AN UPGRADE† FOR "GEOMETRIC CATEGORIES AND O-MINIMAL STRUCTURES"

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This document contains comments, corrections, improvements and updates to the above-mentioned paper [Duke Math. J. **84** (1996), 497–540], co-authored with Lou van den Dries. We thank all who have alerted us to errors, and shall continue to post updates as appropriate.

Clarification

We stress that the paper is an exposition of the basic *mathematics* of o-minimal structures on the real field, specialized for geometers; we did not provide any *history* of the subject. It is well known to model-theorists—but less so to geometers—that seminal work in abstract model-theoretic o-minimality by Pillay and Steinhorn [PS] and Knight, *et al.* [KPS] was crucial in getting o-minimality off the ground. For historical context and original sources, the reader might begin by consulting [4] (especially the notes at the end of chapters) or the expository paper [D].

Errors

Page 516. In B.3.(1) and (2), interchange "$\mathfrak{S}_n$" and "$\mathfrak{S}_m$".

Page 520. The first displayed formula in the proof of B.11 should be:

$$(a,b) \in TA(\lambda)$$
$$\Leftrightarrow$$
$$a \in A(\lambda) \ \ \& \ \ \exists \epsilon > 0 \ \big[\, (\epsilon, a, \pi_\lambda(a)) \in U_\lambda \ \ \& \ \ b \in (D((f_\lambda)_{(\epsilon,a)})(\pi_\lambda(a))).\mathbb{R}^k \, \big]$$

Page 524. Replace the first sentence of the paragraph preceding C.11 by: "The next result was established by E. Bierstone, P. Milman and W. Pawłucki for the subanalytic category [private correspondence, 1995]."

Page 525. The third paragraph of the proof of C.11 should begin as follows: "Now suppose that $d < n$. Replacing A with $\mathrm{cl}(\tau_n(A))$ (τ_n as in §3), we reduce to the case that A is compact."

† This material is not intended for publication; please do not refer to this document as a preprint.

‡ Speaking also for van den Dries, but Miller is responsible for any errors in this document.

Whitney stratification. There is a gap in the proof of C^p Whitney stratification (1.19); the last sentence of the alleged proof is, in general, false. Now, it is not all that difficult to fix the proof of part (1), that is, C^p Whitney stratification of *sets*, but we ran into some difficulty in attempting to do the same with part (2) (stratification of *maps*); we have been informed that there should an alternate way of constructing a correct proof based on the notion of "canonical stratification"—see e.g. [Ł]—but we have not yet verified for ourselves that this works.

Improvements

The proof of the C^p zeroset theorem (C.11) can be streamlined somewhat. After reducing to the case that A is equal to the closure of the graph of a bounded C^p map $\psi : U \to \mathbb{R}^e$, belonging to $\mathfrak{S}$, with $e = n - d$ and $\emptyset \neq U \subseteq \mathbb{R}^d$ open in $\mathbb{R}^d$, proceed as follows.

Inductively, there exists $g \in C^p_{\mathfrak{S}}(\mathbb{R}^d)$ with $Z(g) = \mathrm{bd}(U)$. For $(x, y) \in \mathbb{R}^d \times \mathbb{R}^e$ put

$$G(x,y) := \begin{cases} ||y - \psi(x)||g(x), & x \in U \\ g(x), & \text{otherwise.} \end{cases}$$

Note that G belongs to $\mathfrak{S}$, and is continuous and C^p off $Z(G) = (\mathrm{bd}(U) \times \mathbb{R}^e) \cup A$. Applying C.10, there exists $F_1 \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(F_1) = (\mathrm{bd}(U) \times \mathbb{R}^e) \cup A$. As before, it now suffices to find $F_2 \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ such that $\Gamma(\psi) \subseteq Z(F_2)$ and $F_2(x, y) \neq 0$ for all $(x, y) \in (\mathrm{bd}(U) \times \mathbb{R}^e) \setminus A$.

Now, $A \setminus \Gamma(\psi) = \mathrm{fr}(\Gamma(\psi))$ is closed, so by the inductive assumptions and 4.7, there exists $h \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(h) = A \setminus \Gamma(\psi) = (\mathrm{bd}(U) \times \mathbb{R}^e) \setminus A$. Define $H : \mathbb{R}^d \to \mathbb{R}$ by

$$H(x) := \begin{cases} h(x, \psi(x)), & x \in U \\ 0, & x \in \mathbb{R}^d \setminus U. \end{cases}$$

By C.10, there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\phi \circ H \in C^p_{\mathfrak{S}}(\mathbb{R}^d)$. Put

$$F_2(x, y) := \phi(h(x, y)) - \phi(H(x))$$

for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^e$. $\square$

Updates

Newer o-minimal structures. See [DS1], [DS2], [KRS], [LR], [RSS], [RSW], [S].

Failure of analytic and C^∞ cell decomposition.

By [RSW], there exist nowhere-analytic functions $f : \mathbb{R} \to \mathbb{R}$ such that $(\mathbb{R}_{\mathrm{an}}, f)$ is o-minimal and has C^∞ cell decomposition (cf. the Remark following 1.8).

By [LR], there exist functions $f : \mathbb{R} \to \mathbb{R}$ such that $(\mathbb{R}, +, \cdot, f)$ is o-minimal and C^∞ cell decomposition fails (cf. Remark following 4.1).

References (including updates)

[2] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb. (3) **12**, Springer-Verlag, Berlin, 1987.

[2.5] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Ergeb. Math. Grenzgeb. (3) **36**, Springer-Verlag, Berlin, 1998 (translated from [2] and revised by the authors).

[4] ———, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note Series **248**, Cambridge University Press, 1998.

[D] ———, *o-Minimal structures*, Logic: From Foundations to Applications, Oxford Sci. Publ., Oxford University Press, New York, 1996, pp. 137–185.

[6] L. van den Dries, A. Macintyre and D. Marker, *Logarithmic-exponential power series*, J. London Math. Soc. (2) **56** (1997), 417–434.

[DS1] L. van den and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, 4377–4421.

[DS2] ———, *The field of reals with multisummable series and the exponential function*, Proc. London Math. Soc. **81**, 513–565.

[KPS] J. Knight, A. Pillay and S. Steinhorn, *Definable sets in ordered structures* II, Trans. Amer. Math. Soc. **295** (1986), 593–605.

[KRS] T. Kaiser, J.-P. Rolin and P. Speissegger, *Transition maps at non-resonant hyperbolic singularities are o-minimal*, J. Reine Angew. Math. **636** (2009), 1–45.

[LR] O. Le Gal and J.-P. Rolin, *An o-minimal structure which does not admit C^∞ cellular decomposition*, Ann. Inst. Fourier (Grenoble) **59** (2009), 543–562.

[15] T. Loi, *Whitney stratification of sets definable in the structure* $\mathbb{R}_{\exp}$, Banach Center Publications **33**, 1996, pp. 401–409.

[15.5] ———, *Verdier and strict Thom stratifications in o-minimal structures*, Illinois J. Math. **42** (1998), 347–356 (an alternative, perhaps even preferable, to [15]).

[Ł] S. Łojasiewicz, *Triangulations of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 449–474.

[PS] A. Pillay and S. Steinhorn, *Definable sets in ordered structures* I, Trans. Amer. Math. Soc. **295** (1986), 565–592.

[RSS] J.-P. Rolin, F. Sanz and R. Schäfke, *Quasi-analytic solutions of analytic ordinary differential equations and o-minimal structures*, Proc. Lond. Math. Soc. (3) **95** (2007), 413–442.

[RSW] J.-P. Rolin, P. Speissegger and A. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751–777.

[24.5] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Prog. in Math., vol. 150, Birkhäuser, Boston, 1997 ([24] is the abstract for this book).

[S] P. Speissegger, *The Pfaffian closure of an o-minimal structure*, J. Reine Angew. Math. **508** (1999), 189–211.

[27] A. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), 1051–1094.

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