7. Techniques of Integration

Review of Basic Integrals

Constant, Powers

1. \[ \int k \, dx = kx + C. \]

2. \[ \int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|x| + C, & n = -1 \end{cases} \]

Exponentials

3. \[ \int e^x \, dx = e^x + C. \]

4. \[ \int a^x \, dx = \frac{a^x}{\ln a} + C, \quad a \neq 1, \ a > 0. \]

Trigonometric Functions

5. \[ \int \sin x \, dx = -\cos x + C. \]

6. \[ \int \cos x \, dx = \sin x + C. \]

7. \[ \int \sec^2 x \, dx = \tan x + C. \]

8. \[ \int \csc^2 x \, dx = -\cot x + C. \]

9. \[ \int \sec x \tan x \, dx = \sec x + C. \]

10. \[ \int \csc x \cot x \, dx = -\csc x + C. \]

11. \[ \int \tan x \, dx = \ln|\sec x| + C. \]

12. \[ \int \cot x \, dx = \ln|\sin x| + C. \]

13. \[ \int \sec x \, dx = \ln|\sec x + \tan x| + C. \]

14. \[ \int \csc x \, dx = \ln|\csc x - \cot x| + C. \]

Algebraic Functions

15. \[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C. \]

16. \[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C. \]

17. \[ \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C. \]

18. \[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C. \]

Hyperbolic Functions

19. \[ \int \sinh x \, dx = \cosh x + C. \]

20. \[ \int \cosh x \, dx = \sinh x + C. \]
7.1 Integration by Parts

The Formula

(a) Indefinite integrals: \( \int u \, dv = uv - \int v \, du. \)

(b) Definite integrals: \( \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du. \)

**Rules of Thumb**

1. The ILATE Rule. A rule of thumb for the choice of \( u \) and \( dv \) is to choose \( u \) by whichever function comes first in this list:

   - I : inverse trigonometric functions: \( \tan^{-1} x, \sin^{-1} x \), etc.
   - L : logarithmic functions: \( \ln x \), etc.
   - A : algebraic functions: \( x^2, 3x^5 \), etc.
   - T : trigonometric functions: \( \sin x, \tan x \), etc.
   - E : exponential functions: \( e^x \), etc.

Then make \( dv \) the other function. You can remember the list by the mnemonic ILATE. The reason for this is that functions longer down in the list have easier antiderivatives than the functions above them.

\[
\text{[Hw 7.1.3]} \quad \int x \cos 5x \, dx. \\
\text{[Hw 7.1.9]} \quad \int \ln(2x + 1) \, dx.
\]

2. In some cases repeated integration by parts is needed.

\[
\text{[Hw 7.1.7]} \quad \int x^2 \sin \pi x \, dx. \\
\text{[Hw 7.1.13]} \quad \int (\ln x)^2 \, dx.
\]

\[
\text{[Hw 7.1.15]} \quad \int e^{2\theta} \sin 3\theta \, d\theta.
\]

3. In some cases a substitution is needed or preferred before integration by parts.

\[
\text{[Hw 7.1.27]} \quad \int \cos x \ln(\sin x) \, dx. \\
\text{[Hw 7.1.29]} \quad \int \cos(\ln x) \, dx.
\]

\[
\text{[Hw 7.1.33]} \quad \int \sin \sqrt{x} \, dx. \\
\text{[Hw 7.1.35]} \quad \int_{\sqrt{\pi}/2}^{\pi} \theta^3 \cos(\theta^2) \, d\theta.
\]
7.2 Trigonometric Integrals

Useful Identities

Pythagorean Identities

1. \( \sin^2 x + \cos^2 x = 1. \)
2. \( 1 + \tan^2 x = \sec^2 x. \)
3. \( 1 + \cot^2 x = \csc^2 x. \)

Half-Angle Identities

4. \( \sin^2 \frac{x}{2} = \frac{1}{2} \left(1 - \cos 2x \right). \)
5. \( \cos^2 \frac{x}{2} = \frac{1}{2} \left(1 + \cos 2x \right). \)

Product Identities

6. \( \sin mx \cos nx = \frac{1}{2} \left[ \sin(m + n)x + \sin(m - n)x \right]. \)
7. \( \sin mnx \sin nx = -\frac{1}{2} \left[ \cos(m + n)x - \cos(m - n)x \right]. \)
8. \( \cos mx \cos nx = \frac{1}{2} \left[ \cos(m + n)x + \cos(m - n)x \right]. \)
Type 1 ($\int \sin^m x \cos^n x \, dx$)

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx.$$ 

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \sin^n x \sin x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx.$$ 

Then substitute $u = \cos x$.

(c) If the powers of both sine and cosine are even, use the half-angle identities. It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x.$$

**Examples**

[Hw 7.2.13] $\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx$.  

[Hw 7.1.17] $\int \cos^2 x \tan^3 x \, dx$.  


Type 2 ($\int \tan^m x \sec^n x \, dx$)

(a) If the power of secant is even ($n = 2k, \ k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx.$$ 

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx.$$ 

Then substitute $u = \sec x$.

**Examples**

[Hw 7.2.21] $\int \sec^2 x \tan x \, dx$.

Type 3 ($\int \sin mx \cos nx \, dx, \int \sin mx \sin nx \, dx, \int \cos mx \cos nx \, dx$)

Use product identities.

**Examples**

[Hw 7.2.41] $\int \sin 5x \sin 2x \, dx$. 
7.3 Trigonometric Substitution

The Rules

<table>
<thead>
<tr>
<th>Radical</th>
<th>Substitution</th>
<th>Restriction on $\theta$</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2 - x^2}$</td>
<td>$x = a \sin \theta$</td>
<td>$-\pi/2 \leq \theta \leq \pi/2$</td>
<td>$a \cos \theta$</td>
</tr>
<tr>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$x = a \tan \theta$</td>
<td>$-\pi/2 &lt; \theta &lt; \pi/2$</td>
<td>$a \sec \theta$</td>
</tr>
<tr>
<td>$\sqrt{x^2 - a^2}$</td>
<td>$x = a \sec \theta$</td>
<td>$0 \leq \theta \leq \pi, \theta \neq \pi/2$</td>
<td>$a \tan \theta$</td>
</tr>
</tbody>
</table>

Examples

[HW 7.3.25] $\int \frac{1}{\sqrt{9x^2 + 6x - 8}} \, dx$.  [HW 7.3.29] $\int x \sqrt{1 - x^4} \, dx$. 
7.4 Integration of Rational Functions by Partial Fractions

Find the Partial Fraction Expansion of a Rational Function $f(x) = p(x)/q(x)$

Step 1. If $f(x)$ is improper, that is, if $p(x)$ is of degree at least that of $q(x)$, divide $p(x)$ by $q(x)$, obtaining

$$f(x) = \text{a polynomial} + \frac{N(x)}{D(x)}.$$  

Step 2. Factor $D(x)$ into a product of linear and irreducible quadratic factors with real coefficients.

Step 3. For each factor of the form $(ax + b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}.$$  

Step 4. For each factor of the form $(ax^2 + bx + c)^m$, expect the decomposition to have the terms

$$\frac{B_{1x} + C_1}{(ax^2 + bx + c)} + \frac{B_{2x} + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_{mx} + C_m}{(ax^2 + bx + c)^m}.$$  

Step 5. Set $N(x)/D(x)$ equal to the sum of all the terms found in Step 3 and 4.

Step 6. Multiply both sides of the equation found in Step 5 by $D(x)$ and solve for the unknown constants. This can be done by either of the two methods: (1) equate coefficients of like powers or (2) assign convenient values to the variables $x$ (recommended).

Rationalizing Substitutions

For integrands involving $\sqrt{ax + b}$, it might be effective to use the substitution $u = \sqrt{ax + b}$.

**Examples**

[Hw 7.4.31] \[\int \frac{1}{x^3 - 1} \, dx.\]
7.8 Improper Integrals

Type 1: Infinite Intervals

- **Definition.**
  
  (a) If \( \int_a^t f(x) \, dx \) exists for every number \( t \geq a \), then
  \[
  \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx,
  \]
  provided this limit exists (as a finite number).

  (b) If \( \int_t^b f(x) \, dx \) exists for every number \( t \leq b \), then
  \[
  \int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx,
  \]
  provided this limit exists (as a finite number).

  The improper integrals \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^b f(x) \, dx \) are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

  (c) If both \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^a f(x) \, dx \) are convergent, then we define
  \[
  \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx.
  \]

- **The p-Rule.** \( \int_1^\infty \frac{1}{x^p} \, dx \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

- **A Comparison Test.** Suppose that \( f \) and \( g \) are continuous functions with \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \).

  (a) If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \) is convergent.

  (b) If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is divergent.

- **Examples**

  [Hw 7.8.13] \( \int_{-\infty}^\infty xe^{-x^2} \, dx \).

  [Hw 7.8.21] \( \int_1^\infty \frac{\ln x}{x} \, dx \).
Type 2: Discontinuous Integrands

- **Definition.**
  
  (a) If $f$ is continuous on $[a, b)$ and is discontinuous at $b$, then
  \[
  \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx,
  \]
  if this limit exists (as a finite number).
  
  (b) If $f$ is continuous on $(a, b]$ and is discontinuous at $a$, then
  \[
  \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx,
  \]
  if this limit exists (as a finite number).
  
  The improper integrals $\int_a^b f(x) \, dx$ is called *convergent* if the corresponding limit exists and *divergent* if the limit does not exist.
  
  (c) If $f$ has a discontinuity at $c$, where $a < c < b$, and both $\int_a^c f(x) \, dx$ and $\int_c^b f(x) \, dx$ are convergent, then we define
  \[
  \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
  \]

**Examples**

[Hw 7.8.37] $\int_{-1}^1 \frac{e^x}{e^x - 1} \, dx.$

[Hw 7.8.39] $\int_0^2 z^2 \ln z \, dz.$
8. Further Applications of Integration

8.1 Arc Length

(a) If the curve is given by \( y = f(x), \ a \leq x \leq b, \) and \( f' \) is continuous on \([a, b]\), then the length of the curve is

\[
L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

(b) If the curve is given by \( x = g(y), \ c \leq y \leq d, \) and \( g' \) is continuous on \([c, d]\), then the length of the curve is

\[
L = \int_c^d \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy.
\]

[Examples]

[Hw 8.1.9] Find the length of the curve \( x = \frac{1}{3} \sqrt{y} (y - 3), \ 1 \leq y \leq 9. \)
8.2 Area of a Surface of Revolution

(a) Let \( y = f(x), \ a \leq x \leq b, \) determines a smooth curve in the upper half of the \( xy \)-plane. The area of the surface generated by revolving the curve about the \( x \)-axis is given by

\[
S = 2\pi \int_a^b y \, ds = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx.
\]

(b) Let \( x = g(y), \ c \leq y \leq d, \) determines a smooth curve in the right half of the \( xy \)-plane. The area of the surface generated by revolving the curve about the \( y \)-axis is given by

\[
S = 2\pi \int_c^d x \, ds = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2} \, dy.
\]

Examples

[Hw 8.2.11] Find the area of the surface obtained by rotating the curve \( x = \frac{1}{3} (y^2 + 2)^{3/2}, \ 1 \leq y \leq 2 \) about the \( x \)-axis.

[Hw 8.2.15] Find the area of the surface obtained by rotating the curve \( x = \sqrt{a^2 - y^2}, \ 0 \leq y \leq a/2 \) about the \( y \)-axis.