Evaluate \( \int_C xyz^2 \, ds \), where \( C \) is the line segment from \((-1,5,0)\) to \((1,6,4)\).

**Solution:** First, we find a parameterization for \( C \):

From (8) on pg 1038, \( \vec{r}(t) = (1-t)\langle -1,5,0 \rangle + t\langle 1,6,4 \rangle \)

\[ = \langle t-1, 5+5t, 4t \rangle \]

\[ = \langle 2t-1, 5t, 4t \rangle \]

so we get

\[ x(t) = 2t-1 \]
\[ y(t) = 5t \]
\[ z(t) = 4t \]

\( 0 \leq t \leq 1 \)

From here, we calculate \( ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \)

\[ = \sqrt{2^2 + 1^2 + 4^2} \, dt \]

\[ = \sqrt{21} \, dt \]

Thus,

\[
\int_C xyz^2 \, ds = \int_0^1 (2t-1)(5t)(4t)^2 \sqrt{21} \, dt
\]

\[ = \int_0^1 (2t^2+9t-5)(16t^2) \sqrt{21} \, dt \]

\[ = \sqrt{21} \int_0^1 (32t^4 + 144t^3 - 80t^2) \, dt \]

\[ = \sqrt{21} \left( \frac{2t^5}{5} + 36t^4 - \frac{80t^3}{3} \right) \bigg|_0^1 \]

\[ = \sqrt{21} \left( \frac{32}{5} + 36 - \frac{80}{3} \right) \]

\[ = \frac{256 \sqrt{21}}{15} \approx 72.09919 \]
16.2.40 Find the work done by the force field \( \vec{F}(x,y) = x \sin y \vec{i} + y \vec{j} \) on a particle that moves along the parabola \( y = x^2 \) from \((-1,1)\) to \((2,4)\).

**Solution:** Using equation (12) on p. 1041, and definition 13 on p. 1042,

\[
W = \int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot d\vec{t}
\]

Thus, we must find a parameterization for \( C \), the parabola from \((-1,1)\) to \((2,4)\).

This is given by \( x(t) = t \) for \(-1 \leq t \leq 2\). Thus, we get \( y(t) = t^2 \).

\[
W = \int_{-1}^{2} \vec{F}(x(t), y(t)) \cdot \vec{v}(t) \, dt
\]

\[
= \int_{-1}^{2} (x \sin y \vec{i} + y \vec{j}) \cdot (\vec{i} + 2t \vec{j}) \, dt
\]

\[
= \int_{-1}^{2} (t \sin t^2 + 2t^3) \, dt
\]

\[
= \frac{1}{2} \cos t^2 + \frac{1}{6} t^4 \bigg|_{-1}^{2}
\]

\[
= \frac{1}{2} \cos 4 + \frac{1}{6} \cos 1 + \frac{15}{2}
\]

16.3.8 Is \( \vec{F}(x,y) = (xy \cos(xy) + \sin(xy)) \vec{i} + (x^2 \cos(xy)) \vec{j} \) a conservative vector field? If so, find a function \( f \) so that \( \nabla f = \vec{F} \).

**Solution:** First, we set \( P(x,y) = xy \cos(xy) + \sin(xy) \) and check if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

\( Q(x,y) = x^2 \cos(xy) \)

\[
\frac{\partial P}{\partial y} = x \cos(xy) - xy \sin(xy) \cdot x + \cos(xy) \cdot x
\]

\[
= 2x \cos(xy) - x^2 y \sin(xy)
\]

\[
\frac{\partial Q}{\partial x} = 2x \cos(xy) - x^2 \sin(xy) \cdot y
\]

So yes, \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). We also see that the domain of \( \vec{F} \) is \( \mathbb{R}^2 \), which is simply connected, so by Theorem 6, p. 1050, \( \vec{F} \) is conservative.
16.3.8 (Continued)

So \( f_x = xy \cos(xy) + \sin(xy) \) for some \( f(xy) \). Integrating \( f_y \), we get

\[
\int f_y \, dy = \int x^2 \cos(xy) \, dy = \frac{x^2}{2} \sin(xy) + g(x)
\]

We must now find \( g(x) \). We do this by differentiating what we just found with respect to \( x \):

\[
f_x = \frac{\partial}{\partial x} \left[ x \sin(xy) + g(x) \right]
\]

\[
= \sin(xy) + x \cos(xy) \frac{dy}{dx} + g'(x).
\]

Comparing this to our original \( f_x = \frac{\partial}{\partial x} \left( xy \cos(xy) + \sin(xy) \right) \), we see that \( g'(x) = 0 \), so \( g(x) = k \), for some constant \( k \). Picking \( k = 0 \), which is an arbitrary choice, we get

\[
f(xy) = x \sin(xy)
\]

16.3.22) Find the work done by the force field \( \vec{F}(xy) = e^{-y} \hat{i} - xe^{-y} \hat{j} \)

moving from \((0,1)\) to \((2,0)\).

Solution: We check if \( \vec{F} \) is a conservative vector field:

\[
\frac{\partial P}{\partial y} = e^{-y}, \quad \frac{\partial Q}{\partial x} = -e^{-y}
\]

\[
\frac{\partial P}{\partial y} = -e^{-y} \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]

Also, the domain of \( \vec{F} \) is \( \mathbb{R}^2 \), simply connected. Thus, \( \vec{F} \) is conservative.

As in 16.2.40, we must evaluate \( \int_C \vec{F} \cdot d\vec{s} \), for some curve from \((0,1)\) to \((2,0)\). Since \( \vec{F} \) is conservative, this integral is independent of path \( C \), so we can pick any curve \( C \) from \((0,1)\) to \((2,0)\). We will see that a specific parameterization is not needed.
Following the method from 16.2.10, we parameterize C by
\[ x(t) = 2t \quad \text{and} \quad y(t) = 1 - t \]
Since \( \vec{F} \) is conservative, there is a function \( f \) with \( \nabla f = \vec{F} \).

We will now find \( f \). We know that
\[
\begin{align*}
  f_x &= e^{-y} \\
  f_y &= -xe^{-y}.
\end{align*}
\]

Integrating \( f_x \), we get
\[
\begin{align*}
f(x,y) &= \int f_x \, dx \\
&= \int e^{-y} \, dy \\
&= xe^{-y} + g(y).
\end{align*}
\]

As before,
\[
\begin{align*}
f_y &= \frac{\partial}{\partial y} [xe^{-y} + g(y)] \\
&= xe^{-y} + g'(y).
\end{align*}
\]

So \( g'(y) = 0 \), hence, \( g(y) = k \). So \( f(x,y) = xe^{-y} + k \). Picking \( k = 0 \),
\[
f(x,y) = xe^{-y}.
\]

Thus,
\[
W = \int_{C} \vec{F} \cdot d\vec{r} = \int \nabla f \cdot d\vec{r}
\]
\[
= f(2,0) - f(0,1) \quad \text{(by theorem 2, 10.4.6)}
\]
\[
= 2e^0 - 0e^1
\]
\[
= 2.
\]

Note: since we used theorem 2, we did not need the parameterization.