \cos 4x}{5 \sec^2 5x} = \frac{4 \cos 0}{5 \sec^2 0} = \frac{4}{5} \]

(b) \[ \lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = -\frac{1}{6} \]

(c) \[ \lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\ln x^{1/x}} = \lim_{x \to \infty} e^{\frac{1}{x} \ln x} = e^{\lim_{x \to \infty} \frac{\ln x}{x}} = e^{\lim_{x \to \infty} \frac{x}{x} \cdot \frac{\ln x}{x}} = e^0 = 1 \]
5) (20 points) The integral
\[ \int_0^\infty \frac{1}{\sqrt{x(1+x)}} \, dx \]
is improper for two reasons: The interval \([0, \infty)\) is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:
\[ \int_0^\infty \frac{1}{\sqrt{x(1+x)}} \, dx = \int_0^1 \frac{1}{\sqrt{x(1+x)}} \, dx + \int_1^\infty \frac{1}{\sqrt{x(1+x)}} \, dx \]

\[ \Box \]
\[ \int_{\frac{1}{\sqrt{x(1+x)}}} \, dx = 2 \int_{\tan^{-1}u}^1 du = 2 \tan^{-1}u + C = 2 \tan^{-1}\sqrt{x} + C \]

\[ u = \frac{1}{\sqrt{x}} \quad , \quad u^2 = x \]
\[ du = \frac{1}{2\sqrt{x}} \, dx \]
\[ 2du = \frac{1}{\sqrt{x}} \, dx \]

\[ \Box \]
\[ \int_0^\infty \frac{1}{\sqrt{x(1+x)}} \, dx = \int_0^1 \frac{1}{\sqrt{x(1+x)}} \, dx + \int_1^\infty \frac{1}{\sqrt{x(1+x)}} \, dx \]

\[ = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x(1+x)}} \, dx + \lim_{s \to \infty} \int_1^s \frac{1}{\sqrt{x(1+x)}} \, dx \]

\[ = \lim_{t \to 0^+} 2\tan^{-1}\sqrt{x} \bigg|_t^1 + \lim_{s \to \infty} 2\tan^{-1}\sqrt{x} \bigg|_1^s \]

\[ = 2\tan^{-1}1 - 2\lim_{t \to 0^+} \tan^{-1}\sqrt{t} + 2\lim_{s \to \infty} \tan^{-1}\sqrt{s} - 2\tan^{-1}1 \]

\[ = -2\tan^{-1}(0) + 2 \cdot \frac{\pi}{2} \]

\[ = 0 + \pi = \pi \]