15.6.30. Express \( \iiint_E f(x,y,z) \, dV \) as an iterated integral six different ways, where \( E \) is bounded by \( y^2 + z^2 = 9 \), \( x = -2 \), \( x = 2 \).

Solution: The bounding curves are a cylinder around the \( x \)-axis of radius 3, and two planes parallel to the \( YZ \)-plane. The resulting region \( E \) is

Projecting onto the \( XZ \)-plane:

\[
\begin{align*}
\int_{-2}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z) \, dy \, dz \, dx
\end{align*}
\]

Projected onto the \( XY \)-plane:

\[
\begin{align*}
\int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z) \, dz \, dx \, dy
\end{align*}
\]

15.6.46. Consider the hemisphere \( x^2 + y^2 + z^2 = 1 \), \( z \geq 0 \), with density function \( \rho(x,y,z) = \sqrt{x^2 + y^2 + z^2} \). Set up, but do not evaluate, integral expressions for

a.) the mass.

Solution: Recall in 15.6, we do not yet have cylindrical or spherical coordinates, so we must set up this integral in rectangular coordinates.

Projecting onto the \( xy \)-plane, we get

\[
\begin{align*}
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \rho(x,y) \, dy \, dx
\end{align*}
\]

So the mass, \( M \), is found by \( M = \iiint_E \rho(x,y,z) \, dV \).

\[
\begin{align*}
M &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} \rho(x,y,z) \, dz \, dy \, dx
\end{align*}
\]

or

\[
\begin{align*}
M &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} \rho(x,y,z) \, dz \, dy \, dx
\end{align*}
\]
b.) The center of mass.

Solution: With $m$, as in part a, the center of mass, $(\bar{x}, \bar{y}, \bar{z})$ is given by:

\[ \bar{x} = \frac{1}{m} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} x \cdot x^2+y^2+z^2 \, dz \, dy \, dx \]

\[ \bar{y} = \frac{1}{m} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} y \cdot x^2+y^2+z^2 \, dz \, dy \, dx \]

\[ \bar{z} = \frac{1}{m} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} z \cdot x^2+y^2+z^2 \, dz \, dy \, dx \]

For integration $dz \, dy \, dx$, the bounds of the integrals change to the bounds shown in part (a).

C.) The moment of inertia about the $z$-axis, $I_z$.

\[ I_z = \iiint_E (x^2+y^2) \rho(x,y,z) \, dV \]

\[ I_z = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} (x^2+y^2)(x^2+y^2+z^2) \, dz \, dy \, dx \]

15.229 Evaluate $\iiint_E x \, dV$, where $E$ is enclosed by $z = 0$, $z = x+y+5$, $x^2+y^2 = 4$, $x^2+y^2 = 9$.

Solution: The top of the region is $z = x+y+5$.

The bottom of the region is $z = 0$.

Rotating the standard view, we get the graph.

Note: The plane $z = x+y+5$ does not intersect the plane $z = 0$ in the disc $x^2+y^2 = 9$.

This is what ensures that the top of the region is always $z = x+y+5$.

The setup, in cylindrical coordinates, is

\[ \int_{0}^{\pi} \int_{2}^{3 \cos \theta} \int_{0}^{r \cos \theta} \rho r \cos \theta \cdot r \, dz \, dr \, d\theta \]
\[15.7.30\text{(Continued)}\]

\[
\begin{align*}
&= \int_0^{2\pi} \int_{-a}^a r^2 \cos^3 \theta \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_{-a}^a \frac{1}{4} r^4 \cos^3 \theta + \frac{1}{4} r^3 \cos \theta \sin \theta + \frac{3}{2} r^3 \cos \theta \, dr \, d\theta \\
&= \int_0^{2\pi} \frac{65}{4} \cos^3 \theta + \frac{65}{4} \cos \theta \sin \theta + \frac{95}{3} \cos \theta \, d\theta \\
&= \left( \frac{65}{4} \theta + \frac{1}{2} \sin \theta - \frac{65}{4} \cos \theta + \frac{95}{3} \sin \theta \right) \bigg|_0^{2\pi} \\
&= \frac{65\pi}{4} \quad (0 + 0) - (0 + 0) \\
&= \frac{65\pi}{4}.
\end{align*}
\]

\[15.8.32\]

Let \( H \) be a solid hemisphere of radius \( a \), whose density at any point is proportional to its distance from the center of its base.

A.) Find the mass of \( H \).

**Solution:** If we position the hemisphere so that the base is on the xy-plane with center at \((0,0,0)\), then \( p(x,y,z) = k\sqrt{x^2+y^2+z^2} \). Easier is in spherical coordinates: \( p(\rho,\phi,\theta) = \rho \). Notation: \( \rho \) is a variable, while \( p(\rho,\phi,\theta) \) is the density function.

Then the mass \( M = k \int_0^{\pi/2} \int_0^{2\pi} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \):

\[
\begin{align*}
&= k \int_0^{\pi/2} \int_0^{2\pi} \left[ \frac{1}{4} \rho^4 \sin \phi \right]_0^a \, d\phi \, d\theta \\
&= k \int_0^{\pi/2} \left[ \frac{1}{4} a^4 \sin \phi \right]_0^\pi \, d\phi \, d\theta \\
&= k \int_0^{\pi/2} \left[ -\frac{1}{4} a^4 \cos \phi \cdot 2\pi \right]_0^\pi \\
&= k \left( -\frac{1}{4} a^4 \cos \phi \right) \\
&= k \frac{2\pi}{4} a^4 \\
&= k \frac{\pi}{2} a^4.
\end{align*}
\]
b.) Find the center of mass of $H$.

Solution: By symmetry we expect $\bar{x} = \bar{y} = 0$. Indeed:

$$\bar{x} = \frac{1}{m} \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \cos \phi \sin \theta \cos \theta \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{m} \int_0^{2\pi} \int_0^\pi \frac{1}{2} \rho^2 \sin \phi \cos \phi \, d\rho \, d\phi$$

$$= \frac{1}{m} \int_0^{2\pi} \int_0^\pi \frac{1}{2} \rho^2 \cos \phi \sin \phi \, d\rho \, d\phi$$

$$= \frac{1}{m} \int_0^{2\pi} \int_0^\pi \frac{1}{2} \rho^2 \cos \phi \sin \phi \, d\rho \, d\phi$$

$$= \frac{2a^2\pi}{5 \cos^2 \theta} \left[ \frac{\tan \theta}{\tan^2 \theta} \right]_0^{2\pi}$$

$$= \frac{4a^2}{5} \left( \frac{1}{2} - 0 \right)$$

$$= \frac{2a^2}{5}$$

So the center of mass is $(0, 0, \frac{2a^2}{5})$.

C.) Find the moment of inertia of $H$ about its $z$ axis.

Solution: $I_z = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin^2 \phi (\sin^2 \phi + \cos^2 \phi) \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

$$= k \int_0^{2\pi} \int_0^\pi \int_0^a \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta$$

$$= k \int_0^{2\pi} \int_0^\pi \frac{1}{3} \rho^4 \sin^3 \phi \, d\rho \, d\phi$$

$$= k \frac{a^6 \pi}{3} \int_0^\pi \sin^3 \phi \, d\phi$$

$$= k \frac{a^6 \pi}{3} \int_0^\pi (1 - \cos^2 \phi) \sin \phi \, d\phi$$

$$= k \frac{a^6 \pi}{3} \int_0^\pi (1 - u^2) \, du$$

$$= k \frac{a^6 \pi}{3} \left[ \left( u - \frac{1}{3} u^3 \right) \right]_0^1$$

$$= k \frac{a^6 \pi}{3} \left( 1 - \frac{1}{3} \right)$$

$$= k \frac{2a^6 \pi}{9}$$
Evaluate the integral by changing to spherical coordinates.

\[
\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3) \, dz \, dy
\]

We see the projection onto the xy-plane is:

the top of the region is given by \( z = \sqrt{a^2-x^2-y^2} \)
the bottom of the region is given by \( z = -\sqrt{a^2-x^2-y^2} \)

Thus, the region is a sphere, of radius \( a \).

\[
\int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} z(x^2 + y^2 + z^2) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{0}^{\pi} \left[ \frac{\rho^3}{3} \right]_{0}^{a} \sin \phi \, d\phi \, d\theta
\]

\[
= \frac{a^4 \pi}{3} \int_{0}^{\pi} \sin \phi \cos \phi \, d\phi
\]

\[
= \frac{a^4 \pi}{3} \left[ \frac{1}{2} \sin^2 \phi \right]_{\theta}^{\pi}
\]

\[
= \frac{a^4 \pi}{3} (0 - 0)
\]

\[
= 0
\]