ON THE REALISATION PROBLEM FOR MAPPING DEGREE SETS

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Abstract. The set of degrees of maps $D(M,N)$, where M,N are closed oriented n-manifolds, always contains 0 and the set of degrees of self-maps $D(M)$ always contains 0 and 1. Also, if $a,b \in D(M)$, then $ab \in D(M)$; a set $A \subseteq \mathbb{Z}$ so that $ab \in A$ for each $a,b \in A$ is called multiplicative. On the one hand, not every infinite set of integers (containing 0) is a mapping degree set [NWW] and, on the other hand, every finite set of integers (containing 0) is the mapping degree set of some 3-manifolds [CMV]. We show the following:

(i) Not every multiplicative set A containing $0,1$ is a self-mapping degree set.
(ii) For each $n \in \mathbb{N}$ and $k \geq 3$, every $D(M,N)$ for n-manifolds M and N is $D(P,Q)$ for some $(n+k)$-manifolds P and Q.

As a consequence of (ii) and [CMV], every finite set of integers (containing 0) is the mapping degree set of some n-manifolds for all $n \neq 1,2,4,5$.

1. Introduction

Let M,N be two closed oriented manifolds of the same dimension. The degree of a map $f\colon M \to N$, denoted by $\deg(f)$, is probably one of the oldest and most fundamental concepts in topology. The *set of degrees of maps* from M to N is defined by

$$D(M,N) := \{d \in \mathbb{Z} \mid \exists\ f\colon M \to N,\ \deg(f) = d\}.$$

When $M = N$, the *set of degrees of self-maps* $D(M,M)$ is denoted by $D(M)$.

The following question, about realising subsets of integers as mapping degree sets, has been circulating for some years, but formally appeared only recently:

Problem 1.1. [NWW, Problem 1.1] *Given a set $A \subseteq \mathbb{Z}$ with $0 \in A$, are there closed oriented manifolds M and N such that $D(M,N) = A$?*

On the one hand, a negative answer has been given for infinite sets:

Theorem A. [NWW, Theorem 1.3] *There exists an infinite set $A \subseteq \mathbb{Z}$ with $0 \in A$ which is not $D(M,N)$, for any closed oriented n-manifolds M,N.*

On the other hand, using 3-manifolds, which are connected sums of non-trivial circle bundles over hyperbolic surfaces, and their products, it was proved in [NWW, Theorems

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1.7 and 1.9] that many finite subsets of integers are mapping degree sets, including finite arithmetic progressions (containing 0) and finite geometric progressions of positive integers starting from 1 (and containing 0). So, the following natural problems arose:

Problem 1.2. [NWW, Problem 1.3] *Suppose A is a finite set of integers containing 0. Is $A = D(M, N)$, for some closed oriented n-manifolds M and N?*

Problem 1.3. [NWW, Problem 1.6] *Can every arithmetic progression containing 0 be realised as $D(M, N)$, for some closed oriented n-manifolds M, N?*

Problem 1.4. [NWW, Problem 1.8] *Together with 0, can every geometric progression of integers be realised as $D(M, N)$, for some closed oriented n-manifolds M, N?*

Recently, Costoya, Muñoz and Viruel gave a complete positive answer to Problem 1.2 in a stronger form:

Theorem B. [CMV, Theorem A]. *If A is a finite set of integers containing 0, then $A = D(M, N)$ for some closed oriented connected 3-manifolds M, N.*

It is rather surprising that all 3-manifolds used in Theorem B are just connected sums of non-trivial circle bundles over hyperbolic surfaces. Further realisability results are shown in [CMV], including that any finite set of integers containing 0 is the mapping degree set of some simply connected $(4k-1)$-manifolds for $k > 3$ [CMV, Theorem C].

The self-mapping degree $D(M)$, a subset of integers associated with a given closed oriented manifold M, is very interesting from a number theoretic point of view. We say that a set $A \subseteq \mathbb{Z}$ is *multiplicative*, if $a, b \in A$ implies that $ab \in A$. When A is finite and multiplicative, then clearly $A \subseteq \{-1, 0, 1\}$. Note that $D(M)$ is a multiplicative set containing $\{0, 1\}$: 0 is realised by a constant map, 1 is realised by the identity, and if $a, b \in D(M)$, then $ab \in D(M)$, which is realised by a self-map $g \circ f$ of M, where g and f are self-maps of M realising a and b respectively.

Similarly to Problem 1.1, the following question has been also circulating over the years:

Problem 1.5. *Suppose A is a multiplicative set of integers containing $\{0, 1\}$. Is there a closed oriented manifold M with $D(M) = A$?*

It is worth mentioning that, although any finite mapping degree set is the mapping degree set of some 3-manifolds, the corresponding statement is not true for infinite self-mapping degrees, as explained below:

Example 1.6. *Recall that $D(\mathbb{CP}^n) = \{k^n \mid k \in \mathbb{Z}\}$, where $\mathbb{CP}^n$ is the n-dimensional complex projective plane. However, for $n > 2$, one can check, with some number theoretic arguments, that $\{k^n \mid k \in \mathbb{Z}\}$ is not the self-mapping degree set $D(M)$ of any 3-manifold M, according to* [SWWZ].

Although the proof of the claim $D(\mathbb{CP}^n) = \{k^n \mid k \in \mathbb{Z}\}$ in Example 1.6 should be well-known, we will include it at the end of paper, since we could not locate a precise reference.

2. Results and questions

Our first result in this note is a negative answer to Problem 1.5:

Theorem 2.1. *There exists a multiplicative set A containing $\{0, 1\}$ which is not $D(M)$ for any closed oriented n-manifold M.*

Since higher dimensional manifolds are richer than those in lower dimensions, it is natural to expect that $D(M, N)$, for any pair of n-manifolds M, N, is realised by a pair of higher dimensional manifolds. The next result supports this expectation:

Theorem 2.2. *For each $n \in \mathbb{N}$ and $k \geq 3$, every mapping degree set A of n-manifolds is the mapping degree set of some $(n+k)$-manifolds. Moreover there are infinitely many pairs of $(n+k)$ manifolds realising A. The same is true for self-mapping degree sets.*

Combining Theorem B and Theorem 2.2, we extend Theorem B in all dimensions ≥ 6. An analog for self-mapping degree sets also follows from the proof of Theorem 2.2.

Theorem 2.3. *For each positive integer $n \neq 1, 2, 4, 5$, every finite set of integers containing 0 is the mapping degree set of some n-manifolds.*

For each positive integer $n \neq 1, 2$, every finite multiplicative set containing $\{0, 1\}$ is the self-mapping degree set of some n-manifold.

We believe that Theorem 2.3 also holds for $n = 4, 5$, but we do not have a proof yet. Also, for the sake of completeness, note that the mapping degree sets of 1- and 2-dimensional manifolds are very special: they are either $\mathbb{Z}$ or $\{-k, -k+1, ..., -1, 0, 1, ..., k-1, k\}$ (see [NWW, Example 1.5]).

Constructing specific non-realisable sets seems to be a more subtle problem:

Question 2.4. *Find a concrete subset of integers containing 0 which is not a mapping degree set, and a concrete multiplicative subset of integers containing $\{0, 1\}$ which is not a self-mapping degree set.*

Löh and Uschold proved that each $D(M, N)$ is recursively enumerable [LU, Proposition A.1]. Theorem 2.1 and [LU, Proposition A.1] together imply that there must exist non-recursively enumerable multiplicative sets. In Question 2.4 we do hope to find a non-realisable subset of integers that can be written down explicitly without using any non-constructive existence result.

There are many examples of manifolds M with $D(M)$ an infinite arithmetic progression (see [SWWZ] and its references). So far, we do not know any $D(M)$ which is an infinite geometric progression together with $\{0, 1\}$. Parallel to Problems 1.3 and 1.4, we ask the following realisation question for (multiplicative) arithmetic and geometric progressions as self-mapping degree sets:

Question 2.5.

(1) *Is every infinite arithmetic progression together with* $\{0,1\}$ *a self-mapping degree set?*
(2) *Is there an infinite geometric progression together with* $\{0,1\}$ *which is a self-mapping degree set?*

3. Proofs

3.1. **Proof of Theorem 2.1.** Let $\mathcal{P}$ be the set of all prime numbers. For a subset $S \subseteq \mathcal{P}$, let

$$\Pi(S) = \{p_1^{a_1}...p_k^{a_k} \mid p_1,...,p_k \in S, a_1,...,a_k \in \mathbb{Z}_{\geq 0}\} \cup \{0,1\}.$$

We have the following two elementary lemmas:

Lemma 3.1. $\Pi(S)$ *is a multiplicative set.*

Proof. Let α, $\beta \in \Pi(S)$.

Case 1. $\alpha = 0$ or $\beta = 0$. Then $\alpha\beta = 0 \in \Pi(S)$.

Case 2. $\alpha, \beta \geq 1$. Write

$$\alpha = p_1^{a_1}...p_k^{a_k}, \beta = q_1^{b_1}...q_l^{b_l}, \text{ where } p_1,...,p_k,q_1,...,q_l \in S \text{ and } a_1,...,a_k,b_1,...,b_l \in \mathbb{Z}_{\geq 0}.$$

Then $\alpha\beta = p_1^{a_1}...p_k^{a_k}q_1^{b_1}...q_l^{b_l} \in \Pi(S)$. □

Lemma 3.2. $\Pi(S) \cap \mathcal{P} = S$, *for any subset* $S \subseteq \mathcal{P}$.

Proof. In one direction, note that $S \subseteq \Pi(S)$, hence $S \subseteq \Pi(S) \cap \mathcal{P}$.

Next, we show that $\Pi(S) \cap \mathcal{P} \subseteq S$: Let $p \in \Pi(S) \cap \mathcal{P}$. Since $p \in \mathcal{P}$, we have that $p \neq 0, 1$. Then $p \in \Pi(S)$ means that $p = p_1^{a_1}...p_k^{a_k}$, where $p_1,...,p_k \in S$ and $a_1,...,a_k \in \mathbb{N}$. But p is a prime, and all $p_1,...,p_k$ are primes, hence $p = p_i$, for some $i = 1,...,k$. Thus $p \in S$. □

Let now $\mathbb{P}(\mathcal{P})$ be the set of all subsets of $\mathcal{P}$. By Lemma 3.1, there is a map

$$f \colon \mathbb{P}(\mathcal{P}) \to \{\text{multiplicative subsets of } \mathbb{Z} \text{ containing } \{0,1\}\}$$

defined by

$$f(S) = \Pi(S)$$

for each $S \in \mathbb{P}(\mathcal{P})$. By Lemma 3.2, we have

$$f(S) \cap \mathcal{P} = \Pi(S) \cap \mathcal{P} = S.$$

Hence, f is injective. Since $\mathbb{P}(P)$ is uncountable, we conclude that $f(\mathbb{P}(\mathcal{P}))$ is also uncountable. Thus, the set that consists of all multiplicative subsets of $\mathbb{Z}$ containing $\{0,1\}$ is uncountable.

On the other hand, according to a theorem of M. Mather [Ma], there are only countably many homotopy classes of closed orientable n-manifolds. It is easy to verify that if two closed orientable n-manifolds M and M' are homotopy equivalent, then $D(M) = D(M')$. So the subsets of $\mathbb{Z}$ which can be realised as sets of self mapping degrees $D(M)$ for some closed oriented n-manifolds M are only countably many, for all positive integers n. This completes the proof of Theorem 2.1.

3.2. **Proof of Theorem 2.2.** We will prove the following precise version of Theorem 2.2:

Theorem 3.3. *Let M, N be closed oriented n-manifolds. For any integer $k \geq 3$, there exist infinitely many closed oriented hyperbolic k-manifolds W such that*

$$D(M, N) = D(M \times W, N \times W).$$

The proof of Theorem 3.3 is based on several results about closed oriented hyperbolic manifolds and the following specific form of [NWW, Theorem 4.6] (or [Ne, Theorem 1.4] for self-maps):

Theorem 3.4. *Let M, N be two closed oriented manifolds of dimension n and W be a closed oriented manifold of dimension k. Suppose*

(i) *W does not admit maps of non-zero degree from direct products $W_1 \times W_2$, where $\dim W_1, \dim W_2 > 0$ and $\dim W_1 + \dim W_2 = k$, and*
(ii) *for any map $M \to W$, the induced homomorphism $H_k(M; \mathbb{Q}) \to H_k(W; \mathbb{Q})$ is trivial.*

Then $D(M \times W, N \times W) = D(M, N) \cdot D(W)$. In particular, if in addition

(iii) *$D(W) = \{0, 1\}$,*

then $D(M \times W, N \times W) = D(M, N)$.

Hence, in order to prove Theorem 3.3 we need to find hyperbolic manifolds W satisfying (i)–(iii) in Theorem 3.4. First of all, hyperbolic manifolds satisfy (i), that is, they do not admit maps of non-zero degree from products [KL]. Next, given a closed oriented hyperbolic k-manifold L, with fundamental class $[L] \in H_k(L)$, its simplicial volume satisfies $\|L\| = \|[L]\|_1 > 0$ and $|\deg(f)|\|L\| \leq \|L\|$ for each map $f\colon L \to L$ [Th, 6.1.4, 6.1.2], thus $D(L) \subseteq \{-1, 0, 1\}$. (Here, $\| \cdot \|_1$ denotes the ℓ^1-semi-norm, which in top degree is the simplicial volume.) Since $0, 1$ always belong to $D(L)$, we need $-1 \notin D(L)$. This is indeed the case quite often, as observed by S. Weinberger (see [Mu, Section 3] and [Ne, Section 3.1]).

To deal with (ii) and (iii) of Theorem 3.4, we need the following two facts:

Lemma 3.5. *For each $k \geq 3$, there are infinitely many closed oriented hyperbolic k-manifolds $\{L_i\}$ such that $D(L_i) = \{0, 1\}$ and the volume of L_i is unbounded as i tends to infinity.*

Proof. By a result of Belolipesky and Lubotszky [BL, Theorem 1.1], for each $k \geq 2$ and any finite group Γ, there exists a closed oriented hyperbolic k-manifold L such that $\mathrm{Isom}(L) \cong \Gamma$, where $\mathrm{Isom}(L)$ is the full isometry group of L. By [Th, Theorem 6.4], every map $f\colon L \to L$ of $|\deg(f)| = 1$ is homotopic to an isometry when $k \geq 3$. Note that each orientation reversing isometry must has even order. Hence, if Γ is of odd order, then we have $\deg(f) = 1$ and $D(L) = \{0, 1\}$.

Now let L_i be a closed oriented hyperbolic k-manifold such that $\mathrm{Isom}(L_i) \cong \mathbb{Z}_{2i+1}$, the cyclic group of order $2i+1$. Then, the family $\{L_i\}$ contains infinitely many hyperbolic k-manifolds with $D(L_i) = \{0, 1\}$. The volume of L_i is unbounded as i tends to infinity. Indeed, if $k > 3$, this follows directly from H. C. Wang's theorem [Wa] that there are only finitely many hyperbolic k-manifolds with volume bounded by a fixed number $r > 0$. If $k = 3$, we have the following equality about hyperbolic volumes

$$V(L_i) = (2i+1)V(L_i/\mathbb{Z}_{2i+1}).$$

By a result of Meyerhoff [Me], the volumes of 3-dimensional hyperbolic orbifolds have a lower bound $C > 0$, hence $V(L_i) > (2i+1)C$ is unbounded as i tends to infinity. □

Lemma 3.6. *Let M and W be closed oriented manifolds of dimensions n and k respectively, and $\{\alpha_1, ..., \alpha_n\}$ be a basis of $H_k(M;\mathbb{Q})$ such that*

(i) *each α_i is the image of a homology class in $H_k(M;\mathbb{Z})$, and*
(ii) $\|W\| > \max\{\|\alpha_i\|_1 | i = 1, ..., k\}$.

Then for any map $M \to W$, the induced homomorphism $H_k(M;\mathbb{Q}) \to H_k(W;\mathbb{Q})$ is trivial.

Proof. Let $f\colon M \to W$ be a map. Let $[W]$ be the fundamental class of W. Then for any i, as integer homology classes, we have $H_k(f)(\alpha_i) = d_i[W]$ for some $d_i \in \mathbb{Z}$. By our assumption on $\|W\|$ and the functoriality of the ℓ^1-semi-norm (cf. [Gr, p.8]), we obtain

$$\|\alpha_i\|_1 \geq \|H_k(f)(\alpha_i)\|_1 = |d_i|\|W\| > |d_i|\|\alpha\|_1.$$

Thus $d_i = 0$ holds and $H_k(f)(\alpha_i) = 0$. Since $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of $H_k(M;\mathbb{Q})$ and $H_k(f)$ is linear, $H_k(f)$ must be trivial. □

Now we finish the proof of Theorem 3.3: Let $\{\alpha_1, ..., \alpha_n\}$ be a basis of $H_k(M;\mathbb{Q})$ chosen as in Lemma 3.6 (i). Recall that for each closed oriented hyperbolic manifold, its hyperbolic volume is proportional to the simplicial volume [Th, Prop. 6.1.4]. In Theorem 3.4, we can take W to be some closed oriented hyperbolic k-manifold L_i given by Lemma 3.5 such that Lemma 3.6 (ii) is satisfied as well, that is,

$$\|W\| > \max\{\|\alpha_i\|_1 | i = 1, ..., n\}.$$

Then W satisfies conditions (i) (ii) and (iii) of Theorem 3.4 (by Lemma 3.5, Lemma 3.6 and [KL]). Finally, we note that there are infinitely many choices for W (by Lemma 3.5), thus we conclude Theorem 3.3.

Remark 3.7. *Note that for $k > n$ (i.e. in all but finitely many dimensions for each n), condition (ii) of Theorem 3.4 is automatically satisfied and thus Lemma 3.6 and the second part of Lemma 3.5 about volumes is not needed in the proof of Theorem 3.3.*

3.3. **Proof of Theorem 2.3.** Let A be a finite set of integers containing 0. Then A is the mapping degree set of some 3-manifolds, by Theorem B. For each $n = 3 + k$, where $k \geq 3$, A is also the mapping degree set of some n-manifolds, by Theorem 2.2.

For the second part of Theorem 2.3, note that there are only two finite multiplicative sets containing $\{0,1\}$, namely $\{0,1\}$ itself and $\{-1,0,1\}$. The first set is realised by a closed hyperbolic n-manifold L as in Lemma 3.5. The second set is realised by $L\#\overline{L}$, where $\overline{L}$ is the manifold L with the opposite orientation, since $L\#\overline{L}$ admits a degree -1 self-map which is realised by the reflection about the $(n-1)$-sphere of the connected sum. Finally note that by [Gr] we have $||L\#\overline{L}|| = ||L|| + ||\overline{L}|| = 2||L|| > 0$. So $D(L\#\overline{L})$ is finite.

3.4. **A proof for $D(\mathbb{CP}^n) = \{k^n \mid k \in \mathbb{Z}\}$ (Example 1.6).** Without a precise reference for this known fact, we provide a proof: If t is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$, then $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[t]/(t^{n+1})$. From this, one derives that $D(\mathbb{CP}^n) \subseteq \{k^n \mid k \in \mathbb{Z}\}$. Below we show that for each integer k, there is a map

$$f_k\colon \mathbb{CP}^n \to \mathbb{CP}^n$$

of degree k^n. We may assume that $k \neq 0$.

If $k > 0$, define

$$f_k[z_0 : z_1 : ... : z_n] = [z_0^k : z_1^k : ... : z_n^k],\ [z_0 : z_1 : ... : z_n] \in \mathbb{CP}^n.$$

Note that for $\mathbb{CP}^1 = \{[z_0 : z_1 : 0 : 0 : ... : 0]\} \subseteq \mathbb{CP}^n$, the map $f_k|_{\mathbb{CP}^1}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ has degree k. Since $[\mathbb{CP}^1]$ is a generator of $H_2(\mathbb{CP}^n;\mathbb{Z})$, we have that $f_k\colon H_2(\mathbb{CP}^n;\mathbb{Z}) \to H_2(\mathbb{CP}^n;\mathbb{Z})$ is given by multiplication by k, and thus, by algebraic duality, $f_k^*\colon H^2(\mathbb{CP}^n;\mathbb{Z}) \to H^2(\mathbb{CP}^n;\mathbb{Z})$ is given by $f_k^*(t) = kt$. By the ring structure of $H^*(\mathbb{CP}^n;\mathbb{Z})$, we have $\deg(f_k) = k^n$.

If $k < 0$, define

$$f_k[z_0 : z_1 : ... : z_n] = [\bar{z_0}^{-k} : \bar{z_1}^{-k} : ... : \bar{z_n}^{-k}].$$

The map $f_k|_{\mathbb{CP}^1}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ has still degree k, and thus, as above, we have $\deg(f_k) = k^n$.

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