15.3.20) Find the volume of the solid under the surface \( z = 2x + y^2 \) and above the region bounded by \( z = y^2 \) and \( x = y^3 \).

Solution: \( y^2 = y^3 \) at \( y = 0 \) and \( y = 1 \). \( R = \{(x,y) : y^2 \leq x \leq y^3, 0 \leq y \leq 1\} \).

\[
V = \iint_R (2x+y^2) \, dA = \int_0^1 \int_{y^2}^{y^3} (2x+y^2) \, dx \, dy
\]

\[
= \int_0^1 \left[ x^2 + xy^2 \right]_{y^2}^{y^3} \, dy
\]

\[
= \int_0^1 (y^6 - y^5) \, dy
\]

\[
= \frac{2}{5} y^5 - \frac{1}{6} y^6 - \frac{1}{7} y^7 \bigg|_0^1
\]

\[
= \frac{2}{5} - \frac{1}{6} - \frac{1}{7} = \frac{19}{210} \approx 0.090476
\]

15.3.46) Evaluate \( \int_0^\pi \int_0^{\sqrt{\pi}} \cos(x^2) \, dx \, dy \) by reversing the order of integration.

Solution: \( \{(x,y) : y \leq x \leq \pi, 0 \leq y \leq \sqrt{\pi}\} \)

\[
\int_0^\pi \int_{y}^{\pi} \cos(x^2) \, dx \, dy = \int_0^{\sqrt{\pi}} \int_y^{\pi} \cos(x^2) \, dx \, dy
\]

\[
= \int_0^{\sqrt{\pi}} \left. x \cos(x^2) \right|_0^\pi \, dy
\]

\[
= \frac{1}{2} \sin(x^2) \bigg|_0^{\sqrt{\pi}} = \frac{1}{2} \sin(\pi) - \frac{1}{2} \sin(0) = 0
\]
Use polar coordinates to find the volume inside the sphere 
\[ x^2 + y^2 + z^2 = 16 \] and outside the cylinder \( x^2 + y^2 = 4 \).

Solution: For \( f(x,y) = \sqrt{16-x^2-y^2}, \ g(x,y) = -\sqrt{16-x^2-y^2} \), \( R = \{(x,y) : 4 \leq x^2+y^2 \leq 16\} \), the volume in question is given by

\[
V = \iint_R f(x,y) - g(x,y) \, dA
\]

\[
= \iint_R 2 \sqrt{16-x^2-y^2} \, dA
\]

In polar coordinates, \( x^2 + y^2 = r^2 \) and \( R = \{(r,\theta) : 2 \leq r \leq 4, \ 0 \leq \theta \leq 2\pi\} \).

So,

\[
V = \int_0^{2\pi} \int_2^4 2 \sqrt{16-r^2} \ r \, dr \, d\theta
\]

\[
= \frac{2\pi}{3} \left[ (16-r^2)^{3/2} \right]_2^4
\]

\[
= \frac{2\pi}{3} \left[ 12 \right]
\]

\[
= \frac{2\pi}{3} \cdot 12^2 \cdot \theta \bigg|_0^{2\pi}
\]

\[
= 16\sqrt{3} \cdot 2\pi
\]

\[
= \sqrt{32\pi^3} \approx 174.124739
\]
Evaluate \( \int \int_{\frac{-a^2-y^2}{x^2}} x^2 y \, dx \, dy \) by converting to polar coordinates.

Solution: In polar coordinates, \( \{(x, y) : -\sqrt{a^2-y^2} \leq x \leq 0, 0 \leq y \leq a\} \) is given as \( \{(r, \theta) : 0 \leq r \leq a, \frac{\pi}{2} \leq \theta \leq \pi\frac{3}{2}\} \).

So \( \int \int_{\frac{-a^2-y^2}{x^2}} x^2 y \, dx \, dy = \int \int_{\frac{-a^2-y^2}{x^2}} r^2 \cos^2 \theta \cdot r \sin \theta \cdot r \, dr \, d\theta \)

\[
= \int_{\frac{\pi}{2}}^{\pi} \cos \theta \sin \theta \cdot \frac{1}{2} r^5 \bigg|_0^a \, d\theta
\]

\[
= \frac{1}{5} a^5 \int_{\frac{\pi}{2}}^{\pi} \cos^2 \theta \sin \theta \, d\theta
\]

\[
= \frac{1}{5} a^5 \cdot \left(-\frac{1}{3} \cos^3 \theta \right)_{\frac{\pi}{2}}^{\pi}
\]

\[
= \frac{1}{5} a^5 \cdot \frac{1}{3}
\]

\[
= \frac{\sqrt{a^5}}{15}
\]
15.5.14) Find the center of mass of the lamina whose boundary consists of the semicircles \( y = \sqrt{1-x^2} \) and \( y = \sqrt{4-x^2} \) together with the portions of the \( x \)-axis that join them if the density at any point is proportional to its distance from the origin.

Solution: The lamina is most easily described in polar coordinates by \( R = \{ (r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq \pi/2 \} \). So the density is given (in polar coordinates) by \( p(r, \theta) = \frac{k}{r} \), where \( k \) is the constant of proportionality.

So the mass of the lamina is given by

\[
m = \iint_R p \, dA = \int_0^{\pi/2} \int_1^2 \frac{k}{r} \cdot r \, dr \, d\theta = \int_0^{\pi/2} k \int_1^2 1 \, dr \, d\theta = k \left( \frac{2-1}{2} \right) \left( \frac{\pi}{2} - 0 \right) = \pi k.
\]

Then

\[
\bar{x} = \frac{1}{m} \iint_R x \cdot p \, dA = \frac{1}{m} \int_0^{\pi/2} \int_1^2 r \cos \theta \cdot \frac{k}{r} \cdot r \, dr \, d\theta = \frac{k}{m} \int_0^{\pi/2} \cos \theta \cdot \frac{1}{2} r^2 \, d\theta = \frac{k}{2m} \left[ \left( \frac{1}{2} r^2 \right) \right]_1^2 \int_0^{\pi/2} \cos \theta \, d\theta = \frac{3k}{2m} \left[ \sin \pi - \sin 0 \right] = 0
\]

\[
= 0
\]
And \[ \bar{y} = \frac{1}{m} \iint_{R} y \cdot p \cdot dA = \frac{1}{m} \int_{0}^{\pi} \int_{0}^{2} r \sin \theta \cdot \frac{k}{r} \cdot r \, dr \, d\theta \]
\[ = \frac{k}{m} \int_{0}^{\pi} \sin \theta \cdot \frac{1}{2} r^2 \bigg|_{0}^{2} \, d\theta \]
\[ = \frac{3k}{2m} \left[ -\cos \theta \right]_{0}^{\pi} \]
\[ = \frac{3k}{2m} \cdot 2 \]
\[ = \frac{3k}{m} = \frac{3k}{2\pi k} = \frac{3}{2\pi} \text{.} \]

So the center of mass of the lamina is at

\[ (\bar{x}, \bar{y}) = (0, \frac{3}{2\pi}) \text{.} \]

Remark: Note that

1. The center of mass is independent of the constant of proportionality \( k \) whereas the moment about the \( x \)-axis, \( M_x = 3k \), is not. Why is that? i.e. What is the physical meaning of these quantities?

2. The center of mass of this object isn't even on the lamina.