A TRICHOTOMY FOR EXPANSIONS OF $\mathbb{R}_{\mathrm{an}}$ BY TRAJECTORIES OF ANALYTIC PLANAR VECTOR FIELDS

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Caveat. This note has become seriously out of date due, among other things, to recent work by Philipp Hieronymi [*Defining the set of integers in expansions of the real field by a closed discrete set.* Proc. Amer. Math. Soc. 138 (2010), no. 6, 21632168. MR2596055]. In particular, the paragraph after the proof of Corollary 4 should be struck. Rather than update this particular note, we are are actively working on an actual submittable paper. See also [Miller, Chris *Expansions of o-minimal structures on the real field by trajectories of linear vector fields.* Proc. Amer. Math. Soc. 139 (2011), no. 1, 319330. MR2729094] for related current developments.

We are interested in expanding o-minimal structures on the real field by trajectories[1] of definable vector fields. This note[2] is a preliminary report on some progress. We do not attempt to state and prove results as efficiently as possible or in the greatest generality.

The reader is assumed to be familiar with o-minimal expansions of the real field; see [4–6] for basic references and surveys. We also need a fair amount of ODE theory, some quite basic, some less so; see *e.g.* [14].[3] An important source of inspiration was the essentially expository [1, Ch. 5]. See [8, 9, 12] for some related material, and [11] for discussion of a more extensive context into which this paper fits.

Throughout, "definable" means "parametrically definable"; "analytic" means "real analytic". Given structures $\mathfrak{R}_1$ and $\mathfrak{R}_2$, we write $\mathfrak{R}_1 = \mathfrak{R}_2$ if they are interdefinable. The open ball about the origin of radius $r > 0$ is denoted by $B_r(0)$. As usual, we abuse notation and write 0 instead of $(0, 0)$.

Before we can state our main result, we must introduce the cast of characters.

The real field $(\mathbb{R}, +, \cdot\,)$ is denoted by $\overline{\mathbb{R}}$. Recall that a subset of $\mathbb{R}^n$ is definable in $\overline{\mathbb{R}}$ if and only if it is semialgebraic; see *e.g.* [4, Ch. 2].

The expansion $(\overline{\mathbb{R}}, \mathbb{Z})$ of $\overline{\mathbb{R}}$ by the set of all integers is denoted by PH, short for "(real) projective hierarchy". A subset of $\mathbb{R}^n$ is definable in PH if and only if it is projective in the sense of descriptive set theory. Every Borel subset of $\mathbb{R}^n$ is definable in PH; then so are

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Some details of the main result of this note were fleshed out, with help from the participants, at a workshop held at The Fields Institute (Toronto), May 25–27, 2004. We thank the Institute for its support.

[1] A.k.a. phase curves, solution curves, orbits, ... As is often done in the literature, we sometimes identify trajectories with their images.

[2] More precisely, an unpublished note; **not** a preprint. Comments are welcome.

[3] This book omits the proofs of some of the theorems that we need, but it is much easier reading than most of the sources to which it refers, so we use it for now.

all projections of Borel sets, complements of projections of Borel sets, and so on. Hence, PH is quite wild from the standpoint of real-analytic geometry (or even geometric measure theory, for that matter). All structures that come under consideration in this note are reducts (in the sense of definability) of PH. See *e.g.* [7, Ch. V] for basic definitions and facts about the projective hierarchy; also interesting is [4, Ch. 1.2.6].

The expansion of $\overline{\mathbb{R}}$ by all restricted analytic functions is denoted by $\mathbb{R}_{\mathrm{an}}$. A subset of $\mathbb{R}^n$ is definable in $\mathbb{R}_{\mathrm{an}}$ if and only if it is globally (sometimes called "finitely") subanalytic. In particular, if γ is a trajectory of an analytic planar vector field, then every compact connected subset of γ is definable in $\mathbb{R}_{\mathrm{an}}$. See [3,6] for more information about $\mathbb{R}_{\mathrm{an}}$.

The Pfaffian closure of $\mathbb{R}_{\mathrm{an}}$ is denoted by $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$. We do not need the precise definition; we need only know that $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$ is an o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$ such that if $g\colon \mathbb{R}^2 \to \mathbb{R}$ is definable in $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$ and $f\colon (a,b) \to \mathbb{R}$ satisfies $f'(t) = g(t, f(t))$ for all $t \in (a,b)$, then f is definable in $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$. Note that we may regard the graph of f as a trajectory of the definable (in $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$) vector field $(x,y) \mapsto (1, g(x,y))\colon (a,b) \times \mathbb{R} \to \mathbb{R}^2$. See [13,15] for details and more information.

For $a + ib \in \mathbb{C}$, x^{a+ib} denotes the map $t \mapsto t^a(\cos b \log t, \sin b \log t)\colon \mathbb{R}^{>0} \to \mathbb{R}^2$, *i.e.*, the restriction to the positive real line of the complex power function z^{a+ib} taken with respect to an appropriate branch of $\log z$. For $a \in \mathbb{R}$, identify x^{a+i0} with the real power function x^a. Note that $(\overline{\mathbb{R}}, x^{a+ib}) = (\overline{\mathbb{R}}, x^a, x^{ib})$. Since x^{a+ib} is analytic, every compact connected subset of its graph, and of its image, is definable in $\mathbb{R}_{\mathrm{an}}$. The image of x^{a+ib} is a trajectory of the linear vector field with matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ (re-parameterize by e^s, $s \in \mathbb{R}$). Hence, if $ab \neq 0$, then the image of x^{a+ib} is the logarithmic spiral $S_\omega := \{\, (e^{(1+i\omega)t} : t \in \mathbb{R} \,\}$ where $\omega = |a/b|$. We collect a few more easy facts.

1. **Lemma.** *Let $\omega > 0$. Put $\alpha = e^{2\pi/\omega}$ and $\alpha^{\mathbb{Z}} = \{\, \alpha^k : k \in \mathbb{Z} \,\}$.*

(1) $(\overline{\mathbb{R}}, S_\omega) = (\overline{\mathbb{R}}, x^{1+i\omega}) = (\overline{\mathbb{R}}, x^{i\omega})$.
(2) *For $r > 0$, $(\overline{\mathbb{R}}, x^{i\omega}\restriction(0,r)) = (\overline{\mathbb{R}}, x^{i\omega}) = (\overline{\mathbb{R}}, x^{i\omega}\restriction(r,\infty))$.*
(3) *For $r > 0$, $(\overline{\mathbb{R}}, S_\omega \cap B_r(0)) = (\overline{\mathbb{R}}, S_\omega) = (\overline{\mathbb{R}}, S_\omega \setminus B_r(0))$.*
(4) $(\overline{\mathbb{R}}, x^{i\omega})$ *defines* $\alpha^{\mathbb{Z}}$.
(5) $(\overline{\mathbb{R}}, x^{i\omega}\restriction[1,\alpha], \alpha^{\mathbb{Z}})$ *defines* $x^{i\omega}$.
(6) $(\mathbb{R}_{\mathrm{an}}, x^{i\omega}) = (\mathbb{R}_{\mathrm{an}}, \alpha^{\mathbb{Z}}) = (\mathbb{R}_{\mathrm{an}}, S_\omega)$.

Proof. (1). For the first equality, parameterize S_ω by distance to the origin.
(2). For every $s > 0$, we have $s^{i\omega} = \lim_{t\to 0^+}(st)^{i\omega}/t^{i\omega} = \lim_{t\to+\infty}(st)^{i\omega}/t^{i\omega}$.
(3). This is immediate from (1) and (2).
(4). $\alpha^{\mathbb{Z}} \times \{0\} = S_\omega \cap (\mathbb{R}^{>0} \times \{0\})$.
(5). $z = t^{i\omega}$ iff there exist $a \in \alpha^{\mathbb{Z}}$ and $b \in [1,\alpha)$ such that $t = ab$ and $z = b^{i\omega}$.
(6). $x^{i\omega}\restriction[1,\alpha]$ is analytic, hence definable in $\mathbb{R}_{\mathrm{an}}$. Now apply (1), (4) and (5). □

For $0 \neq \omega \in \mathbb{R}$, $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$ is clearly not o-minimal, but it turns out to be as well behaved as one could reasonably expect:

2. **Proposition.** *Let $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ definable in $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$. Then there is a finite partition $\mathcal{M}$ of $\mathbb{R}^n$ into embedded, analytic,* not necessarily connected *submanifolds of $\mathbb{R}^n$, each definable in $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$, and there is a* countable *partition $\mathcal{C}$ of*

$\mathbb{R}^n$ *into analytic cells, each definable in* $\mathbb{R}_{\mathrm{an}}$, *such that* $\mathcal{C}$ *is compatible with* $\mathcal{M}$, *and* $\mathcal{M}$ *is compatible with* $\mathcal{A}$.

Proof. By Lemma 1.6, there exists $\alpha > 1$ such that $(\mathbb{R}_{\mathrm{an}}, x^{i\omega}) = (\mathbb{R}_{\mathrm{an}}, \alpha^{\mathbb{Z}})$. By [3], $\mathbb{R}_{\mathrm{an}}$ defines no irrational power functions. Now see [11, §8.6, Remark]. □

In particular, every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$ is the union of an open set and finitely many discrete sets[4], so $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$ is certainly a proper reduct of PH.

We are now ready to state the main result of this note.

3. **Theorem.** *Let* $F = (F_1, F_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be analytic such that* $F^{-1}(0) = \{0\}$; $\lambda_1, \lambda_2 \in \mathbb{C}$ *be the eigenvalues of the Jacobian matrix of* F *at* 0; $\gamma\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ *be differentiable such that* $\gamma' = F \circ \gamma$ *and* $\lim_{t\to+\infty} \gamma(t) = 0$; *and* $\gamma_r := \gamma([r, \infty))$ *for* $r \in \mathbb{R}$.

(a) *If* $0 \neq \lambda_1$ *is imaginary, then* $(\overline{\mathbb{R}}, \gamma_r) = \mathrm{PH}$ *for every* $r \in \mathbb{R}$.
(b) *If* $\lambda_1 = a + ib$ *for* $a, b \in \mathbb{R} \setminus \{0\}$, *then* $(\mathbb{R}_{\mathrm{an}}, \gamma_r) = (\mathbb{R}_{\mathrm{an}}, x^{ia/b})$ *for every* $r \in \mathbb{R}$.
(c) *If* $0 \neq \lambda_1 \in \mathbb{R}$, *then* γ_r *is definable in* $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$—*so* $(\mathbb{R}_{\mathrm{an}}, \gamma_r)$ *is o-minimal—for every* $r \in \mathbb{R}$.

The theorem is trichotomous with respect to the assumption that at least one of λ_1, λ_2 is nonzero (recall that λ_1, λ_2 are either both real or they are complex conjugates). Loosely speaking, the theorem says that things go either as nicely as possible or as unpleasantly as possible. Naturally, the question arises: Does the trichotomy hold without the assumptions on the eigenvalues? At present, we do not know; work is ongoing.

Note. Only for temporary expositional convenience do we take F to be analytic on all of $\mathbb{R}^2$ and γ to be defined on all of $\mathbb{R}$. All results, appropriately modified, hold for F analytic on an open neighborhood U of 0 and γ defined on an open subinterval of $\mathbb{R}$.

4. **Corollary.** *Let* $F_1, \dots, F_n\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be analytic such that for* $k = 1, \dots, n$, $F_k^{-1}(0) = \{0\}$ *and the Jacobian of* F_k *at* 0 *has a nonzero eigenvalue. Then there is a finite* $\Omega \subseteq \mathbb{R} \setminus \{0\}$ *such that for any collection* $\mathcal{T}$ *of trajectories of* $F_1 \dots, F_n$, *each having* 0 *as a limit point, at least one of the following holds*:

(a) $(\mathbb{R}_{\mathrm{an}}, \mathcal{T}) = \mathrm{PH}$.
(b) *There exist* $\emptyset \neq \Omega_{\mathcal{T}} \subseteq \Omega$ *and a polynomially bounded o-minimal reduct* $\mathfrak{R}_{\mathcal{T}}$ *of* $(\mathbb{R}_{\mathrm{an}}, \mathcal{T})$ *such that* $(\mathbb{R}_{\mathrm{an}}, \mathcal{T}) = \big(\mathfrak{R}_{\mathcal{T}}, (x^{i\omega})_{\omega\in\Omega_{\mathcal{T}}}\big)$.
(c) $(\mathbb{R}_{\mathrm{an}}, \mathcal{T})$ *is a reduct of* $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$ (*and thus is o-minimal*).

Proof. Let Ω be the set of all $|a/b|$ such that $a + ib$ is an eigenvalue of some F_k and $ab \neq 0$. Let $\mathcal{T}$ be a collection of trajectories of $F_1, \dots, F_n$, each having 0 as a limit point. Suppose that neither (a) nor (c) holds; we show that (b) holds. It suffices to assume that each F_k has a trajectory that belongs to $\mathcal{T}$ and show that (b) holds with $\Omega_{\mathcal{T}} = \Omega$. Since (a) fails, none of the eigenvalues of the F_k are purely imaginary. Since (c) fails, $\Omega \neq \emptyset$. If none of the eigenvalues of the F_k are real, then put $\mathfrak{R}_{\mathcal{T}} = \mathbb{R}_{\mathrm{an}}$. Finally, suppose that $1 \leq m < n$ and $F_1, \dots, F_m$ are the F_k having real eigenvalues. The expansion $\mathfrak{R}_{\mathcal{T}}$ of $\mathbb{R}_{\mathrm{an}}$ by the trajectories in $\mathcal{T}$ of any of $F_1, \dots, F_m$ is a reduct of $\mathcal{P}(\mathbb{R}_{\mathrm{an}})$, and thus is o-minimal. Suppose, toward a contradiction, that $\mathfrak{R}_{\mathcal{T}}$ is not polynomially bounded. By growth dichotomy [10], it defines

[4] Actually, $(\mathbb{R}_{\mathrm{an}}, x^{i\omega})$ is d-minimal, but we don't want to make an issue of this right now.

the real exponential function e^x. Then $(\mathbb{R}_{\mathrm{an}}, \mathcal{T})$ is an expansion of $(\overline{\mathbb{R}}, e^x, x^{i\omega})$ for some $\omega \neq 0$. But $(\overline{\mathbb{R}}, e^x, x^{i\omega})$ defines *complex* exponentiation—$x^{1/\omega}$ is definable in $(\overline{\mathbb{R}}, e^x)$ and $e^{x+iy} = e^x((e^y)^{i\omega})^{1/\omega}$—hence also $\mathbb{Z}$, contradicting that (a) does not hold. $\square$

We do not know if the corollary is a trichotomy. Very little is known about the structures of case (b) unless $\mathfrak{R}_{\mathcal{T}}$ defines no irrational powers and there exists $\omega \in \Omega_{\mathcal{T}}$ such that $\Omega_{\mathcal{T}} \subseteq \mathbb{Q}.\omega$, in which case something similar to Proposition 2 holds (for the same reasons). Indeed, we do not yet understand structures $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ where $\alpha, \beta > 1$ and β is not a rational power of α, except that $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ defines sets that are somewhere both dense and codense, *e.g.*, the product group $\alpha^{\mathbb{Z}} \cdot \beta^{\mathbb{Z}}$. Note that if r is irrational, then $(\overline{\mathbb{R}}, x^r, x^{i\omega})$ defines the groups $\alpha^{\mathbb{Z}}$ and $\alpha^{r\mathbb{Z}}$ ($\alpha = e^{2\pi/\omega}$).

We now begin the proof of Theorem 3.

5. **Lemma** ([12])**.** *A structure on* $\overline{\mathbb{R}}$ *defines* $\mathbb{Z}$ *iff it defines the range of a sequence* $(a_k)_{k\in\mathbb{N}}$ *of real numbers such that* $\lim_{k\to+\infty}(a_{k+1} - a_k) \in \mathbb{R} \setminus \{0\}$.

Proof of Theorem 3(a). Suppose that $0 \neq \lambda_1$ is imaginary and let $r \in \mathbb{R}$. We show that $(\overline{\mathbb{R}}, \gamma_r)$ defines $\mathbb{Z}$. Since F is analytic, there exists $\delta > 0$ such that the Poincaré return map P of F exists on $(0, \delta)$, extends analytically to $(-\delta, \delta)$ and $P'(0) = 1$; see *e.g.* [14, pg. 218]. Since $\lim_{t\to+\infty} \gamma(t) = 0$, there is a neighborhood of 0 that is disjoint from any closed trajectories. Hence, P is not the identity, so there exist $c \neq 0$ and an integer $N > 1$ such that $P(x) = x + cx^N + o(x^N)$ as $x \to 0^+$. Moreover, after shrinking δ, we have $0 < P(x) < x$ for all $x \in (0, \delta)$. Choose any a_0 such that $0 < a_0 < \delta$ and $(a_0, 0) \in \gamma_r$. Inductively, put $a_{k+1} = P(a_k)$ for $k \in \mathbb{N}$; then $\gamma_r \cap \big((0, a_0] \times \{0\}\big) = \{\, (a_k, 0) : k \in \mathbb{N} \,\}$. Hence, the set $\{\, a_k^{1-N} : k \in \mathbb{N} \,\}$ is definable in $(\overline{\mathbb{R}}, \gamma_r)$. Routine computation[5] yields $\lim_{k\to+\infty}(a_{k+1}^{1-N} - a_k^{1-N}) = c(N-1) \neq 0$. Apply Lemma 5. $\square$

Proof of Theorem 3(b). Suppose that $\lambda_1 = a + ib$ for nonzero $a, b \in \mathbb{R}$. Put $\omega = |a/b|$ and let $r \in \mathbb{R}$. We show that $(\mathbb{R}_{\mathrm{an}}, \gamma_r) = (\mathbb{R}_{\mathrm{an}}, x^{i\omega})$. This is essentially immediate from ODE theory, and is easy to explain informally: "Near the origin, F is analytically equivalent to its linear part"; see *e.g.* [2, §24]. This means that there is an open neighborhood U of 0 and an analytic isomorphism $H \colon U \to V \subseteq \mathbb{R}^2$ such that H maps trajectories of $F{\restriction}U$ onto trajectories of $T{\restriction}V$, where T is the linear vector field with matrix $\left(\begin{smallmatrix} -1 & \omega \\ -\omega & -1 \end{smallmatrix}\right)$. By shrinking U, both H and H^{-1} are definable in $\mathbb{R}_{\mathrm{an}}$. Hence, by increasing r and replacing γ_r with $H(\gamma_r)$, we may assume that γ_r is a (half) trajectory of T. Then there exist $\epsilon > 0$ and $0 \neq z_0 \in \mathbb{C}$ such that $S_\omega \cap B_\epsilon(0) = \{\, \gamma(t) z_0 : t > r \,\}$. Apply Lemma 1. $\square$

Proof of Theorem 3(c). Here, we need neither the analyticity of F nor any special properties of $\mathbb{R}_{\mathrm{an}}$ (other than o-minimality), *so we drop all earlier assumptions and start over.*

Let $\mathfrak{R}$ be an o-minimal structure on $\overline{\mathbb{R}}$; "definable" means "definable in $\mathfrak{R}$". Cells and decompositions are taken with respect to $\mathfrak{R}$. By passing to its Pfaffian closure, we may assume that the following holds.

[5] Credit goes to Kobi Peterzil for noticing, during the workshop at Fields mentioned in the first-page footnotes, that this *is* the routine computation to check.

6. **Proposition** ([13, Prop. 7]). *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be definable, I be an open subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be differentiable such that $f'(t) = g(t, f(t))$ for all $t \in I$. Then f is definable.*

Note. There are no assumptions on g other than definability.

A subset A of $\mathbb{R}^2$ is a **spiral** (around the origin) if $0 \notin A$ and there exist continuous functions $\rho, \theta\colon \mathbb{R} \to \mathbb{R}$ such that $A = \{\, \rho(t)e^{i\theta(t)} : t \in \mathbb{R}\,\}$, $\rho > 0$, $\lim_{t\to+\infty} \rho(t) = 0$, and θ is either ultimately increasing and unbounded above, or ultimately decreasing and unbounded below.

Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 and definable such that $F^{-1}(0) = \{0\}$. Let γ and γ_r be as before.

7. **Theorem** (the "fundamental alternative"; *cf.* [1, pp. 84-85]). *Either $\gamma(\mathbb{R})$ is a spiral or every γ_r is definable.*

Assuming this for the moment, we have the following generalization of Theorem 3(c):

8. **Proposition.** *If F is C^2 in a neighborhood of 0 and the Jacobian of F at 0 has a nonzero real eigenvalue, then γ_r is definable for every r.*

Proof. By the fundamental alternative, we need only show that $\gamma(\mathbb{R})$ is not a spiral, which is immediate from ODE theory. We give only a brief outline here.

Assume that both eigenvalues have the same sign. Then, near the origin, F is C^1-diffeomorphic to its linear part [14, Theorem, pg. 127]; in particular, F has a spiralling trajectory around 0 if and only the same is true of its linear part. Now apply the classification of linear fields [14, §1.5 Cases I and II].

Assume that the eigenvalues have distinct signs. By the center manifold theorem [14, pg. 116], at least one trajectory of F has 0 as a limit point and is not a spiral. Hence, no trajectory of F is a spiral. □

Remark. The map $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x,y) \mapsto \begin{cases} \left(-x - 2y/\log(x^2+y^2),\ -y + 2x/\log(x^2+y^2)\right), & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$$

shows that Proposition 8 does not hold with C^1 in place of C^2. (*Hint.* Switch to polar coordinates.)

We now proceed toward the proof of the fundamental alternative.

9. **Lemma.** *Let I be an open subinterval of $\mathbb{R}$.*

(1) *If $F_1 \circ \gamma{\restriction}I = 0$, then $\gamma_2{\restriction}I$ is definable.*
(2) *If $F_1 \circ \gamma{\restriction}I < 0$ or $F_1 \circ \gamma{\restriction}I > 0$, then $\gamma_1{\restriction}I$ is strictly monotone and $\gamma_2 \circ \gamma_1^{-1}\colon \gamma_1(I) \to \mathbb{R}$ is definable.*
(3) *If $F_1 \circ \gamma{\restriction}I$ is of constant sign, then $\gamma(I)$ is definable.*

Proof. (1). If $F_1 \circ \gamma{\restriction}I = 0$, then $\gamma_1'{\restriction}I = 0$, so for some $c \in \mathbb{R}$ we have $\gamma_2' = F_2(c, \gamma_2)$ on I. Apply Proposition 6.

(2). Suppose that either $F_1 \circ \gamma \restriction I > 0$ or $F_1 \circ \gamma \restriction I < 0$. Then γ_1 is strictly monotone on I with differentiable compositional inverse $\gamma_1^{-1}\colon \gamma_1(I) \to \mathbb{R}$. Put

$$g(x,y) = \begin{cases} F_2(x,y)/F_1(x,y), & F_1(x,y) \neq 0 \\ 0, & F_1(x,y) = 0. \end{cases}$$

Then $(\gamma_2 \circ \gamma_1^{-1})'(t) = g(t, \gamma_2 \circ \gamma_1^{-1}(t))$ for all $t \in I$. By Proposition 6, $\gamma_2 \circ \gamma_1^{-1}\colon \gamma_1(I) \to \mathbb{R}$ is definable.

(3). This is immediate from (1) and (2). $\square$

10. **Lemma.** *Let J be a compact subinterval of $\mathbb{R}$. Then $\gamma(J)$ is definable.*

Proof. Take a C^1-cell decomposition of $\mathbb{R}^2$ compatible with $F_1^{-1}(0)$. Since γ' has no zeros and $\gamma(J)$ is compact and connected, the intersection of each cell of the decomposition with $\gamma(J)$ has only finitely many connected components. Apply Lemma 9. $\square$

11. **Lemma.** *Let $C \subseteq \mathbb{R}^2$ be a 1-dimensional C^1-cell such that $0 \in \mathrm{fr}(C)$. Then there exists r such that one of the following holds*:

(1) $\gamma_r \cap C = \emptyset$.
(2) $\gamma_r = B_\delta(0) \cap C$ *for some* $\delta > 0$.
(3) $\gamma_r \cap C$ *is (the range of) a sequence* $(P_n)_{n\in\mathbb{N}}$ *of points such that* $\lim_{n\to+\infty} P_n = 0$ *and* $\gamma(\mathbb{R})$ *is clockwise transverse to* C *at each* P_n.
(4) $\gamma_r \cap C$ *is a sequence* $(P_n)_{n\in\mathbb{N}}$ *of points such that* $\lim_{n\to+\infty} P_n = 0$ *and* $\gamma(\mathbb{R})$ *is counterclockwise transverse to* C *at each* P_n.

Proof. We are done if (1) holds, so assume otherwise, *i.e.*, $\gamma_r \cap C \neq \emptyset$ for every r. Since $\lim_{t\to+\infty} \gamma(t) = 0$, we have $B_\epsilon(0) \cap C \cap \gamma(\mathbb{R}) \neq \emptyset$ for every $\epsilon > 0$.

By curve selection, there is a definable C^1 parametrization $\phi\colon (a,\infty) \to \mathbb{R}^2$ of C such that $\lim_{t\to+\infty} \phi(t) = 0$. By increasing a, we may assume that ϕ' has no zeros and $(-\phi_2', \phi_1')\cdot(F\circ\phi)$ is of constant sign.

If $(-\phi_2', \phi_1') \cdot (F \circ \phi) = 0$, then F is tangent to C at every point of C. Since $\gamma(\mathbb{R})$ is tangent to F at every point of $\gamma(\mathbb{R})$ and F is C^1, we have $\gamma_r \subseteq C$ for every r such that $\gamma(r) \in C$. Hence, (2) holds.

If $(-\phi_2', \phi_1') \cdot (F \circ \phi) < 0$, then F is clockwise transverse to C at every point of C, so $\gamma(\mathbb{R})$ is clockwise transverse to C at every point of $\gamma(\mathbb{R}) \cap C$. Hence, (3) holds. Similarly, (4) holds if $(-\phi_2', \phi_1') \cdot (F \circ \phi) > 0$. $\square$

Proof of the fundamental alternative. For $(x,y) \in \mathbb{R}^2$, put $\sigma(x,y) = xF_2(x,y) - yF_1(x,y)$. Note that σ is definable and the sign of $\sigma(\gamma(t))$ is that of the angular derivative of γ at t.

Suppose that $\sigma \geq 0$ on some open ball B about 0. Let R be such that $\gamma_R \subseteq B$. Then the function $t \mapsto \int_R^t \sigma \circ \gamma\colon [R, +\infty) \to \mathbb{R}$ is increasing, so $\lim_{t\to+\infty} \int_R^t \sigma \circ \gamma$ exists in $[0, +\infty]$. If $\lim_{t\to+\infty} \int_R^t \sigma\circ\gamma = +\infty$, then γ is a spiral. If $\lim_{t\to+\infty} \int_R^t \sigma\circ\gamma \in \mathbb{R}$, then there exists $u \in S^1$ such that $\lim_{t\to+\infty} \gamma(t)/\|\gamma(t)\| = u$. By rotation, we may assume that $u = (1,0)$. Take a C^1-cell decomposition of $\mathbb{R}^2$ compatible with $\mathrm{bd}(F_1^{-1}(0))$. Then there exists $\epsilon > 0$ such that $\mathrm{bd}(F_1^{-1}(0)) \cap B_\epsilon(0) \setminus \big((-\infty, 0] \times \{0\}\big)$ is a finite disjoint union of 1-dimensional C^1-cells $C_1, \ldots, C_d$ such that for each $j = 1, \ldots, d$, we have $0 \in \mathrm{fr}(C_j)$ and $C_j \cap \mathrm{bd}(B_\epsilon(0)) \neq \emptyset$. Fix one for the moment, say $C := C_1$. Now, $\big((-\infty, 0] \times \{0\}\big) \cup C$ disconnects $B_\epsilon(0)$, so by Lemma 11 (and the intermediate value theorem), there exists r such that γ_r is either contained in C or is disjoint from C. Since this is true for each C_j, $F_1 \circ \gamma$ is ultimately of constant sign. Apply Lemmas 9 and 10 to finish.

By a similar argument, we are done if $\sigma \leq 0$ on an open ball about 0.

Finally, assume that for every $\epsilon > 0$ there exist points $P_\epsilon, Q_\epsilon \in B_\epsilon(0) \cap \gamma(\mathbb{R})$ such that $\sigma(P_\epsilon) > 0$ and $\sigma(Q_\epsilon) < 0$. By curve selection and cell decomposition, there exist disjoint 1-dimensional C^1-cells $C, D \subseteq \mathbb{R}^2$ such that $\{0\} = \mathrm{fr}(C) \cap \mathrm{fr}(D)$, $\sigma{\restriction}C > 0$ and $\sigma{\restriction}D < 0$. Then there exists $\epsilon > 0$ such that $B_\epsilon(0) \setminus \mathrm{cl}(C \cup D)$ consists of two disjoint connected nonempty open sets V, W such that γ is disjoint from one of V or W. The rest of the argument is now similar to that of the case that σ is nonnegative in a neighborhood of 0. $\square$

We have now established Theorem 3. Reflection on the proof shows the way toward possible extension of the trichotomy. By the fundamental alternative, the real issue is: What can be said about an expansion $(\mathfrak{R}, \gamma_r)$ of an o-minimal structure $\mathfrak{R}$ on $\overline{\mathbb{R}}$ by a spiralling half-trajectory γ_r of a definable-in-$\mathfrak{R}$ planar vector field (say, at least C^1) if $(\mathfrak{R}, \gamma_r)$ does *not* define $\mathbb{Z}$? In this generality, the question is quite daunting—recall that we don't even know how to handle $(\overline{\mathbb{R}}, x^{a+ib})$ if a is irrational and $b \neq 0$—so for now we are attempting to remove the assumptions on the eigenvalues in Theorem 3.

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