Here's the exact reference of this (see below)

# Séries Formelles

# et

# Combinatoire Algébrique

9iéme Colloque

14 Juillet – 18 Juillet 1997

# Formal Power Series

# and

# Algebraic Combinatorics

9th Conference

July 14 – July 18, 1997

UNIVERSITÄT WIEN

*Conference Proceedings — Volume 3* of 3.

(eds.: P. Kirschenhofer, C. Krattenthaler, D. Krob, and H. Prodinger), pp. 403–417, FPSAC'97 (1997)

# NEW INFINITE FAMILIES OF EXACT SUMS OF SQUARES FORMULAS, JACOBI ELLIPTIC FUNCTIONS, AND RAMANUJAN'S TAU FUNCTION

Exact reference: Stephen C. Milne, "New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function" pp. 403–417 of Formal Power Series and Algebraic Combinatorics, 9th Conference, July 14 – July 18, 1997, Universität Wien, Conference Proceedings — Volume 3 of 3, (eds.: P. Kirschenhofer, C. Krattenthaler, D. Krob, and H. Prodinger), FPSAC'97 (1997).


STEPHEN C. MILNE

The Ohio State University


16 June 1996 (revised 20 April 1997)


1991 *Mathematics Subject Classification.* Primary 11E25, 33E05; Secondary 05A15, 33D70.

*Key words and phrases.* Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, Schur functions, multiple basic hypergeometric series, $C_\ell$ nonterminating $_6\phi_5$ summation theorem.

S. C. Milne was partially supported by National Security Agency grant MDA 904-93-H-3032

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

abstract:
In this paper we give two infinite families of explicit exact formulas that generalize Jacobi's (1829) 4 and 8 squares identities to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's tau function $\tau(n)$, when $n$ is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. We have also obtained many additional infinite families of identities in this same setting that are analogous to the $\eta$-function identities in Appendix I of Macdonald (1972). A special case of our methods yields a proof of the two Kac–Wakimoto (1994) conjectured identities involving representing a positive integer by sums of $4n^2$ or $4n(n+1)$ triangular numbers, respectively. Our 16 and 24 squares identities were originally obtained via multiple basic hypergeometric series, Gustafson's $C_\ell$ nonterminating ${}_6\phi_5$ summation theorem, and Andrews' basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for $n^2$ or $n(n+1)$ squares, respectively. Our sums of more than 8 squares identities are not the same as the formulas of Mathews (1895), Glaisher (1907), Ramanujan (1916), Mordell (1917, 1919), Hardy (1918, 1920), Kac and Wakimoto (1994), and many others.

résumé:
Dans ce travail on présente deux familles infinies de formules exactes qui généralisent les identités à 4 (resp. 8) carrés de Jacobi (1829) en identités à $4n^2$ (resp. $4n(n+1)$) carrés, sans utiliser des formes cuspoides. Notre identité à 24 carrés nous donne une formule différente de la formule de Ramanujan pour la fonction $\tau(n)$ pour $n$ impair. Ces résultats apparaissent dans le cadre des fonctions elliptiques de Jacobi, fractions continues de Jacobi, déterminants de Hankel (ou Turán), séries de Fourier, séries de Lambert, principe d'inclusion/exclusion, développeme t de déterminants selon Laplace et fonctions de Schur. Dans ce cadre nous avons également obtenu plusieurs familles supplémentaires d'identités qui correspondent à des identités pour la fonction $\eta$ dans l'appendice I de Macdonald (1972). Dans un cas particulier notre méthode démontre les deux identités traitant les représentations d'un entier positif par une somme de $4n^2$ (resp $4n(n+1)$) nombres triangulaires, conjecturées par Kac-Wakimoto en 1994. Nos identités à 16 et à 24 carrés ont d'abord été obtenues par les séries hypergéométriques basiques multiples, le théorème de sommation pour une série ${}_6\phi_5$ non-terminante de type $C_\ell$ de Gustafson, et la démonstration d'Andrews utilisant des séries hypergéométriques basiques des identités à 4 et à 8 carrés de Jacobi. Nous avons employé (ailleurs) des techniques de symétrie et de fonctions de Schur à cette approche originale afin de démontrer l'existence de familles infinies semblables d'identités à $n^2$ et à $n(n+1)$ carrés. Nos identités à plus de 8 carrés ne sont pas les mêmes que les formules de Mathews (1895), Glaisher (1907), Ramanujan (1916), Mordell (1917,1919), Hardy (1918,1920), Kac et Wakimoto (1994) et beaucoup d'autres.

## 1. Introduction

In this paper we give infinite families of explicit exact formulas involving either squares or triangular numbers, two of which generalize Jacobi's [27] 4 and 8 squares identities to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's [56] tau function $\tau(n)$, when $n$ is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. This background material is contained in [5, 6, 18, 19, 25–29, 33, 38, 59, 60, 62, 70, 71]. Further details of the proofs of all our infinite families of identities appear in [46, 47]. Some of this work has already been announced in [44].

The problem of representing an integer as a sum of squares of integers has had a long and interesting history, which is surveyed in [20] and chapters 6-9 of [9]. The review article [63] presents many questions connected with representations of integers as sums of squares. Direct applications of sums of squares to lattice point problems and crystallography can be found in [17]. One such example is the computation of the constant $Z_N$ that occurs in the evaluation of a certain Epstein zeta function, needed in the study of the stability of rare gas crystals, and in that of the so-called *Madelung constants* of ionic salts.

The $s$ squares problem is to count the number $r_s(n)$ of integer solutions $(x_1, \dots, x_s)$ of the diophantine equation

$$x_1^2 + \cdots + x_s^2 = n, \tag{1}$$

in which changing the sign or order of the $x_i$'s give distinct solutions.

Diophantus (325-409 A.D.) knew that no integer of the form $4n-1$ is a sum of two squares. Girard conjectured in 1632 that $n$ is a sum of two squares if and only if all prime divisors $q$ of $n$ with $q \equiv 3 \pmod 4$ occur in $n$ to an even power. Fermat in 1641 gave an "irrefutable proof" of this conjecture. Euler gave the first known proof in 1749. Early explicit formulas for $r_2(n)$ were given by Legendre in 1798 and Gauss in 1801. It appears that Diophantus was aware that all positive integers are sums of four integral squares. Bachet conjectured this result in 1621, and Lagrange gave the first proof in 1770.

Jacobi in his famous *Fundamenta Nova* [27] of 1829 introduced elliptic and theta functions, and utilized them as tools in the study of (1). Motivated by Euler's work on 4 squares, Jacobi knew that the number $r_s(n)$ of integer solutions of (1) was also determined by

$$\vartheta_3(0,-q)^s := 1 + \sum_{n=1}^{\infty}(-1)^n r_s(n) q^n, \tag{2}$$

where $\vartheta_3(0,q)$ is the $z=0$ case of the theta function $\vartheta_3(z,q)$ in [70, pp. 464] given by

$$\vartheta_3(0,q) := \sum_{j=-\infty}^{\infty} q^{j^2}. \tag{3}$$

Jacobi then used his theory of elliptic and theta functions to derive remarkable identities for the $s = 2, 4, 6, 8$ cases of $\vartheta_3(0,-q)^s$. He immediately obtained elegant explicit formulas for $r_s(n)$, where $s = 2, 4, 6, 8$.

We recall Jacobi's identities for $s = 4$ and 8 in

**Theorem 1.1 (Jacobi).**

$$\vartheta_3(0,-q)^4 = 1 - 8\sum_{r=1}^{\infty}(-1)^{r-1}\frac{rq^r}{1+q^r} = 1 + 8\sum_{n=1}^{\infty}(-1)^n \Big[\sum_{\substack{d\mid n,\, d>0 \\ 4\nmid d}} d\Big] q^n, \tag{4}$$

*and*

$$\vartheta_3(0,-q)^8 = 1+16\sum_{r=1}^{\infty}(-1)^r\frac{r^3q^r}{1-q^r} = 1+16\sum_{n=1}^{\infty}\Big[\sum_{d|n,d>0}(-1)^d d^3\Big]q^n. \qquad (5)$$

*Consequently, we have*

$$r_4(n) = 8\sum_{\substack{d|n,d>0\\ 4\nmid d}} d \qquad \text{and} \qquad r_8(n) = 16\sum_{d|n,d>0}(-1)^{n+d}d^3, \qquad (6)$$

*respectively.*

In general it is true that

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n), \qquad (7)$$

where $\delta_{2s}(n)$ is a divisor function and $e_{2s}(n)$ is a function of order substantially lower than that of $\delta_{2s}(n)$. If $2s = 2,4,6,8$, then $e_{2s}(n) = 0$, and (7) becomes Jacobi's formulas for $r_{2s}(n)$, including (6). On the other hand, if $2s > 8$ then $e_{2s}(n)$ is never 0. The function $e_{2s}(n)$ is the coefficient of $q^n$ in a suitable "cusp form". The difficulties of computing (7), and especially the non-dominate term $e_{2s}(n)$, increase rapidly with $2s$. The modular function approach to (7) and the cusp form $e_{2s}(n)$ is discussed in [59, pp. 241–244]. For $2s > 8$ modular function methods such as those in [21, 23, 24, 36, 53, 58], or the more classical elliptic function approach of [7, 31, 32], are used to determine general formulas for $\delta_{2s}(n)$ and $e_{2s}(n)$ in (7). Explicit, exact examples of (7) have been worked out for $2 \leq 2s \leq 32$. Similarly, explicit formulas for $r_s(n)$ have been found for (odd) $s < 32$. Alternate, elementary approaches to sums of squares formulas can be found in [39, 61, 64–67].

We next consider classical analogs of (4) and (5) corresponding to the $s = 8$ and 12 cases of (7).

Glaisher [15, pp. 210] utilized elliptic function methods, rather than modular functions, to prove

**Theorem 1.2 (Glaisher).**

$$\vartheta_3(0,-q)^{16} = 1 + \tfrac{32}{17}\sum_{y_1,m_1\geq 1}(-1)^{m_1}m_1^7q^{m_1y_1} \qquad (8a)$$

$$- \tfrac{512}{17}\, q\,(q;q)_\infty^8\,(q^2;q^2)_\infty^8 \qquad (8b)$$

*where we have*

$$(q;q)_\infty := \prod_{r\geq 1}(1-q^r). \qquad (9)$$

Glaisher took the coefficient of $q^n$ to obtain $r_{16}(n)$. The same formula appears in [59, Eqn. (7.4.32), pp. 242].

In order to find $r_{24}(n)$, Ramanujan [56, Entry 7, table VI], see also [59, Eqn. (7.4.37), pp. 243], first proved

**Theorem 1.3 (Ramanujan).** *Let* $(q;q)_\infty$ *be defined by* (9). *Then*

$$\vartheta_3(0,-q)^{24} = 1 + \tfrac{16}{691}\sum_{y_1,m_1\geq 1}(-1)^{m_1}m_1^{11}q^{m_1y_1} \tag{10a}$$

$$-\tfrac{33152}{691}\,q\,(q;q)_\infty^{24} - \tfrac{65536}{691}\,q^2\,(q^2;q^2)_\infty^{24}. \tag{10b}$$

One of the main motivations for this paper was to generalize Theorem 1.1 to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms such as (8b) and (10b), while still utilizing just sums of products of at most $n$ Lambert series similar to either (4) or (5), respectively. This is done in Theorems 2.2 and 2.4 below. Here, we state the $n=2$ cases, which determine different formulas for 16 and 24 squares.

**Theorem 1.4.**

$$\vartheta_3(0,-q)^{16} = 1 - \tfrac{32}{3}\,(U_1+U_3+U_5) + \tfrac{256}{3}\,\left(U_1U_5 - U_3^2\right), \tag{11}$$

*where*

$$U_s \equiv U_s(q) := \sum_{r=1}^{\infty}(-1)^{r-1}\frac{r^sq^r}{1+q^r} = \sum_{n=1}^{\infty}\Big[\sum_{d|n,d>0}(-1)^{d+n/d}d^s\Big]q^n \tag{12}$$

$$= \sum_{y_1,m_1\geq 1}(-1)^{y_1+m_1}m_1^sq^{m_1y_1}.$$

Analogous to Theorem 1.3, we have

**Theorem 1.5.**

$$\vartheta_3(0,-q)^{24} = 1 + \tfrac{16}{9}\,(17G_3+8G_5+2G_7) + \tfrac{512}{9}\,\left(G_3G_7 - G_5^2\right), \tag{13}$$

*where*

$$G_s \equiv G_s(q) := \sum_{r=1}^{\infty}(-1)^{r}\frac{r^sq^r}{1-q^r} = \sum_{n=1}^{\infty}\Big[\sum_{d|n,d>0}(-1)^{d}d^s\Big]q^n \tag{14}$$

$$= \sum_{y_1,m_1\geq 1}(-1)^{m_1}m_1^sq^{m_1y_1}.$$

An analysis of (10b) depends upon Ramanujan's [56] tau function $\tau(n)$ defined by

$$q\,(q;q)_\infty^{24} := \sum_{n=1}^{\infty}\tau(n)q^n. \tag{15}$$

For example, $\tau(1)=1$, $\tau(2)=-24$, $\tau(3)=252$, $\tau(4)=-1472$, $\tau(5)=4830$, $\tau(6)=-6048$, and $\tau(7)=-16744$. Ramanujan [56, Eqn. (103)] conjectured, and Mordell [52] proved that $\tau(n)$ is multiplicative.

In the case where $n$ is an odd integer [in particular an odd prime], equating (10a-b) and (13) yields two formulas for $\tau(n)$ that are different from Dyson's [10] formula. We first obtain

**Theorem 1.6.** *Let $\tau(n)$ be defined by* (15) *and let $n$ be odd. Then*

$$259\tau(n) = \tfrac{1}{2^3\cdot 3^2}\left[17\cdot 691\sigma_3(n) + 8\cdot 691\sigma_5(n) + 2\cdot 691\sigma_7(n) - 9\sigma_{11}(n)\right] \tag{16}$$

$$- \tfrac{691\cdot 2^2}{3^2}\sum_{m=1}^{n-1}\left[\sigma_3^\dagger(m)\sigma_7^\dagger(n-m) - \sigma_5^\dagger(m)\sigma_5^\dagger(n-m)\right],$$

*where*

$$\sigma_r(n) := \sum_{d|n,d>0} d^r \qquad \text{and} \qquad \sigma_r^\dagger(n) := \sum_{d|n,d>0}(-1)^d d^r \tag{17}$$

*Remark.* We can use (16) to compute $\tau(n)$ in $\leq 6n\ln n$ steps when $n$ is an odd integer. This may also be done in $n^{2+\epsilon}$ steps by appealing to Euler's infinite-product-representation algorithm (EIPRA) [3, pp.104 ] applied to $(q;q)_\infty^{24}$ in (15).

A different simplification involving a power series formulation of (13) leads to

**Theorem 1.7.** *Let $\tau(n)$ be defined by* (15) *and let $n \geq 3$ be odd. Then*

$$259\tau(n) = \tfrac{1}{2^3}\sum_{d|n,d>0}(-1)^d d^{11} - \tfrac{691}{2^3\cdot 3^2}\sum_{d|n,d>0}(-1)^d d^3\left(17 + 8d^2 + 2d^4\right) \tag{18a}$$

$$- \tfrac{691\cdot 2^2}{3^2}\sum_{\substack{m_1>m_2\geq 1\\ m_1+m_2\leq n\\ gcd(m_1,m_2)|n}}(-1)^{m_1+m_2}(m_1m_2)^3(m_1^2-m_2^2)^2\sum_{\substack{y_1,y_2\geq 1\\ m_1y_1+m_2y_2=n}} 1. \tag{18b}$$

*Remark.* The inner sum in (18b) counts the number of solutions $(y_1, y_2)$ of the classical linear diophantine equation $m_1y_1 + m_2y_2 = n$. This relates (18a-b) to the combinatorics in sections 4.6 and 4.7 of [62].

In Section 2 we present infinite families of explicit exact formulas that include generalizations of Theorems 1.1, 1.4, and 1.5.

Our methods yield in [46–49] many additional infinite families of identities analogous to the $\eta$-function identities in Appendix I of Macdonald [37]. A special case of our analysis gives a proof in [47] of the two Kac–Wakimoto [30] conjectured identities involving representing a positive integer by sums of $4n^2$ or $4n(n+1)$ triangular numbers, respectively. The $n = 1$ case gives the classical identities of Legendre [34]. See also [5, Eqns. (ii) and (iii), pp. 139].

Theorems 1.4 and 1.5 were originally obtained via multiple basic hypergeometric series [35, 40–43, 45, 50, 51] and Gustafson's [22] $C_\ell$ nonterminating ${}_6\phi_5$ summation theorem combined with Andrews' [2] basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have in [48] applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for $n^2$ or $n(n+1)$ squares, respectively.

Our sums of more than 8 squares identities are not the same as the formulas of Mathews [39], Glaisher [13–16], Sierpinski [61], Uspensky [64–66], Bulygin [7, 8], Ramanujan [56], Mordell [53, 54], Hardy [23, 24], Bell [4], Estermann [11], Rankin [57, 58], Lomadze [36], Walton [69], Walfisz [68], Ananda-Rau [1], van der Pol [55], Krätzel [31, 32], Gundlach [21], and, Kac and Wakimoto [30].

## 2. The $4n^2$ and $4n(n+1)$ squares identities

In order to state our identities we first need the Bernoulli numbers $B_n$ defined by

$$\frac{t}{e^t-1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad \text{for } |t| < 2\pi. \tag{19}$$

We also use the notation $I_n := \{1,2,\dots,n\}$, $\|S\|$ is the cardinality of the set $S$, and $\det(M)$ is the determinant of the $n \times n$ matrix $M$.

The single Hankel determinant form of the $4n^2$ squares identity is

**Theorem 2.1.** *Let $\vartheta_3(0,-q)$ be determined by* (3), *and let $n = 1,2,3,\cdots$. We then have*

$$\vartheta_3(0,-q)^{4n^2} = \left\{(-1)^n 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1}\right\} \cdot \det(g_{r+s-1})_{1\le r,s\le n}, \tag{20}$$

*with*

$$g_i := U_{2i-1} - c_i, \tag{21}$$

*where $U_{2i-1}$ is determined by* (12) *and $c_i$ is defined by*

$$c_i := (-1)^{i-1} \frac{(2^{2i}-1)}{4i} \cdot |B_{2i}|, \qquad \text{for } i = 1,2,3,\cdots, \tag{22}$$

*with $B_{2i}$ the Bernoulli numbers defined by* (19).

The determinant sum form of the $4n^2$ squares identity is

**Theorem 2.2.** *Let $n = 1,2,3,\cdots$. Then*

$$\vartheta_3(0,-q)^{4n^2} = 1 + \sum_{p=1}^{n} (-1)^p 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1} \sum_{\substack{\emptyset \subset S \subseteq I_n \\ \|S\|=p}} \det(M_{n,S}), \tag{23}$$

*where $\vartheta_3(0,-q)$ is determined by* (3), *and $M_{n,S}$ is the $n \times n$ matrix whose $i$-th row is*

$$U_{2i-1}, U_{2(i+1)-1}, \cdots, U_{2(i+n-1)-1}, \quad \text{if } i \in S \quad \text{and} \quad c_i, c_{i+1}, \cdots, c_{i+n-1}, \quad \text{if } i \notin S, \tag{24}$$

*where $U_{2i-1}$ is determined by (12), and $c_i$ is defined by* (22), *with $B_{2i}$ the Bernoulli numbers defined by* (19).

The single Hankel determinant form of the $4n(n+1)$ squares identity is

**Theorem 2.3.** *Let $\vartheta_3(0,-q)$ be determined by* (3), *and let $n = 1,2,3,\cdots$. We then have*

$$\vartheta_3(0,-q)^{4n(n+1)} = \left\{2^{2n^2+3n} \prod_{r=1}^{2n} (r!)^{-1}\right\} \cdot \det(g_{r+s-1})_{1\le r,s\le n}, \tag{25}$$

*with*

$$g_i := G_{2i+1} - a_i, \tag{26}$$

*where $G_{2i+1}$ and $a_i := c_{i+1}$ are determined by* (14) *and* (22), *respectively.*

The determinant sum form of the $4n(n+1)$ squares identity is

**Theorem 2.4.** *Let $n = 1, 2, 3, \cdots$. Then*

$$\vartheta_3(0,-q)^{4n(n+1)} = 1 + \sum_{p=1}^{n}(-1)^{n-p}2^{2n^2+3n}\prod_{r=1}^{2n}(r!)^{-1}\sum_{\substack{\emptyset\subset S\subseteq I_n\\ \|S\|=p}}\det(M_{n,S}), \qquad (27)$$

*where $\vartheta_3(0,-q)$ is determined by* (3), *and $M_{n,S}$ is the $n\times n$ matrix whose $i$-th row is*

$$G_{2i+1}, G_{2(i+1)+1}, \cdots, G_{2(i+n-1)+1}, \quad \text{if } i\in S \quad \text{and} \quad a_i, a_{i+1}, \cdots, a_{i+n-1}, \quad \text{if } i\notin S, \qquad (28)$$

*where $G_{2i+1}$ and $a_i := c_{i+1}$ are determined by* (14) *and* (22), *respectively.*

After seeing an earlier version of this paper, Garvan [12] observed via modular forms that (23) could be written as (20), and suggested the same be done for (27). The paper [46] already contained Theorems 2.1 and 2.3, and many similar results. We give some of these here in Theorems 2.3, 2.5, and 2.7. Garvan also conjectured that the square of the series in (15) could be written as a 3 by 3 Hankel determinant of classical Eisenstein series. This and similar results were subsequently proven in section 9 of [46].

The analysis of the formulas for $r_{4n^2}(N)$ and $r_{4n(n+1)}(N)$ obtained by taking the coefficient of $q^N$ in Theorems 2.2 and 2.4 is analogous to the formulas for $r_{16}(n)$ and $r_{24}(n)$ in [46]. The dominate terms for $r_{4n^2}(N)$ and $r_{4n(n+1)}(N)$ arise from the $p = n$ terms in (23) and (27), respectively. The other terms are all of a strictly decreasing lower order of magnitude. That is, the terms for $r_{4n^2}(N)$ and $r_{4n(n+1)}(N)$ corresponding to the $p$-th terms in (23) and (27) have orders of magnitude $N^{(4np-2p^2-1)}$ and $N^{(4np-2p^2+2p-1)}$, respectively. The dominate $p = n$ cases are consistent with [20, Eqn. (9.20), pp. 122]. Note that this analysis does not apply to the $n = 1$ case of Theorem 2.2.

In order to state the next identity we need the Euler numbers $E_n$ defined by

$$\frac{2e^t}{e^{2t}+1} := \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad \text{for } |t| < \pi/2. \qquad (29)$$

We next have the single Hankel determinant identity in

**Theorem 2.5.** *Let $\vartheta_3(0,-q)$ be determined by* (3), *and let $n = 1, 2, 3, \cdots$. We then have*

$$\vartheta_3(0,q)^{2n(n-1)}\vartheta_3(0,-q)^{2n^2} = \left\{(-1)^n 2^{2n}\prod_{r=1}^{n-1}(2r)!^{-2}\right\}\cdot\det(g_{r+s-1})_{1\le r,s\le n}, \quad (30)$$

*with*

$$g_i := R_{2i-2} - b_i, \qquad (31)$$

*where $R_{2i-2}$ and $b_i$ are defined by*

$$R_{2i-2} \equiv R_{2i-2}(q) := \sum_{r=1}^{\infty}(-1)^{r+1}\frac{(2r-1)^{2i-2}q^{2r-1}}{1+q^{2r-1}}, \qquad \text{for } i = 1, 2, 3, \cdots \quad (32)$$

*and*

$$b_i := (-1)^{i-1}\tfrac{1}{4}\cdot|E_{2i-2}|, \qquad \text{for } i = 1, 2, 3, \cdots, \qquad (33)$$

*with $E_{2i-2}$ the Euler numbers defined by* (29).

The corresponding determinant sum identity is

**Theorem 2.6.** *Let $n = 1, 2, 3, \cdots$. Then*

$$\vartheta_3(0,q)^{2n(n-1)}\vartheta_3(0,-q)^{2n^2} = 1 + \sum_{p=1}^{n}(-1)^p 2^{2n}\prod_{r=1}^{n-1}(2r)!^{-2}\sum_{\substack{\emptyset \subset S \subsetneq I_n \\ \|S\|=p}} \det(M_{n,S}), \quad (34)$$

*where $\vartheta_3(0,-q)$ is determined by (3), and $M_{n,S}$ is the $n \times n$ matrix whose i-th row is*

$$R_{2i-2}, R_{2(i+1)-2}, \cdots, R_{2(i+n-1)-2}, \quad \text{if } i \in S \quad \text{and} \quad b_i, b_{i+1}, \cdots, b_{i+n-1}, \quad \text{if } i \notin S, \quad (35)$$

*where $R_{2i-2}$ and $b_i$ are defined by (32) and (33), respectively, with $E_{2i-2}$ the Euler numbers defined by (29).*

The next identities involve the $z = 0$ case of the theta function $\vartheta_2(z,q)$ in [70, pp. 464], defined by

$$\vartheta_2(0,q) := \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2}. \quad (36)$$

We have

**Theorem 2.7.** *Let $\vartheta_2(0,q)$ be defined by (36), and let $n = 1, 2, 3, \cdots$. We then have*

$$\vartheta_2(0,q)^{4n^2} = \left\{4^{n(n+1)}\prod_{r=1}^{2n-1}(r!)^{-1}\right\} \cdot \det(C_{2(r+s-1)-1})_{1\le r,s\le n}, \quad (37)$$

*and*

$$\vartheta_2(0,q^{1/2})^{4n(n+1)} = \left\{2^{n(4n+5)}\prod_{r=1}^{2n}(r!)^{-1}\right\} \cdot \det(D_{2(r+s-1)+1})_{1\le r,s\le n}, \quad (38)$$

*where $C_{2i-1}$ and $D_{2i+1}$ are defined by*

$$C_{2i-1} \equiv C_{2i-1}(q) := \sum_{r=1}^{\infty}\frac{(2r-1)^{2i-1}q^{2r-1}}{1-q^{2(2r-1)}}, \qquad \text{for } i = 1, 2, 3, \cdots, \quad (39)$$

*and*

$$D_{2i+1} \equiv D_{2i+1}(q) := \sum_{r=1}^{\infty}\frac{r^{2i+1}q^r}{1-q^{2r}}, \qquad \text{for } i = 1, 2, 3, \cdots, \quad (40)$$

*respectively.*

Theorem 2.7 is the first step of our proof [47] of the Kac-Wakimoto conjectures.

We next utilize Schur functions $s_\lambda(x_1, \ldots, x_p)$ to rewrite Theorems 2.2, 2.4, and 2.6. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a partition of nonnegative integers in decreasing order, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \cdots$, such that only finitely many of the $\lambda_i$ are nonzero. The length $\ell(\lambda)$ is the number of nonzero parts of $\lambda$.

Given a partition $\lambda = (\lambda_1, \cdots, \lambda_p)$ of length $\leq p$,

$$s_\lambda(x) \equiv s_\lambda(x_1, \cdots, x_p) := \frac{\det(x_i^{\lambda_j+p-j})}{\det(x_i^{p-j})} \tag{41}$$

is the Schur function [38] corresponding to the partition $\lambda$. (Here, $\det(a_{ij})$ denotes the determinant of a $p \times p$ matrix with $(i,j)$-th entry $a_{ij}$). The Schur function $s_\lambda(x)$ is a symmetric polynomial in $x_1, \cdots, x_p$ with nonnegative integer coefficients. We typically have $1 \leq p \leq n$.

We use Schur functions in (41) corresponding to the partitions $\lambda$ and $\nu$, with

$$\lambda_r := \ell_{p-r+1} - \ell_1 + r - p \quad \text{and} \quad \nu_r := j_{p-r+1} - j_1 + r - p, \quad \text{for } r = 1, 2, \cdots, p, \tag{42}$$

where the $\ell_r$ and $j_r$ are elements of the sets $S$ and $T$, with

$$S := \{\ell_1 < \ell_2 < \cdots < \ell_p\} \qquad \text{and} \qquad S^c := \{\ell_{p+1} < \cdots < \ell_n\}, \tag{43}$$

$$T := \{j_1 < j_2 < \cdots < j_p\} \qquad \text{and} \qquad T^c := \{j_{p+1} < \cdots < j_n\}, \tag{44}$$

where $S^c := I_n - S$ is the compliment of the set $S$. We also have

$$\Sigma(S) := \ell_1 + \ell_2 + \cdots + \ell_p \qquad \text{and} \qquad \Sigma(T) := j_1 + j_2 + \cdots + j_p. \tag{45}$$

Keeping in mind (41)-(45), symmetry and skew-symmetry arguments, row and column operations, and the Laplace expansion formula [28, pp. 396–397] for a determinant, we now rewrite Theorem 2.2 as

**Theorem 2.8.** *Let $n = 1, 2, 3, \cdots$. Then*

$$\begin{aligned}\vartheta_3(0,-q)^{4n^2} =& 1 + \sum_{p=1}^{n} (-1)^p 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1} \sum_{\substack{y_1,\cdots,y_p \geq 1 \\ m_1 > m_2 > \cdots > m_p \geq 1}} (-1)^{y_1+\cdots+y_p} \\ & \cdot (-1)^{m_1+\cdots+m_p} q^{m_1y_1+\cdots+m_py_p} \prod_{1 \leq r < s \leq p} (m_r^2 - m_s^2)^2 \\ & \cdot (m_1 m_2 \cdots m_p) \sum_{\substack{\emptyset \subset S, T \subseteq I_n \\ \|S\| = \|T\| = p}} (-1)^{\Sigma(S)+\Sigma(T)} \cdot \det(D_{n-p,S^c,T^c}) \\ & \cdot (m_1 m_2 \cdots m_p)^{2\ell_1+2j_1-4} s_\lambda(m_1^2, \cdots, m_p^2)\, s_\nu(m_1^2, \cdots, m_p^2),\end{aligned} \tag{46}$$

*where $\vartheta_3(0,-q)$ is determined by (3), the sets $S$, $S^c$, $T$, $T^c$ are given by (43)-(44), $\Sigma(S)$ and $\Sigma(T)$ by (45), the $(n-p)\times(n-p)$ matrix $D_{n-p,S^c,T^c} := \left[c_{(\ell_{p+r}+j_{p+s}-1)}\right]_{1 \leq r,s \leq n-p}$, where the $c_i$ are determined by (22), with the $B_{2i}$ in (19), and $s_\lambda$ and $s_\nu$ are the Schur functions in (41), with the partitions $\lambda$ and $\nu$ given by (42).*

We next rewrite Theorem 2.4 as

**Theorem 2.9.** *Let $n = 1,2,3,\cdots$. Then*

$$\begin{aligned}\vartheta_3(0,-q)^{4n(n+1)} =1+\sum_{p=1}^{n}(-1)^{n-p}2^{2n^2+3n}\prod_{r=1}^{2n}(r!)^{-1}\sum_{\substack{y_1,\ldots,y_p\geq 1\\ m_1>m_2>\cdots>m_p\geq 1}}(-1)^{m_1+\cdots+m_p} \qquad (47)\\ \cdot q^{m_1y_1+\cdots+m_py_p}(m_1m_2\cdots m_p)^3\prod_{1\leq r<s\leq p}\left(m_r^2-m_s^2\right)^2\\ \cdot\sum_{\substack{\emptyset\subset S,T\subseteq I_n\\ \|S\|=\|T\|=p}}(-1)^{\Sigma(S)+\Sigma(T)}\cdot\det(D_{n-p,S^c,T^c})\\ \cdot(m_1m_2\cdots m_p)^{2\ell_1+2j_1-4}s_\lambda(m_1^2,\ldots,m_p^2)\,s_\nu(m_1^2,\ldots,m_p^2),\end{aligned}$$

*where the same assumptions hold as in* Theorem 2.8, *except that the $(n-p)\times(n-p)$ matrix $D_{n-p,S^c,T^c} := \left[a_{(\ell_{p+r}+j_{p+s}-1)}\right]_{1\leq r,s\leq n-p}$, where the $a_i := c_{i+1}$ are determined by* (22).

We now rewrite Theorem 2.6 as

**Theorem 2.10.** *Let $n = 1,2,3,\cdots$. Then*

$$\begin{aligned}\vartheta_3(0,q)^{2n(n-1)}\vartheta_3(0,-q)^{2n^2} =1+\sum_{p=1}^{n}2^{2n}\prod_{r=1}^{n-1}(2r)!^{-2}\sum_{\substack{y_1,\ldots,y_p\geq 1\\ m_1>m_2>\cdots>m_p\geq 1 (odd)}}(-1)^{\frac{1}{2}(p+m_1+\cdots+m_p)} \qquad (48)\\ \cdot q^{m_1y_1+\cdots+m_py_p}\prod_{1\leq r<s\leq p}\left(m_r^2-m_s^2\right)^2\\ \cdot\sum_{\substack{\emptyset\subset S,T\subseteq I_n\\ \|S\|=\|T\|=p}}(-1)^{\Sigma(S)+\Sigma(T)}\cdot\det(D_{n-p,S^c,T^c})\\ \cdot(m_1m_2\cdots m_p)^{2\ell_1+2j_1-4}s_\lambda(m_1^2,\ldots,m_p^2)\,s_\nu(m_1^2,\ldots,m_p^2),\end{aligned}$$

*where the same assumptions hold as in* Theorem 2.6, *except that the $(n-p)\times(n-p)$ matrix*

$$D_{n-p,S^c,T^c} := \left[b_{(\ell_{p+r}+j_{p+s}-1)}\right]_{1\leq r,s\leq n-p}, \qquad (49)$$

*where the $b_i$ are determined by* (33), *with the $E_{2i-2}$ in* (29).

We close this section with some comments about the above theorems.

In order to prove Theorem 2.2 we first compare the Fourier and Taylor series expansions of the Jacobi elliptic function $f_1(u,k) := \mathrm{sc}(u,k)\mathrm{dn}(u,k)$, where $k$ is the modulus. An analysis similar to that in [5, 6, 71] leads to the relation $U_{2m-1}(-q) = c_m + d_m$, for $m = 1,2,3,\cdots$, where $U_{2m-1}(-q)$ and $c_m$ are defined by (12) and (22), respectively, and $d_m$ is given by $d_m = ((-1)^m z^{2m}/2^{2m+1})\cdot(sd/c)_m(k^2)$, where $z := {}_2F_1(1/2,1/2;1;k^2) = 2K(k)/\pi \equiv 2K/\pi$, with $K(k)\equiv K$ the complete elliptic integral of the first kind in [70, pp. 498] and $(sd/c)_m(k^2)$ is the coefficient of $u^{2m-1}/(2m-1)!$ in the Taylor series expansion of $f_1(u,k)$ about $u=0$.

An inclusion/exclusion argument then reduces the $q \mapsto -q$ case of (23) to finding suitable product formulas for the $n \times n$ Hankel determinants $\det(d_{i+j-1})$ and $\det(c_{i+j-1})$. Row and column operations immediately imply that

$$\det(d_{i+j-1}) = (z^{2n^2}(-1)^n/2^{2n^2+n}) \det((sd/c)_{i+j-1}(k^2)). \tag{50}$$

From Theorem 7.9 of [6, pp.26] we have $z = \vartheta_3(0,q)^2$, where $q = exp(-\pi K(\sqrt{1-k^2})/K(k))$. Setting $z = \vartheta_3(0,q)^2$ in (50) and then taking $q \mapsto -q$ produces the $\vartheta_3(0,-q)^{4n^2}$ in (23).

The proof of Theorem 2.2 is complete once we show that

$$\det((sd/c)_{i+j-1}(k^2)) = \prod_{r=1}^{2n-1}(r!) \quad \text{and} \quad \det(c_{i+j-1}) = 2^{-(2n^2+n)} \cdot \prod_{r=1}^{2n-1}(r!). \tag{51}$$

By a classical result of Heilermann [25, 26], more recently presented in [29, Theorem 7.14, pp. 244–246], Hankel determinants whose entries are the coefficients in a formal power series $L$ can be expressed as a certain product of the "numerator" coefficients of the associated Jacobi continued fraction $J$ corresponding to $L$, provided $J$ exists. Modular transformations, followed by row and column operations, reduce the evaluation of $\det((sd/c)_{i+j-1}(k^2))$ in (51) to applying Heilermann's formula to Rogers' [60] $J$-fraction expansion of the Laplace transform of $\mathrm{sc}(u,k)\mathrm{dn}(u,k)$. The evaluation of $\det(c_{i+j-1})$ can be done similarly, starting with $\mathrm{sc}(u,k)$ and the relation $\mathrm{sc}(u,0) = \tan(u)$.

The proof of Theorem 2.4 is similar to Theorem 2.2, except that we start with $\mathrm{sc}^2(u,k)\mathrm{dn}^2(u,k)$. For Theorem 2.6, we start with $\mathrm{nc}(u,k)$.

Our proofs of the Kac–Wakimoto conjectures do not require inclusion/exclusion, and the analysis involving Schur functions is simpler than in (46) and (47).

We have [46] written down the $n = 3$ cases of Theorems 2.8 and 2.9 which yield explicit formulas for 36 and 48 squares, respectively.

## References


1. K. Ananda-Rau, *On the representation of a number as the sum of an even number of squares*, J. Madras Univ. Sect. B **24** (1954), 61–89.
2. G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441–484.
3. ———, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, NSF CBMS Regional Conference Series, vol. 66, 1986.
4. E. T. Bell, *On the number of representations of 2n as a sum of 2r squares*, Bull. Amer. Math. Soc. **26** (1919), 19–25.
5. B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1991.
6. ———, *Ramanujan's theory of theta-functions*, Theta Functions From the Classical to the Modern (M. Ram Murty, ed.), vol. 1 of CRM Proceedings & Lecture Notes, American Mathematical Society, Providence, RI, 1993, pp. 1–63.
7. V. Bulygin, *Sur une application des fonctions elliptiques au problème de représentation des nombres entiers par une somme de carrés*, Bull. Acad. Imp. Sci. St. Petersbourg Ser. VI **8** (1914), 389–404 (in French);B. Boulyguine, *Sur la représentation d'un nombre entier par une somme de carrés*, Comptes Rendus Paris **158** (1914), 328–330. (in French)

8. ——— (B. Boulyguine), *Sur la représentation d'un nombre entier par une somme de carrés*, Comptes Rendus Paris **161** (1915), 28–30. (in French)
9. L. E. Dickson, *History of the Theory of Numbers*, vol. 2, Chelsea, New York, 1966.
10. F. J. Dyson, *Missed opportunities*, Bull. Amer. Math. Soc. **78** (1972), 635–653.
11. T. Estermann, *On the representations of a number as a sum of squares*, Acta Arith. **2** (1936), 47–79.
12. F. Garvan; Private Communication, March 1997.
13. J. W. L. Glaisher, *On the numbers of representations of a number as a sum of 2r squares, where 2r does not exceed eighteen*, Proc. London Math. Soc. (2) **5** (1907), 479–490.
14. ———, *On the representations of a number as the sum of two, four, six, eight, ten, and twelve squares*, Quart. J. Pure and Appl. Math. Oxford **38** (1907), 1–62.
15. ———, *On the representations of a number as the sum of fourteen and sixteen squares*, Quart. J. Pure and Appl. Math. Oxford **38** (1907), 178–236.
16. ———, *On the representations of a number as the sum of eighteen squares*, Quart. J. Pure and Appl. Math. Oxford **38** (1907), 289–351.
17. M. L. Glasser and I. J. Zucker, *Lattice Sums*, vol. 5 of Theoretical Chemistry:Advances and Perspectives (H. Eyring and D. Henderson, eds.), Academic Press, New York, 1980, pp. 67–139.
18. I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, New York, 1983.
19. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed., Academic Press, San Diego, 1980; Translated from the Russian by Scripta Technica, Inc., and edited by A. Jeffrey.
20. E. Grosswald, *Representations of integers as sums of squares*, Springer-Verlag, New York, 1985.
21. K.-B. Gundlach, *On the representation of a number as a sum of squares*, Glasgow Math. J. **19** (1978), 173–197.
22. R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very well-poised on semi-simple Lie algebras*, Ramanujan International Symposium on Analysis (Dec. 26th to 28th, 1987, Pune, India) (N. K. Thakare, ed.), 1989, pp. 187–224.
23. G. H. Hardy, *On the representation of a number as the sum of any number of squares, and in particular of five or seven*, Proc. Nat. Acad. Sci., U.S.A. **4** (1918), 189–193.
24. ———, *On the representation of a number as the sum of any number of squares, and in particular of five*, Trans. Amer. Math. Soc. **21** (1920), 255–284.
25. J. B. H. Heilermann, *De transformatione serierum in fractiones continuas*, Dr. Phil. Dissertation, Royal Academy of Münster, 1845.
26. ———, *Über die Verwandlung der Reihen in Kettenbrüche*, J. Reine Angew. Math. **33** (1846), 174–188. (in German)
27. C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomonti. Sumptibus fratrum Bornträger, 1829; reprinted in Jacobi's Gesammelte Werke, vol. 1, Reimer, Berlin, 1881–1891, pp. 49–239; reprinted by Chelsea, New York, 1969; Now distributed by The American Mathematical Society, Providence, RI.
28. N. Jacobson, *Basic Algebra I*, W. H. Freeman and Co., San Francisco, CA, 1974.
29. W. B. Jones and W. J. Thron, *Continued Fractions:Analytic Theory and Applications*, Encyclopedia of Mathematics and Its Applications (G.-C. Rota, ed.), vol. 11, Addison-Wesley, London, 1980; Now distributed by Cambridge University Press, Cambridge.
30. V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Lie theory and geometry. In honor of Bertram Kostant (Jean-Luc Brylinski, Ranee Brylinski, Victor Guillemin and Victor Kac, eds.), vol. 123 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1994, pp. 415–456.
31. E. Krätzel, *Über die Anzahl der Darstellungen von natürlichen Zahlen als Summe von 4k Quadraten*, Wiss. Z. Friedrich-Schiller-Univ. Jena **10** (1960/61), 33–37. (in German)
32. ———, *Über die Anzahl der Darstellungen von natürlichen Zahlen als Summe von $4k+2$ Quadraten*, Wiss. Z. Friedrich-Schiller-Univ. Jena **11** (1962), 115–120. (in German)

33. D. F. Lawden, *Elliptic Functions and Applications*, vol. 80 of Applied Mathematical Sciences, Springer-Verlag, New York, 1989.
34. A. M. Legendre, *Traité des Fonctions Elliptiques et des Intégrales Euleriennes*, t. III, Huzard-Courcier, Paris, 1828, pp. 133–134. (in French)
35. G. M. Lilly and S. C. Milne, *The $C_\ell$ Bailey Transform and Bailey Lemma*, Constr. Approx. **9** (1993), 473–500.
36. G. A. Lomadze, *On the representation of numbers by sums of squares*, Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **16** (1948), 231–275. (in Russian; Georgian summary)
37. I. G. Macdonald, *Affine root systems and Dedekind's $\eta$-function*, Invent. Math. **15** (1972), 91–143.
38. ———, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, Oxford, 1995.
39. G. B. Mathews, *On the representation of a number as a sum of squares*, Proc. London Math. Soc. **27** (1895-6), 55–60.
40. S. C. Milne, *An elementary proof of the Macdonald identities for $A_\ell^{(1)}$*, Adv. in Math. **57** (1985), 34–70.
41. ———, *Basic hypergeometric series very well-poised in $U(n)$*, J. Math. Anal. Appl. **122** (1987), 223–256.
42. ———, *Classical partition functions and the $U(n+1)$ Rogers-Selberg identity*, Discrete Math. **99** (1992), 199–246.
43. ———, *The $C_\ell$ Rogers-Selberg Identity*, SIAM J. Math. Anal. **25** (1994), 571–595.
44. ———, *New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function*, Proc. Nat. Acad. Sci., U.S.A. **93** (1996), 15004–15008.
45. ———, *Balanced ${}_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series*, Adv. in Math. , in press.
46. ———, *Sums of squares, Jacobi elliptic functions, and Schur functions*, in preparation.
47. ———, *Sums of squares, Jacobi elliptic functions and continued fractions, and Schur functions*, in preparation.
48. ———, *Sums of squares, Schur functions, and multiple basic hypergeometric series*, in preparation.
49. ———, *Applications of computer algebra to sums of squares*, in preparation.
50. S. C. Milne and G. M. Lilly, *The $A_\ell$ and $C_\ell$ Bailey transform and lemma*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 258–263.
51. ———, *Consequences of the $A_\ell$ and $C_\ell$ Bailey Transform and Bailey Lemma*, Discrete Math. **139** (1995), 319–346.
52. L. J. Mordell, *On Mr Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1917), 117–124.
53. ———, *On the representation of numbers as the sum of 2r squares*, Quart. J. Pure and Appl. Math. Oxford **48** (1917), 93–104.
54. ———, *On the representations of a number as a sum of an odd number of squares*, Trans. Cambridge Philos. Soc. **22** (1919), 361–372.
55. B. van der Pol, *The representation of numbers as sums of eight, sixteen and twenty-four squares*, Nederl. Akad. Wetensch. Proc. Ser. A **57**; = Nederl. Akad. Wetensch. Indag. Math. **16** (1954), 349–361.
56. S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184; reprinted in Collected Papers of Srinivasa Ramanujan, Chelsea, New York, 1962, pp. 136–162.
57. R. A. Rankin, *On the representations of a number as a sum of squares and certain related identities*, Proc. Cambridge Philos. Soc. **41** (1945), 1–11.
58. ———, *On the representation of a number as the sum of any number of squares, and in particular of twenty*, Acta Arith. **7** (1962), 399–407.
59. ———, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.
60. L. J. Rogers, *On the representation of certain asymptotic series as convergent continued fractions*, Proc. London Math. Soc (2) **4** (1907), 72–89.

61. W. Sierpinski, *Wzór analityczny na pewna funkcje liczbowa (Une formule analytique pour une fonction numérique)*, Wiadomości Matematyczne Warszawa **11** (1907), 225–231. (in Polish)
62. R. P. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth & Brooks Cole, Belmont, CA, 1986.
63. O. Taussky, *Sums of Squares*, Amer. Math. Monthly **77** (1970), 805–830.
64. J. V. Uspensky, *Sur la représentation des nombres par les sommes des carrés*, Communications de la Société mathématique de Kharkow série 2 **14** (1913), 31–64. (in Russian)
65. ———, *Note sur le nombre de représentations des nombres par une somme d'un nombre pair de carrés*, Bulletin de l'Académie des Sciences de l'URSS. Leningrad. (Izvestija Akademii Nauk Sojuza Sovetskich Respublik. Leningrad.) Serie 6 **19** (1925), 647–662. (in French)
66. ———, *On Jacobi's arithmetical theorems concerning the simultaneous representation of numbers by two different quadratic forms*, Trans. Amer. Math. Soc. **30** (1928), 385–404.
67. J. V. Uspensky and M. A. Heaslet, *Elementary Number Theory*, McGraw-Hill Book Co., New York, 1939.
68. A. Z. Walfisz, *On the representation of numbers by sums of squares:asymptotic formulas*, Uspehi Mat. Nauk (N.S.) (6) **52** (1952), 91–178 (in Russian); English transl. in Amer. Math. Soc. Transl. (2) **3** (1956), 163–248.
69. J. B. Walton, *Theta series in the Gaussian field*, Duke Math. J. **16** (1949), 479–491.
70. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.
71. I. J. Zucker, *The summation of series of hyperbolic functions*, SIAM J. Math. Anal. **10** (1979), 192–206.

Department of Mathematics, The Ohio State University, Columbus, Ohio, 43210
*E-mail address*: milne@math.ohio-state.edu