HARMONIC EXPONENTIAL TERMS ARE POLYNOMIAL

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Abstract. Let n be a positive integer and f belong to the smallest ring of functions $\mathbb{R}^n \to \mathbb{R}$ that contains all real polynomial functions of n variables and is closed under exponentation. Then there exists $d \in \mathbb{N}$ such that for all $m \in \{0, \dots, n-1\}$ and $c \in \mathbb{R}^m$, if $x \mapsto f(c, x)\colon \mathbb{R}^{n-m} \to \mathbb{R}$ is harmonic, then it is polynomial of degree at most d. In particular, f is polynomial if it is harmonic.

Throughout, n ranges over the nonnegative integers, $\mathbb{N}$.

Let $\mathcal{E}_n$ be the smallest ring of functions $\mathbb{R}^n \to \mathbb{R}$ that contains all real polynomial functions of n variables and is closed under exponentation (with respect to base e). We identify $\mathcal{E}_0$ with $\mathbb{R}$. Put $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$. We refer to elements of $\mathcal{E}$ as (real) **exponential terms**. (For readers acquainted with basic first-order logic, $\mathcal{E}_n$ consists of the functions $\mathbb{R}^n \to \mathbb{R}$ given by n-ary terms in the structure $(\mathbb{R}, +, -, \cdot, e^x, (r)_{r\in\mathbb{R}})$, with constants regarded as nullary functions.)

If $U \subseteq \mathbb{R}^n$ is open, then a function $f\colon U \to \mathbb{R}$ is **harmonic** if it is C^2 (twice continuously differentiable) and $\Delta f = 0$, where Δ denotes the Laplace operator $\sum_{k=1}^n \partial^2/\partial x_k^2$. Note that Δ is linear.

Every affine linear function $\mathbb{R}^n \to \mathbb{R}$ is harmonic. More generally, for $n \geq 2$, there are infinitely many harmonic polynomials $\mathbb{R}^n \to \mathbb{R}$ of each degree. If j and k are distinct positive integers bounded above by n, then $e^{x_j}\sin x_k$ and $e^{x_j}\cos x_k$ are harmonic functions $\mathbb{R}^n \to \mathbb{R}$, as are all $\mathbb{R}$-linear combinations of such.

Here is the main result of this note.

Theorem. *For all $f \in \mathcal{E}_n$ there exists $d \in \mathbb{N}$ such that for all $m \in \{0, \dots, n\}$ and $c \in \mathbb{R}^m$, if $x \mapsto f(c, x)\colon \mathbb{R}^{n-m} \to \mathbb{R}$ is harmonic, then it is polynomial of degree at most d.*

As we shall see later, the crucial point is to establish the case $m = 0$, that is, every harmonic exponential term is polynomial. In order to motivate our proof of this, we illustrate some of the main ideas by first considering some special cases. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be polynomial. Differential calculus yields

$$\Delta(fe^g) = e^g(f|\nabla g|^2 + \Delta f + 2\nabla f \cdot \nabla g + f\Delta g)$$

where ∇ indicates the gradient, $|\ |$ indicates the Euclidean norm and $\cdot$ indicates scalar product. Hence, if fe^g is harmonic, then

$$(*) \qquad -f|\nabla g|^2 = \Delta f + 2\nabla f \cdot \nabla g + f\Delta g.$$

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A routine formal argument via degree yields $f|\nabla g|^2 = 0$, and so either $f = 0$ or g is constant. Hence, fe^g is polynomial. Now let $J \in \mathbb{N}$ and $f_1, \dots, f_J, g_1, \dots, g_J$ be polynomial. Suppose that $\sum_{j=1}^J f_j e^{g_j}$ is harmonic and $\{e^{g_1}, \dots, e^{g_J}\}$ is algebraically independent. By calculus and linearity of Δ, each $f_j e^{g_j}$ is harmonic—hence polynomial—and so $\sum_{j=1}^J f_j e^{g_j}$ is polynomial.

It is natural to try to generalize the above argument, starting with arbitrary $f, g \in \mathcal{E}_n$. The formal complexity of f is less than that of fe^g, so if fe^g is harmonic and g is constant, then we could conclude inductively that fe^g is polynomial. But in order to show that g must be constant if $f \neq 0$, we would have to deal with equation $(*)$, and it is not immediately clear how to do so in this generality. Indeed, relative to extant facts about exponential terms, this will be the most critical part of the proof of the Theorem.

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We now proceed directly toward the proof of the theorem, postponing further discussion. Fix $n \in \mathbb{N}$. In order to avoid potential trivialities, let $n \geq 2$. (Every solution on $\mathbb{R}$ to $y'' = 0$ is affine linear.) Routine induction on complexity yields that all elements of $\mathcal{E}_n$ are (real-)analytic and that $\mathcal{E}_n$ is closed under taking partial derivatives. Thus, $\mathcal{E}_n$ is a differential domain in the usual way, and we can employ techniques of formal differential algebra.

We rely heavily on some work of van den Dries [3]; for convenience, we adopt some of the notation used there. Put $R_{-1} = \mathbb{R}$. Let R_0 be the set of all real polynomial functions $\mathbb{R}^n \to \mathbb{R}$. Put $A_0 = \{\, g \in R_0 : g(0) = 0 \,\}$. Inductively, put $R_k = R_{k-1}[\, e^g : g \in A_{k-1} \,]$ and let A_k be the set of all finite sums $\sum_{j=1}^J f_j e^{g_j}$ (J ranging over $\mathbb{N}$) such that if $J \neq 0$, then each $f_j \in R_{k-1} \setminus \{0\}$ and $g_1, \dots, g_J$ are pairwise distinct elements of $A_{k-1} \setminus \{0\}$. A routine induction on k yields that each R_k is contained in $\mathcal{E}_n$ and is closed under partial differentiation. A routine induction on complexity of terms yields $\mathcal{E}_n \subseteq \bigcup_{k \in \mathbb{N}} R_k$. Hence, the case $m = 0$ of the Theorem is equivalent to showing that for all $k \in \mathbb{N}$, every harmonic element of R_k lies in R_0.

Important. In [3], the R_k and A_k are defined as formal algebraic objects, but by [3, 4.2], the natural interpretation as functions $\mathbb{R}^n \to \mathbb{R}$ is an exponential-ring isomorphism; this has important consequences for us. To illustrate, each element of A_k has a unique representation $\sum_{j=1}^J f_j e^{g_j}$ (as described above).

Remark. In [3], elements of $\mathcal{E}_n$ would be called "exponential polynomial functions" (with respect to $(\mathbb{R}, +, \cdot, 0, 1, e^x)$), but we prefer "exponential terms" in order to avoid any confusion with $\mathbb{R}[x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}]$.

We shall employ some basic facts from differential calculus; proofs are exercises.

— If $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$, then $\Delta(fe^g) = e^g(\Delta f + 2\nabla f \cdot \nabla g + f\Delta g + f|\nabla g|^2)$.

— If $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$, then fe^g is harmonic iff $f|\nabla g|^2 + \Delta f + 2\nabla f \cdot \nabla g + f\Delta g = 0$.

— If $f_1, \dots, f_J, g_1, \dots, g_J \in C^2(\mathbb{R}^n, \mathbb{R})$ and $\{e^{g_1}, \dots, e^{g_J}\}$ is $\mathbb{Z}$-linearly independent over

$$\mathbb{Z}\,[f_j, \nabla f_j, \Delta f_j, \nabla g_j, \Delta g_j : j = 1, \dots, J\,],$$

then $\sum_{j=1}^J f_j e^{g_j}$ is harmonic iff each $f_j e^{g_j}$ is harmonic.

Next is a key technical result.

Lemma. *Let $k \in \mathbb{N}$, $f \in R_k \setminus \{0\}$ and $g \in A_k \setminus \{0\}$. Then fe^g is not harmonic.*

Proof. We have already established this for $k = 0$ (that is, f and g are polynomial). Assume now that $k > 0$. By [3, 1.7] (and [3, 4.2]), there is a finite $\mathcal{P} \subseteq A_{k-1}$ such that $\{\, e^p : p \in \mathcal{P} \,\}$ is algebraically independent over R_{k-1} and $f, g \in R_{k-1}[\, e^p, e^{-p} : p \in \mathcal{P} \,]$.

For ease of notation, we first give details for the case that $\mathcal{P}$ contains only one element, p. We have $f = \sum_{j\in\mathbb{Z}} f_j e^{jp}$ and $g = \sum_{j\in\mathbb{Z}} g_j e^{jp}$, with each $f_j, g_j \in R_{k-1}$, and only finitely many of them are nonzero. Since $g \in A_k \setminus \{0\}$, it is not in R_{k-1} (recall the uniqueness of representations), and so there exist nonzero $j \in \mathbb{Z}$ such that $g_j \neq 0$. If necessary, we replace p with $-p$ and re-index the sum so that there exist $j > 0$ with $g_j \neq 0$. Put $\gamma = \max\{\, j \in \mathbb{Z} : g_j \neq 0 \,\}$ and $\phi = \max\{\, j \in \mathbb{Z} : f_j \neq 0 \,\}$. Note that $\gamma > 0$. Suppose, toward a contradiction, that fe^g is harmonic; then $f|\nabla g|^2 + \Delta f + 2\nabla f \cdot \nabla g + f\Delta g = 0$. Put $\alpha = \phi + 2\gamma$. By basic differential algebra using that R_{k-1} is a differential domain over which e^p is algebraically independent, and letting i, j and ℓ range over $\mathbb{Z}$, we obtain

$$\begin{aligned} 0 =& 2\sum_{i+j=\alpha} (\nabla f_i + i f_i \nabla p) \cdot (\nabla g_j + j g_j \nabla p) \\ &+ \sum_{i+j+\ell=\alpha} f_i (\nabla g_j + j g_j \nabla p) \cdot (\nabla g_\ell + \ell g_\ell \nabla p) \\ &+ \Delta f_\alpha + 2\alpha \nabla f_\alpha \cdot \nabla p + \alpha^2 f_\alpha |\nabla p|^2 + \alpha f_\alpha \Delta p \\ &+ \sum_{i+j=\alpha} f_i (\Delta g_j + 2j \nabla g_j \cdot \nabla p + j^2 g_j |\nabla p|^2 + j g_j \Delta p). \end{aligned}$$

Now, $\alpha > \phi$, so $f_\alpha = 0$. And if $i + j = \alpha$, then $i + j > \phi + \gamma$, so $f_i = 0$ or $g_j = 0$. Thus, the only nonzero terms occur in the second line when $i = \phi$ and $j = \ell = \gamma$, yielding $f_\phi |\nabla g_\gamma + \gamma g_\gamma \nabla p|^2 = 0$. Since $f_\phi \neq 0$, we have $\nabla g_\gamma + \gamma g_\gamma \nabla p = 0$, hence also $0 = e^{\gamma p}(\nabla g_\gamma + \gamma g_\gamma \nabla p) = \nabla(g_\gamma e^{\gamma p})$. Thus, $g_\gamma e^{\gamma p}$ is constant, contradicting the independence of e^p over R_{k-1} (because $\gamma \neq 0$ and $g_\gamma \neq 0$). Hence, fe^g is not harmonic, as was to be shown.

The argument for the case that $\mathcal{P}$ contains more than one element is essentially the same, but with extra clerical details: Fix $p_0 \in \mathcal{P}$, take the f_j and g_j in $R_{k-1}[\, e^p, e^{-p} : p \in \mathcal{P} \setminus \{p_0\} \,]$, and proceed similarly as above. (The underlying idea is that, by independence, we can think of e^{p_0} as a distinguished variable with an associated notion of degree.) □

Proof of Theorem. Let $f \in \mathcal{E}_n$. We must find $d \in \mathbb{N}$ such that for all $m \in \mathbb{N} \cap [0, n]$ and $c \in \mathbb{R}^m$, if $x \mapsto f(c, x) \colon \mathbb{R}^{n-m} \to \mathbb{R}$ is harmonic, then it polynomial of degree at at most d. It suffices to fix $m \in \mathbb{N}$ and find such a d for m. The result is trivial for $m = n$, so let $m < n$.

First, assume that $m = 0$ and f is harmonic. We show that f is polynomial. Let k be minimal such that $f \in R_k$; we show that $k = 0$. Toward a contradiction, assume that $k > 0$. By [3, 1.7], there is a finite $\mathcal{P} \subseteq A_{k-1}$ of minimal cardinality N such that $f \in R_{k-1}[\, e^p, e^{-p} : p \in \mathcal{P} \,]$ and $\{\, e^p : p \in \mathcal{P} \,\}$ is algebraically independent over R_{k-1}. By minimality of k, we have $N > 0$.

Suppose $N = 1$, say, $\mathcal{P} = \{p\}$. There exists $J \in \mathbb{N}$ such that $f = \sum_{j=-J}^{J} f_j e^{jp}$ with each $f_j \in R_{k-1}$. By the minimality of k, we have $f \neq f_0$, and so there exists $\ell \in \mathbb{Z}$ such that

$0 < |\ell| \leq J$ and $f_\ell \neq 0$. As f is harmonic,

$$0 = \Delta f = \sum_{j\in\mathbb{Z}} \Delta(f_j e^{jp}) = \sum_{j\in\mathbb{Z}} e^{jp}[\Delta f + 2\nabla f \cdot \nabla p + f(\Delta p + |\nabla p|^2)].$$

It follows from the independence of e^p over R_{k-1} that $f_\ell e^{\ell p}$ is harmonic, contradicting the Lemma.

If $N > 1$, then fix any $p_0 \in \mathcal{P}$ and take the $f_j \in R_{k-1}[\,e^p, e^{-p} : p \in \mathcal{P} \setminus \{p_0\}\,]$. Observe that $f \neq f_0$ and proceed as above. (This ends the proof of the case $m = 0$.)

Assume now that $0 < m < n$. (This part of the proof requires some familiarity with definability over the real field; see, e.g., van den Dries and Miller [4] for basics.) For $c \in \mathbb{R}^m$, let f_c denote the function $x \mapsto f(c,x)\colon \mathbb{R}^{n-m} \to \mathbb{R}$. Note that $f_c \in \mathcal{E}_{n-m}$. The set $C := \{\, c \in \mathbb{R}^m : \Delta f_c = 0 \;\;\&\;\; f_c \neq 0 \,\}$ is definable (without parameters) in $(\mathbb{R}, +, \cdot, f)$. Let $c \in C$. By the case $m = 0$, f_c is polynomial, and so

$$\deg f_c = \lim_{r\to+\infty} \frac{r\frac{d}{dr}\max\{\,|f(c,x)| : |x| = r\,\}}{\max\{\,|f(c,x)| : |x| = r\,\}}.$$

Thus, the set $D := \{\,\deg f_c : c \in C\,\}$ is a discrete subset of $\mathbb{R}$ definable in $(\mathbb{R}, +, \cdot, f)$. As f is definable in the o-minimal (by Wilkie [8]) structure $(\mathbb{R}, +, \cdot, e^x)$, so is D, which is thus finite by o-minimality. Put $d = \max D$. (This ends the proof of the Theorem.) □

Harmonic functions on open sets are analytic, and harmonic functions on $\mathbb{R}^n$ have infinite radius of convergence (see, e.g., Axler et al. [1]). Partial derivatives of harmonic functions are harmonic. Hence:

Corollary. *If $u\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic and there exists $N \in \mathbb{N}$ such that, for each $j = 1, \ldots, n$, the N-th partial derivative of u with respect to the j-th variable lies in $\mathcal{E}_n$, then u is polynomial. In particular, if $\nabla u \in \mathcal{E}_n^n$, then u is polynomial.*

We conclude with some discussion of context, motivation and such.

In the early 1990s, author Miller realized that if $u\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic and $(\mathbb{R}, +, \cdot, u)$ is o-minimal, then u is polynomial. (The result fails over $\mathbb{R}^2 \setminus \{0\}$: consider $\log(x^2 + y^2)$, which is definable in $(\mathbb{R}, +, \cdot, e^x)$.) The proof is very easy relative to classical complex analysis, but works only for $n = 2$, and so the result was never submitted for publication. We sketch the proof. The map $F := (\partial u/\partial x, -\partial u/\partial y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is definable in $(\mathbb{R}, +, \cdot, u)$. By o-minimality, its zero set has only finitely many connected components. Identify F with a function $f\colon \mathbb{C} \to \mathbb{C}$; then the zero set of f has only finitely many connected components. By Cauchy-Riemann equations, f is complex differentiable. Hence, by the "Big" Picard Theorem, f is a complex polynomial, and so F is a real polynomial map. Basic calculus now yields that u is polynomial. With only slightly more work (but we omit details), the result can be extended somewhat: If $u\colon \mathbb{R}^{2m} \to \mathbb{R}$ is the real part of a holomorphic $f\colon \mathbb{C}^m \to \mathbb{C}$ and $(\mathbb{R}, +, \cdot, u)$ is o-minimal, then u is polynomial. The question arises: *If $n \geq 3$, $u\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic and $(\mathbb{R}, +, \cdot, u)$ is o-minimal, must u be polynomial?* Degeneracy occurs if $(\mathbb{R}, +, \cdot, u)$ is polynomially bounded, as then u is polynomial by the Harmonic Liouville Theorem. Assume now that $(\mathbb{R}, +, \cdot, u)$ is o-minimal and not polynomially bounded. By Growth Dichotomy [6], $(\mathbb{R}, +, \cdot, u)$ defines the function e^x. Thus, it is natural that we should first attempt to show that u is polynomial if it is definable in $(\mathbb{R}, +, \cdot, e^x)$, beginning with $u \in \mathcal{E}_n$. Given the Theorem, we can revise the question: *If $n \geq 3$, $u\colon \mathbb{R}^n \to \mathbb{R}$ is* *harmonic and* $(\mathbb{R},+,\cdot,u)$ *is o-minimal, must* u *be an exponential term?* As of this writing, even the case that $n=3$ and u is definable in $(\mathbb{R},+,\cdot,e^x)$ is open. Work is ongoing.

Remark. It seems that the analytic geometry (as opposed to the function theory) of harmonic functions $\mathbb{R}^3\to\mathbb{R}$ is poorly understood; see, e.g., De Carli and Hudson [2] and Enciso and Peralta-Salas [5].

Let $\mathfrak{R}$ be an o-minimal expansion of the real field and $g\colon\mathbb{R}^n\to\mathbb{R}$ be definable. We are interested in definable solutions to the Poisson equation $\Delta y=g$. The question of existence is subtle, but suppose we have such a solution f; then so is $f+p$ for every n-ary harmonic polynomial p (by linearity of Δ). We would like for there to be no other solutions, which is equivalent to $\mathfrak{R}$ not defining any nonpolynomial total n-ary harmonic functions. This is true for $n=2$ by Miller's result mentioned earlier, and the Theorem yields another partial result: Modulo harmonic polynomials, there is at most one solution in $\mathcal{E}_n$. (Of course, if $g\notin\mathcal{E}_n$, then there is no solution in $\mathcal{E}_n$.)

Model theorists might wonder whether working over $\mathbb{R}$ is necessary, especially given the general setting of [3]. It is easy to see that the conclusion of the Theorem is preserved under elementary equivalence, ("transfer principle"), but more is true: By [3, 4.4] and results from [7], our proofs yield that the Theorem holds over any nontrivial ordered exponential ring $\mathfrak{M}:=(M,<,+,\cdot,0,1,E)$ that satisfies the intermediate value theorem for definable unary functions (equivalently, that $\mathfrak{M}$ is "definably complete"), though the use of o-minimality must be replaced with a model-theoretic compactness argument in order to obtain the uniformity in parameters. However, the utility of this observation is questionable, as we do not know of any examples of such $\mathfrak{M}$ that are not elementarily equivalent to $(\mathbb{R},<,+,\cdot,0,1,e^x)$. And there are limits to generalization: If $c\in\mathbb{C}^n\setminus\{0\}$ is such that $\sum_{j=1}^n c_j^2=0$ (e.g., $c=(1,i,0,\dots,0)$), then $\prod_{j=1}^n e^{c_jz_j}$ is not polynomial—but it is a complex exponential term that is harmonic with respect to complex differentiation. The proof of the Lemma does show that if f and g are n-ary complex exponential terms and $\Delta(fe^g)=0$, then $f=0$ or $\nabla g\cdot\nabla g=0$. Thus, we could state some version of the Theorem (with $m=0$) over the complex exponential field, but it is unclear to us how useful it could be. (Indeed, the notion of being harmonic with respect to complex differentiation seems to not occur in the literature of several complex variables.) More generally, if $(R,+,-,\cdot,0,1,E)$ is a nontrivial exponential differential ring as defined in [3] and $c\in R^n$, then $\Delta(E(c\cdot x))=0$ iff $c\cdot c=0$, and is polynomial iff $c\cdot x=0$. Hence, in order for all of the harmonic terms to be polynomial, the underlying ring must be totally real ("orderable").

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