# STATUS OF THE O-MINIMAL TWO-GROUP QUESTION*

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*August 7, 2003*

In [MS98], we showed that an o-minimal expansion of a dense linearly set without endpoints $(R, <)$ can support, up to definable isomorphism, at most two continuous (with respect to the usual topology) definable groups on $R$. Certainly, the maximal number is easily achieved: Any o-minimal expansion of an ordered field that does not define an exponential function is an example; see [MS98, §4] for details. Hence, a natural question is whether there exists an o-minimal expansion of an ordered exponential field that defines a continuous group that is not definably isomorphic to the additive group of the field. In this generality, the question appears to be out of reach, so we focus on expansions of the real exponential field.

Throughout, let $(\mathbb{R}, <, \oplus)$ be an ordered group definable in an o-minimal expansion $\mathfrak{R}$ of the real exponential field.

**Question.** Is $(\mathbb{R}, \oplus)$ definably isomorphic in $\mathfrak{R}$ to $(\mathbb{R}, +)$?

A variant of the question is true, and the question has a positive answer for every known case. We shall explain.

**Theorem** (Peterzil *et al.* [PSS00])**.** *There is an isomorphism* $\phi\colon (\mathbb{R}, <, \oplus) \to (\mathbb{R}, <, +)$ *such that* $(\mathfrak{R}, \phi)$ *is o-minimal.*

*Remark.* Up to a multiplicative constant, $\phi$ is unique. Hence, $(\mathfrak{R}, \psi)$ is o-minimal for every isomorphism $\psi\colon (\mathbb{R}, <, \oplus) \to (\mathbb{R}, <, +)$.

Fix $\phi$ as in the statement of the theorem.

**Lemma.** *Suppose that for each unary (that is,* $\mathbb{R} \to \mathbb{R}$*) function $f$ definable in* $(\mathfrak{R}, \phi)$ *there is a unary function $g$ definable in $\mathfrak{R}$ such that $f$ is bounded at $+\infty$ by $g$. Then* $(\mathbb{R}, \oplus)$ *is definably isomorphic in $\mathfrak{R}$ to* $(\mathbb{R}, +)$*.*

*Proof.* Let $g$ be a unary function definable in $\mathfrak{R}$ such that $x \mapsto \phi^{-1}(\phi(x)^2)\colon \mathbb{R} \to \mathbb{R}$ is bounded at $+\infty$ by $g$. Every automorphism of $(\mathbb{R}, <, \oplus)$ is of the form $\phi^{-1} \circ cx \circ \phi$ for some $c > 0$, so every automorphism of $(\mathbb{R}, \oplus)$ definable in $\mathfrak{R}$ is also bounded at $+\infty$ by $g$. By [MS98, Theorem A], $\mathfrak{R}$ defines a binary operation $\odot$ such that $(\mathbb{R}, \oplus, \odot)$ is a field (since $g$ witnesses that $\mathfrak{R}$ is not linearly bounded with respect to $\oplus$). By Otero *et al.* [OPP96], $(\mathbb{R}, \oplus, \odot)$ is definably isomorphic in $\mathfrak{R}$ to $(\mathbb{R}, +, \cdot)$. □

Recall that a structure on $\mathfrak{R}$ is **exponentially bounded** if for each unary definable $f$ there is a compositional iterate $\exp_N$ (depending on $f$) of exp such that $f$ is bounded at $+\infty$ by $\exp_N$.

At present, every expansion of the real field known to be o-minimal is exponentially bounded; see Lion *et al.* [LMS03] for further information.

*This is **not** a preprint; please do not refer to it as such.

As an immediate consequence of the lemma:

**Proposition 1.** *If* $(\mathfrak{R}, \phi)$ *is exponentially bounded, then* $(\mathbb{R}, \oplus)$ *is definably isomorphic in* $\mathfrak{R}$ *to* $(\mathbb{R}, +)$.

A structure on $\mathbb{R}$ is **closed under asymptotic integration** if for each ultimately nonzero definable unary function $f$ there is an ultimately differentiable definable unary function $g$ such that $\lim_{t \to +\infty} [g'(t)/f(t)] = 1$.

At present, every expansion of the real exponential field that is known to be o-minimal is closed under asymptotic integration; see Miller and Speissegger [MS02] for further information.

**Proposition 2.** *If* $(\mathfrak{R}, \phi)$ *is closed under asymptotic integration, then* $(\mathbb{R}, \oplus)$ *is definably isomorphic in* $\mathfrak{R}$ *to* $(\mathbb{R}, +)$.

The proof is obtained by combining the lemma, the main result from [MS02], and the proof of Theorem [PSS00]; see [MS02, Proposition 3] for details.

## References


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