Homework Set 3 for Math 5702, Spring 2016

Directions: Since these problems are more difficult than those of HW assignments 1 and 2, you need to only complete 5 problems from this assignment. But you must complete (at least) one of problems 2 through 4, at least one of problems 6 through 8, and at most one of the problems from homework assignment 2.

1. Determine the Weingarten map for a sphere of radius \( r \) at a generic point on the sphere (that is, fix a basis for the tangent space at this point and compute the matrix representation of the Weingarten map in terms of this basis).

For the next three problems, find the normal curvature \( k(\vec{v}) \) for each tangent direction \( \vec{v} \) (within a fixed basis), the principal curvatures, the principal curvature directions, and compute the Gaussian and mean curvatures of the surface at the given point \( p \).

2. \( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \) at the point \( p = (a, 0, 0) \) (this is an ellipsoid).

3. \( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1 \) at the point \( p = (a, 0, 0) \) (a 1-sheeted hyperboloid).

4. \( x_1^2 + \left( \sqrt{x_2^2 + x_3^2} - 2 \right)^2 = 1 \) at the point \( p = (0, 3, 0) \) (this is a torus).

5. Suppose that the principal curvatures of a parameterized surface in \( \mathbb{R}^3 \) are all zero. Show that the surface is a part of a plane.

For the next three problems, find the Gaussian curvature function \( K : M \rightarrow \mathbb{R} \) for the surface.

6. \( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \) (ellipsoid)
7. \( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - x_3 = 0 \) (elliptic paraboloid)

8. \( \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_3 = 0 \) (hyperbolic paraboloid)

9. \( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1 \) (hyperboloid)

10. Let \( M \) be a hypersurface in \( \mathbb{R}^3 \) and let \( p \in M \). Prove that for any \( v, w \in T_p M \)
    \[ L_p(v) \times L_p(w) = K(p) \, v \times w \]

Problems from homework assignment 2

11. Given a unit speed curve of general type in \( \mathbb{R}^3 \) with distinguished Frenet frame \( \vec{t}_1, \vec{t}_2, \vec{t}_3 \), find a vector field \( \omega \) along the curve such that
    \[ \vec{t}_i = \omega \times \vec{t}_i \]
    for \( i = 1, 2, 3 \) (\( \omega \) is called the Darboux vector field of the curve).

12. Suppose that the osculating planes of a curve of general type in \( \mathbb{R}^3 \)
    have a point in common. Show that the curve is a planar curve.

13. Let \( \gamma \) be a curve of general type in \( \mathbb{R}^n, \vec{t}_1, \ldots, \vec{t}_n \) its distinguished Frenet frame, and \( 0 \leq k \leq n \). By the definition of the distinguished Frenet frame, the \( k^{th} \) derivative of \( \gamma \) can be expressed as a linear combination of \( \vec{t}_1, \ldots, \vec{t}_k \) as
    \[ \gamma^{(k)} = c_1 \vec{t}_1 + \ldots + c_k \vec{t}_k \]
    where \( c_1, \ldots, c_k \) are suitable real-valued functions. Show that
    \[ c_k = \| \gamma' \|^k \kappa_1 \kappa_2 \ldots \kappa_{k-1}. \]

    \textit{Hint: induction}

Remark: One student in the class completed both problems 10 and 11 above on homework assignment 2 (which are problems
12 and 13 on HW 2). That student is permitted to complete problem 11 from homework 2 if they want to.