EXAMPLE PROBLEMS #1

9.3.#68) Telescoping series

For the telescoping series, \( \sum_{k=1}^{\infty} (\tan^{-1}(k+1) - \tan^{-1} k) \), find a formula for the \( n \)th term of the sequence of partial sums \( \{S_n\} \). Then evaluate \( \lim_{n \to \infty} S_n \) to obtain the value of the series or state that the series diverges.

Solution: Let \( c_k = \tan^{-1} k \). Then

\[
\sum_{k=1}^{\infty} (\tan^{-1}(k+1) - \tan^{-1} k) = \sum_{k=1}^{\infty} (c_{k+1} - c_k).
\]

So the partial sums are given by

\[
S_n = \sum_{k=1}^{n} (c_{k+1} - c_k)
\]

\[
= (c_2 - c_1) + (c_3 - c_2) + \ldots + (c_n - c_{n-1}) + (c_{n+1} - c_n)
\]

\[
= (c_{n+1} - c_1)
\]

\[
= \tan^{-1}(n+1) - \tan^{-1} 1
\]

\[
= \tan^{-1}(n+1) - \frac{\pi}{4}.
\]

Then

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( \tan^{-1}(n+1) - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\]
9.5.14) Ratio Test

Use the Ratio Test to determine whether the series \( \sum_{k=1}^{\infty} \frac{k!}{k^k} \) converges.

Solution: For \( a_k = \frac{k!}{k^k} \), we have (since \( a_k > 0 \) for all \( k \))

\[
\left| \frac{a_{k+1}}{a_k} \right| = a_{k+1} \cdot a_k^{-1} = \left( \frac{k+1}{k} \right)^k
\]

\[
= \frac{(k+1)!}{k!} \cdot \frac{k^k}{(k+1)^{k+1}} = \left( \frac{k+1}{k} \right)^k
\]

Let \( f(x) = \left( \frac{x}{x+1} \right)^x \) (we’re going to be using l’Hospital’s Rule eventually, so we switch to a differentiable function). Then

\[
\ln f(x) = x \ln \left( \frac{x}{x+1} \right) = \ln \left( \frac{x}{x+1} \right)^x
\]

As \( x \to \infty \), \( \ln \left( \frac{x}{x+1} \right) \to 0 \) and \( x^{-1} \to 0 \). By l’Hospital’s Rule,
\[
\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln x - \ln(x + 1)}{x^{-1}} \\
= H \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-x^{-2}} \\
= \lim_{x \to \infty} \frac{1}{x(x + 1)} \cdot (-x^2) \\
= - \lim_{x \to \infty} \frac{x^2}{x^2 + x} \\
= -1.
\]

So
\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^{\lim_{x \to \infty} \ln f(x)} = e^{-1} = \frac{1}{e}.
\]

Since \( \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = f(k) \) for all natural numbers \( k \) and \( \lim_{x \to \infty} f(x) \) exists, by Theorem 9.1,
\[
\lim_{k \to \infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = \lim_{x \to \infty} f(x) = \frac{1}{e} < 1.
\]

By the Ratio Test, \( \sum_{k=1}^{\infty} \frac{k!}{k^k} \) converges.