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MR1906722 (2003m:11060)

[Milne, Stephen C.](#) (1-OHS)**Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions. (English summary)***Ramanujan J.* **6** (2002), *no. 1*, 7–149.[11E25](#) ([05A15](#) [11B65](#) [11F27](#) [33D67](#) [33D70](#) [33E05](#))

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FEATURED REVIEW.

Let N be a positive integer. A classical (and difficult) problem in number theory is the question: “How many ways are there to write N as a sum of s squares?”, that is, how many integer solutions (x_1, x_2, \dots, x_s) of the equation

$$x_1^2 + x_2^2 + \dots + x_s^2 = N$$

are there? (It is assumed that changing signs or the order of the x_i 's gives distinct solutions. Otherwise, the problem would be even more difficult.) The problem is naturally approached by using generating functions: let $f(q)$ denote the doubly infinite series

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

Then, what we need in order to solve the problem is to extract the coefficient of q^N in $f(q)^s$. (This gives the number in question, multiplied by $(-1)^N$.) Jacobi's triple product identity [see G. Gasper, Jr. and M. Rahman, *Basic hypergeometric series*, Cambridge Univ. Press, Cambridge, 1990; [MR 91d:33034](#)(Sect. 1.6)] tells us that

$$(1) \quad f(q) = (q; q)_{\infty} (q; q^2)_{\infty},$$

where $(a; q)_{\infty} := \prod_{i=0}^{\infty} (1 - aq^i)$. Therefore, what we have to do is to raise the right-hand side of (1) to the s -th power, and extract the coefficient of q^N . Clearly, this will not give a reasonable expression, unless we have some (q -series) identity which allows us to rewrite this s -th power in a form which is more amenable to coefficient extraction. Thus, the problem has a natural place in

the theory of q -series identities [see op. cit., Sect. 8.11]. For example, Jacobi proved that

$$(2) \quad f(q)^4 = 1 + 8 \sum_{r=1}^{\infty} (-1)^r \frac{rq^r}{1+q^r},$$

which, on coefficient extraction, implies that the number of ways to write N as a sum of 4 squares is given by the elegant formula $8 \sum_{d|N, 4 \nmid d} d$. (In particular, it is obvious from this formula that this number is nonzero and, hence, any positive integer N can be written as a sum of 4 squares, a fact apparently already known to Diophantus, but first proved by Lagrange.) Jacobi found similar formulas for 2, 6, and 8 squares.

Although G. E. Andrews [SIAM Rev. **16** (1974), 441–484; MR **50** #5044] had shown that q -series theory provides elegant proofs of Jacobi’s formulas for 2 and 4 squares, otherwise it has been almost exclusively the theory of elliptic and theta functions, and of modular forms, that has been used to derive results on this problem. This latter approach is motivated by the observation that $f(q)$ is in fact a (specialized) Jacobi theta function, namely $\vartheta_3(0, -q)$. Thus, a heavy and well-developed machinery can be applied. It is in fact the method that Jacobi used in his work. Numerous results have appeared in the literature since Jacobi (see the paper under review for an exhaustive listing) for special values of s . However, until recently no results for infinite families of values of s were known.

It is here that Milne’s fundamental work comes in. The main results of the paper under review are formulas for the number of ways to express an integer N as a sum of s squares for $s = 4n^2$, $n = 1, 2, \dots$, and for $s = 4n(n+1)$, $n = 1, 2, \dots$. These formulas embed Jacobi’s formulas for 4, 6, and 8 squares into infinite families. In particular, it is here that for the first time formulas for infinite families of values of s are found.

Milne indicates in the introduction that, originally, he derived his (new) formulas for 16 and 24 squares from results he had found earlier in his work on multiple basic hypergeometric series associated to root systems. (He does not explain that in more detail, but one gets some glimpse from his papers with V. E. Leininger [Discrete Math. **204** (1999), no. 1-3, 281–317; [MR 2000e:33024](#); Methods Appl. Anal. **6** (1999), no. 2, 225–248; [MR 2001m:33036](#)].) His approach was inspired by Andrews’ idea of applying q -series theory, mentioned above, to find formulas for powers of $\vartheta_3(0, -q)$. Once the formulas were found, Milne saw how to extend them to infinite families. In order to do that, he used theta function theory. Thus, *both* approaches were important in the finding of these impressive new results.

To give an idea of the beauty of Milne’s identities, let me present his result for $4n^2$ squares:

We have

$$(3) \quad \vartheta_3(0, -q)^{4n^2} = (-1)^n 2^{2n^2+n} \left(\prod_{r=1}^{2n-1} r!^{-1} \right) \det(g_{r+s-1})_{1 \leq r, s \leq n},$$

with $g_i := U_{2i-1} - c_i$, where U_{2i-1} and c_i are defined by

$$U_{2i-1} := \sum_{r=1}^{\infty} (-1)^{r-1} \frac{r^{2i-1} q^r}{1+q^r},$$

$$c_i := (-1)^{i-1} \frac{2^{2i} - 1}{4i} \cdot |B_{2i}|,$$

with B_{2i} the $2i$ -th Bernoulli number.

Jacobi's formula (2) is the special case $n = 1$. By extracting the coefficient of q^N on the right-hand side of (3), Milne obtains an expression for the number of representations of N as a sum of $4n^2$ squares, in the form of certain multiple sums featuring Schur functions, which generalize Jacobi's.

If we examine identity (3) more closely, we see that it is in fact a Hankel determinant evaluation, where the entries are certain Lambert series, which yields (up to some precise factors) a power of the theta function $\vartheta_3(0, -q)$. Indeed, a large part of the paper is devoted to a(n almost encyclopedic) study of evaluations of Hankel determinants where the entries are various ratios of Jacobi elliptic functions $\operatorname{sn}(n, k)$, $\operatorname{cn}(n, k)$, $\operatorname{dn}(n, k)$, etc. The article has an amazing number of such evaluations, most of them new. They are proved by establishing continued fraction expansions for the Laplace transform of these ratios, upon which the close connection of continued fractions and Hankel determinants immediately yields Hankel determinant evaluations.

There are more cases (other than (3)) where the result of a Hankel determinant evaluation is (essentially) a power of a theta function. One of them yields, upon coefficient extraction, the new result for $4n(n+1)$ squares. However, there is another case which provides a proof of a formula, conjectured by V. G. Kac and M. Wakimoto [in *Lie theory and geometry*, 415–456, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994; [MR 96j:11056](#)], for the number of representations of an integer N as a sum of s triangular numbers, where $s = 4n^2$ or $s = 4n(n+1)$. In this case, this is based on a Hankel determinant evaluation yielding a power of Jacobi's $\vartheta_2(0, q)$.

The coefficient extraction itself is in fact also nontrivial. It requires a careful examination of which of the terms in the expansion that results from coefficient extraction cancel each other. The author accomplishes this using tricky combinatorial arguments.

Let me finally mention that Milne is also able to use his results to make assertions on the asymptotic behaviour of the numbers of representations of N as a sum of s squares (with $s = 4n^2$ or $s = 4n(n+1)$) as $n \rightarrow \infty$.

To conclude this review, it must be mentioned that the conjecture of Kac and Wakimoto (and, in fact, a more general conjecture by those authors) had been independently proven by D. B. Zagier [Math. Res. Lett. **7** (2000), no. 5-6, 597–604; [MR 2001m:11064](#)], using modular forms techniques. Subsequently, K. Ono [J. Number Theory **95** (2002), no. 2, 253–258; [MR 2003f:11062](#)], motivated by Milne's exciting breakthrough, utilized an elementary modular forms involution to translate Zagier's results into elegant corresponding results on the number of representations as sums of squares. The resulting formulas are different from Milne's, and somewhat simpler. The circle was closed by H.-H. Chan and the reviewer [in preparation], who showed by elementary manipulations

how one can pass from Zagier's results directly to Milne's corresponding determinantal results.

However, this is not the end of the story. One way to view Milne's identities (as for example (3)) is to say that $\vartheta_3(0, -q)^{4n^2}$ is a linear combination of products of "generalized Lambert series" (by which I mean a Lambert series plus some constant). In particular, for $n = 2$ we have that $\vartheta_3(0, -q)^{16}$ is a linear combination of products of *two* generalized Lambert series. Inspired by this observation, Chan and K. S. Chua ["Representations of integers as sums of 32 squares", Ramanujan J., to appear] conjecture that *any* even power of $\vartheta_3(0, -q)$ is a linear combination of products of *two* generalized Lambert series, and not only that, they have a rather precise conjecture about what these linear combinations should be. As this conjecture is so fascinating, let me cite one case: let $\sec^2 u = \sum_{k=0}^{\infty} a_{2k} u^{2k} / (2k)!$, and suppose that $A_k / B_k = a_{2k} / 2^{2k+3}$, where $k > 1$ and $\gcd(A_k, B_k) = 1$. Then for any positive integer $t \geq 1$,

$$(4) \quad \vartheta_3(0, -q)^{8t+8} = \sum_{l=1}^t c_l \mathcal{E}_{4t+2-2l}(q) \mathcal{E}_{2l+2}(q),$$

where the c_l 's are certain rational numbers and

$$\mathcal{E}_{2k+2}(q) = A_k - B_k \sum_{j=1}^{\infty} \frac{(-1)^{k+j} j^{2k+1} q^j}{1 - q^j}.$$

Clearly, upon extracting coefficients of q^N from (4), we obtain extremely efficient formulas for the number of ways to write N as a sum of $8t + 8$ squares provided that, in addition, explicit formulas for the c_l 's can be found. Chan and Chua have similar conjectures for the other even powers of $\vartheta_3(0, -q)$.

Reviewed by [Christian Krattenthaler](#)

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