O-MINIMAL TOTAL HARMONIC FUNCTIONS ARE POLYNOMIAL

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An expansion of the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$ is **o-minimal** if every definable set has only finitely many connected components. For definitions and basic facts, see van den Dries and Miller [6] (though some of the exposition there is now out of date). O-minimality can be seen as a wide-ranging generalization of real-algebraic and subanalytic geometry. It has developed to include applications in areas such as transcendental number theory (via the celebrated Pila-Wilkie Theorem [12] and extensions thereof) and Hodge theory (*e.g.*, Bakker *et al.* [2]), but here I address a basic question of o-minimal calculus.

A function $h\colon U \to \mathbb{R}$, with U open in $\mathbb{R}^n$, is **harmonic** if it is twice differentiable and its Laplacian, Δh, is trivial, that is, $\sum_{i=1}^n \partial^2 h/\partial x_i^2 = 0$. For basic facts about harmonic functions, see Axler *et al.* [1].

Theorem. *If $h\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic and $(\overline{\mathbb{R}}, h)$ is o-minimal, then h is polynomial.*

The statement is perhaps not surprising to readers familiar with both o-minimality and harmonic functions. Heuristically, o-minimality tends to rule out oscillatory behavior, and concrete examples from multivariable calculus of nonpolynomial total harmonic functions tend to be visibly oscillatory, *e.g.*,

$$x \mapsto p(x) + \sum_{\substack{1\le j,k\le n\\ j\ne k}} (a_{j,k}e^{x_j}\cos x_k + b_{j,k}e^{x_j}\sin x_k)\colon \mathbb{R}^n \to \mathbb{R}$$

where p is a harmonic polynomial and $a_{j,k}, b_{j,k} \in \mathbb{R}$. But given any sequence $(p_k)_{k\in\mathbb{N}}$ with each p_k a homogeneous harmonic polynomial $\mathbb{R}^n \to \mathbb{R}$ of degree k, the sum of the p_k converges to a harmonic function $\mathbb{R}^n \to \mathbb{R}$ if

$$\lim_{k\to+\infty} \max\{\, |p_k(x)|^{1/k} : |x| = 1 \,\} = 0.$$

(Here and throughout, $|\ \ |$ indicates the euclidean norm.) Hence, there are plenty of exotic examples. And detecting oscillatory behavior is not always straightforward. Harmonic functions are analytic, hence locally nonoscillatory. Indeed, the expansion of $\overline{\mathbb{R}}$ by all restrictions of harmonic functions to compact subanalytic subsets of their domains is o-minimal. (All restrictions of analytic functions to compact subanalytic subsets of their domains are definable in the o-minimal structure $\mathbb{R}_{\text{an}}$.) In particular, if $h\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic, then the expansion of $\overline{\mathbb{R}}$ by all restrictions of h to bounded balls is o-minimal, and so any oscillatory behavior of h must occur "at ∞", say, the number of the (finitely many) connected components of the sets $\{\, x \in \mathbb{R}^n : |x| = r \ \ \&\ \ h(x) = 0 \,\}$ is unbounded as $r \to +\infty$ (which would violate o-minimality; see [6, 4.4]). Moreover, though the planar harmonic

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function $e^{x_1}\sin x_2$ is not definable in any o-minimal structure (consider its zero set), if I is a bounded interval in $\mathbb{R}$ and $\mathfrak{R}$ is an o-minimal expansion of $\overline{\mathbb{R}}$, then the expansion of $\mathfrak{R}$ by the restriction of $e^{x_1}\sin x_2$ to the strip $\mathbb{R}\times I$ is o-minimal. (By Pfaffian closure, $(\mathfrak{R},\exp,\arctan)$ is o-minimal; see Speissegger [13]. It is an exercise that $(\overline{\mathbb{R}},\arctan)$ defines the restriction of the sine function to any bounded interval.)

Before proceeding to the proof of the Theorem, let us consider some corollaries.

Corollary 1. *Let $f,g\colon \mathbb{R}^n\to\mathbb{R}$ be twice differentiable and $\Delta f=\Delta g$. If $(\overline{\mathbb{R}},f,g)$ is o-minimal, then $f-g$ is a harmonic polynomial.*

(The point is that if $\phi\colon\mathbb{R}^n\to\mathbb{R}$ is definable in an o-minimal expansion of $\overline{\mathbb{R}}$, then modulo the n-ary harmonic polynomials, there is at most one definable solution y to the equation $\Delta y=\phi$.)

Proof. The Laplacian is a linear operator, and thus $f-g$ is harmonic. □

The Theorem fails (but not drastically) if there is a point in $\mathbb{R}^n$ where h is not harmonic.

Corollary 2. *Let $n\geq 2$ and $h\colon\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be harmonic. If $(\overline{\mathbb{R}},h)$ is o-minimal and $\varliminf_{x\to 0}|x|^{n-1}h(x)\geq 0$, then there exist $c\in\mathbb{R}$ and a harmonic polynomial $p\colon\mathbb{R}^n\to\mathbb{R}$ such that $h=p+c\,|x|^{2-n}$ if $n>2$, and $h=p+c\log|x|$ if $n=2$.*

Proof. By the Generalized Bôcher Theorem ([1, 9.11]), there exist $c\in\mathbb{R}$ and a harmonic $g\colon\mathbb{R}^n\to\mathbb{R}$ such that $h=g+c\,|x|^{2-n}$ if $n>2$, and $h=g+c\log|x|$ if $n=2$. Observe that g is definable in $(\overline{\mathbb{R}},h,\exp)$, which is o-minimal (by assumption and Pfaffian closure), and apply the Theorem. □

Corollary 3. *If $f\colon\mathbb{R}^{m+n}\to\mathbb{R}$ and $(\overline{\mathbb{R}},f)$ is o-minimal, then there exists $d\in\mathbb{N}$ such that for all $a\in\mathbb{R}^m$, if $x\mapsto f(a,x)\colon\mathbb{R}^n\to\mathbb{R}$ is harmonic, then it is polynomial of degree at most d.*

Proof. If $p\colon\mathbb{R}^n\to\mathbb{R}$ is polynomial, then the function $r\mapsto\max\{\,|p(x)|:|x|=r\,\}$ $(r>0)$ is ultimately differentiable, and if p is not identically 0, then there exists $q\in\mathbb{Q}$ such that

$$\lim_{r\to+\infty}\frac{r\frac{d}{dr}\max\{\,|p(x)|:|x|=r\,\}}{\max\{\,|p(x)|:|x|=r\,\}}=q.$$

By o-minimality, any set of rational numbers definable in $(\overline{\mathbb{R}},f)$ is finite. Thus, the result reduces to the case $m=0$ (that is, the Theorem). □

The statement of the Theorem makes sense over any ordered field. Next is a result for model theorists.

Corollary 4. *Let $\mathfrak{A}$ be an o-minimal expansion of an ordered field $(A,<,+,\cdot\,)$ and $h\colon A^n\to A$ be harmonic and definable.*

1. *If $\mathrm{Th}(\mathfrak{A})$ has a model $\mathfrak{R}$ with underlying set $\mathbb{R}$, then h is polynomial.*
2. *If the set of $\emptyset$-definable elements of A is archimedean as an ordered group, then h is polynomial.*

Proof. (1). There exist $m \in \mathbb{N}$, $b \in A^m$ and a function $f\colon A^{m+n} \to A$ $\emptyset$-definable in $\mathfrak{A}$ such that $h = x \mapsto f(b,x)$ and, for all $a \in A^m$, the function $x \mapsto f(a,x)$ is harmonic. Apply Corollary 3 to the realization of f in $\mathfrak{R}$; then there exists $d \in \mathbb{N}$ such that for each $a \in A^m$, the function $x \mapsto f(a,x)$ is polynomial of degree at most d. Hence, h is polynomial.

(2). By Laskowski and Steinhorn [10], the prime submodel of $\mathfrak{A}$ then embeds elementarily into an o-minimal expansion of $\overline{\mathbb{R}}$. Apply (1). $\square$

We now establish the main result.

Proof of Theorem. Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be harmonic and $(\overline{\mathbb{R}}, h)$ be o-minimal. We show that h is polynomial.

If $n = 1$, then Δ is the ordinary second derivative, and so h is affine linear.

Let $n = 2$. The function $F := (\partial h/\partial x, -\partial h/\partial y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is definable. By the Cauchy-Riemann equations, F is complex differentiable as a map $\mathbb{C} \to \mathbb{C}$. By o-minimality and the Identity Theorem, either F is constant or all level sets of F are finite. By the "Big" Picard Theorem, F is polynomial; by real planar calculus, h is polynomial.

Let $n \geq 3$. For $r > 0$, put $H(r) = \max\{\, |h(x)| : |x| = r \,\}$. By the Harmonic Liouville Theorem, it suffices to show that $H(r)$ is polynomially bounded as $r \to +\infty$. (See, *e.g.*, [1, Exercise 2.7]). Assume that h is not constant. By the Harmonic Maximum Principle, H is positive and strictly increasing. Thus, it suffices now to show that $H(r)/H(r/2)$ is bounded as $r \to +\infty$. Without loss of generality, we may take $h(0) = 0$. Let μ indicate $(n-1)$-dimensional Hausdorff measure. As a special case of a result of Logunov *et al.* [11], there exists $c > 0$ such that

$$(1) \qquad r > 0 \Longrightarrow c\sqrt{\log_2\bigl(H(r)/H(r/2)\bigr)} \leq \mu\{\, x \in \mathbb{R}^n : |x| \leq 2 \ \ \&\ \ h(rx) = 0 \,\}.$$

(For each fixed $r > 0$, the function $h \circ rx$ is harmonic; apply [11, Theorem 1.2] with $\epsilon = 1/2$.) By, *e.g.*, Yomdin and Comte [14, Corollary 5.2], there exists $C \in \mathbb{R}$ such that

$$(2) \qquad r > 0 \Longrightarrow \mu\{\, x \in \mathbb{R}^n : |x| \leq 2r \ \ \&\ \ h(x) = 0 \,\} \leq Cr^{n-1}.$$

(This uses only that the zero set of h has empty interior and is definable in an o-minimal expansion of $\overline{\mathbb{R}}$.) Dividing by r^{n-1}, we obtain

$$(3) \qquad r > 0 \Longrightarrow \mu\{\, x \in \mathbb{R}^n : |x| \leq 2 \ \ \&\ \ h(rx) = 0 \,\} \leq C.$$

Equations (1) and (3) yield that $H(r)/H(r/2)$ is bounded as $r \to +\infty$, as desired. $\square$

Remarks on the proof. An interesting alternate proof for $n = 2$ can be obtained from De Carli and Hudson [5, Theorem 3.2].

It is possible to obtain equation (2) by just o-minimal multivariable calculus, but conceptually, it is easier to see as a consequence of Uniform Bounds on Fibers [6, 4.4] and the Cauchy-Crofton formula (as presented in [14]).

For $n \geq 3$, the crucial point is that $H(r)/H(r/2)$ is bounded as $r \to +\infty$. (Of course, this fails in general for o-minimal real-entire functions, say, $\exp(|x|^2)$.) As H is definable in $(\overline{\mathbb{R}}, h)$, it would be understandable for the reader to think this should not need the substantial geometric measure theory behind equation (1). But currently I do not know of any significantly easier proof. Though the functional analysis of harmonic functions has long been fairly well understood, the analytic geometry has been refractory (see [5] and

Enciso and Peralta-Salas [7, 8] for some details). An earlier and weaker version of [11, Theorem 1.2], due to Logunov, is mentioned in [11]; it would suffice, but also must be regarded as substantial geometric measure theory. (Thus, I might as well use here what appears to be the current state of the art.)

Some open questions. If $h\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic and not polynomial, does $(\overline{\mathbb{R}}, h)$ define $\mathbb{Z}$? (As far as I know, this is open even for $n = 2$.) Results from Hieronymi and Miller [9] might be useful.

What can be said about harmonic $h\colon \{\, x \in \mathbb{R}^n : |x| < 1 \,\} \to \mathbb{R}$ if $(\overline{\mathbb{R}}, h)$ is o-minimal? (Of course, not much new can be said if h extends analytically to some $\{\, x \in \mathbb{R}^n : |x| < 1 + \epsilon \,\}$, as then h is definable in $\mathbb{R}_{\text{an}}$. Hence, one should assume that h has at least one analytic singularity on the unit sphere.)

What can be said about harmonic $h\colon \mathbb{R}^{n-1} \times (0, +\infty) \to \mathbb{R}$ if $(\overline{\mathbb{R}}, h)$ is o-minimal?

In Corollary 4, is h polynomial without assuming that $\operatorname{Th}(\mathfrak{A})$ has a model with underlying set $\mathbb{R}$? (From a model-theoretic perspective, it is undesirable that my proof relies ultimately on working over $\overline{\mathbb{R}}$.)

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In the early 1990s, I conjectured the Theorem and established the planar case ($n = 2$). With just a little more work that relies only on the planar case and some standard o-minimal tricks, h is polynomial if n is even and h is the real (or imaginary) part of a holomorphic function $\mathbb{C}^{n/2} \to \mathbb{C}$. Because the proofs of these special cases are so short and easy, I did not attempt to publish them. And I could not see how to deal with the case $n = 3$, nor with $n = 4$ and h is not the real (or imaginary) part of a holomorphic function $\mathbb{C}^2 \to \mathbb{C}$.

The conjecture then languished until recent joint work ([4]) with a PhD student, Tyler Borgard: We confirmed the case that h is a real exponential term (more precisely, given by a term in the structure $(\mathbb{R}, +, \cdot, -, \exp, (r)_{r\in\mathbb{R}})$) via a proof that extends to any exponential ordered field that satisfies the least upper bound property for unary definable sets. (Some other miscellaneous examples are found in Borgard's PhD thesis [3].)

I began to think again in earnest about the conjecture. As an amusement, I had my friend K. Ball ask ChatGPT whether the conjecture was true. (I had no experience at all with ChatGPT at that point, but Ball had plenty.) To my surprise, ChatGPT answered yes, and that it was known "folklore"—but with no concrete references. Ball then helped me attempt to obtain a proof from ChatGPT, but after a few of its suggested approaches failed, I gave up. (Some seemed as though they might work, but there was always a devil somewhere in the details.) Nevertheless, in trying to understand (and eventually debunk) the suggested approaches, I became aware (more through MathSciNet than ChatGPT) of the paper [11] and its Theorem 1.2. Hence, it seems fair to give ChatGPT *some* credit—but not much.

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