BOOK REVIEW[1][2]

Paul Nevai[3]

King Abdulaziz University, Saudi Arabia

Orthogonal Polynomials and Continued Fractions (From Euler's Point of View). *Sergey Khrushchev.* Encyclopedia of Mathematics, No. 122, Cambridge University Press, 2008. xvi + 478 pp., hardcover, size 228 x 152 mm, weight 0.86 kg, USD 126, EUR 88, GBP 70, ISBN 978-0-521-85419-1.

SHORT VERSION

Gentle Reader, why am I wasting your time here ... you know Sergey ... you know his math ... the conclusion is pretty much obvious ... if you care about OPs or CFs, then you must read this book... if you don't care then most likely you still should read it... thanks you for your attention ... good night.[4]

LONGER VERSION

Is there any person on Earth in my age group who has not written a book on a subject closely related to orthogonal polynomials yet?[5] Let me rephrase it to *...has not written a splendidly marvelous book....* Well, let me tell the Reader right now that Professor Khrushchev is not the exception to the rule.

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[2]Andrei Martínez-Finkelshtein is JAT's Book Review Editor.

[3]`paul@nevai.us`

[4]This book review was originally planned to be completed in 2009. However, unforeseen events, admittedly fully under my control, have kept me from fulfilling the promises I made to Andrei and myself. Mea máxima culpa.

[5]Just for fun, google *orthogonal polynomials djvu* and *continued fractions djvu*, and you will find an amazing number of books on these subjects. MathSciNet lists at least 56 books whose title include the words *CF* and no less than 90 with *OP*. However, surprisingly, *OPCF* is the one and only book that has *all* four words in its title although, at the time of writing this sentence, there were altogether 23 title hits.

In what follows, in view of my close personal and professional friendship with Professor Khrushchev dating back to 1966, I will allow myself to refer to him as *Sergey*. At the same time, this is an implicit warning to the Reader that my brain is hardwired to be incapable to exercise fair (or, for that matter, unfair) criticism when it comes to describing his magnum opus *OPCF*.[6]

Luckily for the gentle Reader, this journal is just a plain vanilla JAT as opposed to some fancy-shmancy journal such as BAMS, so that he[7] is not subjected to a never ending brief history of orthogonal polynomials and continued fractions that would culminate in a one paragraph review of the actual contents of the book under consideration.

PUZZLE. Who discovered[8] Chebyshev polynomials?

ANSWER. I bet you have no idea and neither do I,[9] but, according to Theorem 2.22 on p. 87, William Brouncker was already familiar in 1657 not only with them but also with their second kind sister polynomials.[10] Googling *brouncker chebyshev polynomials* leads to a number of interesting results, including Sergey's 2007 survey paper titled *Orthogonal Polynomials: The First Minutes* published in the Festschrift in honor of another OP guy who shall remain unnamed.[11]

Even a superficial perusal of the table of contents of *OPCF* shows that the proper title should have been CFOP, that is *Continued Fractions and (some) Orthogonal Polynomials*. I have a feeling that Sergey, the businessman, convinced Sergey, the scientist, that OPs sell more books than CFs, and, thereby, the loan on his brand new Opel Zafira Cosmo 1.8 will be sooner paid off. Smart move, Sergey. Thanks to amazon.com, the Reader can browse the TOC himself by going to `tinyurl.com/nnwtz8`, and then clicking on,

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[6] As the Reader will soon find out, this claim is an audacious lie; this review is abundant with unfair criticism.

[7] A modified version of this review, where all masculine references have been replaced by feminine ones, is available upon request. A genderless version is under consideration.

[8] I should perhaps use the verb *invented* or, even better, *introduced*.

[9] To my dismay, not even Ted Rivlin's encyclopedic *Chebyshev Polynomials* or Phil Davis' *The Thread: A Mathematical Yarn*, a superbly entertaining book about Pafnuti Lwowitsch Tschebyscheff (aka Pafnuty Lvovich Chebyshev, Pafnutyij Lvovics Csebisov, etc.), shed any light whatsoever to the origins of CPs.

[10] For the Second Viscount Brouncker of Castle Lyons, or, in short, Bill, see, for instance, `tinyurl.com/m5qc4w`.

[11] Everybody who's anybody knows him anyway.

where else, *click to look inside*.[12]

I wish amazon.com had made *OPCF*'s preface also available but it didn't. What a pity since it not only tells the Reader what the book is all about, but also explains how Sergey, the mathematician turned administrator turned mathematician, ended up studying OPs; see the AMS Notices at `tinyurl.com/lnoxfz` and mathforum.org at `tinyurl.com/nfttlx` for very interesting tidbits. Let me attempt to summarize...

Preface

Sergey learns about OPs from Geronimus' 1958 book but then he gets no insight as to the beauty of the subject until 1987 when he visits Djursholm[13] and ends up writing a paper that was published in 1993. I must admit that I found this paper less than stellar (Sergey gives a "simple" proof of Geronimus' fundamental result on the equality of Schur parameters and Verblunsky coefficients[14] but I am not so sure that his proof is really simple; see Theorem 8.21 in *OPCF*), and it also occurred to me that administrative work might have sucked away his mathematical ingenuity. However, as it turned out, this paper led to Sergey's 2001 paper[15] that was actually completed in 1999, and which, in my opinion, is one of the ten best papers ever written on OPs.[16] The same year, Sergey was unanimously elected to my *OPs Hall of Fame*.

If the reader is dissatisfied with this superficial summary, and, for whatever reason, he is not yet convinced to shell out the current discounted price of $129.86 at amazon.com, including free super saver shipping, then please read the real McCoy that the good people at Cambridge University Press[17] generously allowed me to post at `mw.nevai.us/AT/1.pdf`.

In what follows, it's not going to be a standard book review. Rather, I want to whet the Reader's appetite by my uninvited comments and by more or less randomly picked representative results from each chapter. I want

[12] This is `tinyurl.com/6enhawc`.

[13] The same mansion where I, in 1973, concocted my devious plans to say permanent goodbye to socialism & communism.

[14] Am I the only OPs person on Earth who will never get used to this terminology?

[15] I believe that this is the very first misprint in *OPCF*; it should be 2001a. How many more will the Reader find on this very same p. ix?

[16] In a book published in 2005, another OPs person classified it as a top twelve paper only; how unfair.

[17] Copyright © 2008 Cambridge University Press.

to put it on the record that, I am somewhat ashamed to admit it though, I am not an expert on CFs at all, and, therefore, I risk demonstrating my ignorance more than absolutely necessary. In addition, in view of the juggernautic advances that took place in the past three decades, I am no longer an up-to-date expert in OPs either, but then I am pretty sure that no one else is no matter how much they pretend to be.[18]

Ch. 1. Continued fractions: real numbers

The book starts with a short section on historical background. Whether or not the Reader cares about OPs and CFs, and no matter how (un)sophisticated his mathematics education might be, I guarantee that he will find numerous hugely enjoyable facts and stories on CFs.

For instance, Sergey gives the story how Hippasus of Metapontum "used" continued fractions to find $\sqrt{2}$ and to show that it is irrational.

I told myself, perhaps somewhat arrogantly, that these days one can find all this on the internet and there is no need to add to global warming by printing repetitious information. Indeed, googling *hippasus*, I immediately ended up at `en.wikipedia.org/wiki/Hippasus`; then clicked on `scienceworld.wolfram.com/biography/Hippasus.html`; then went to `mathworld.wolfram.com/PythagorassConstant.html`. After clicking away for several minutes, I realized that I should have started by googling *continued fraction hippasus,* which I did, and I ended up with `tinyurl.com/nmjtea`. Guess what, this is a freebie pdf of the first 10 pages of *OPCF* posted by Cambridge University Press. Touché for Sergey.[19]

Altogether, Section 1.1 lists nine well-known examples that have relations to continued fractions, including discussion of squaring the circle that leads to the *Archimedes-algorithm*[20]

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad \& \quad a_{n+1} = \sqrt{a_n b_{n+1}}\,.$$

[18] With the exception of. . .

[19] Although in the end, I found pages such as `tinyurl.com/lqt6wr` with close to a gazillion proofs similar to Sergey's presentation of Hippasus' argument, none of them were identical to Sergey's; see fig. (1.1) on p. 2 in *OPCF*.

[20] Sergey didn't say it so I add that the literature also refers to this as the *Pfaff-algorithm* or *Borchardt-Pfaff-algorithm*; google it.

Replacing a_n and b_n by their reciprocals, we get

$$b_{n+1} = \frac{a_n + b_n}{2} \quad \& \quad a_{n+1} = \sqrt{a_n b_{n+1}}\ ,$$

which are reminiscent of the sequences defining the arithmetic-geometric mean, that is,

$$b_{n+1} = \frac{a_n + b_n}{2} \quad \& \quad a_{n+1} = \sqrt{a_n b_n}\ .$$

Hence, the question arises if there is any relationship between π and elliptic integrals [smiley]. Since *OPCF* doesn't seem to yield an answer, I'll let the Reader think about it.[21]

Ch. 2. Continued fractions: algebra

So much to read so little time: Euler's algorithm, Lagrange's theorem, Pell's equation, equivalent irrationals, Markov's[22] theory, Jean Bernoulli sequences, and so forth.

But how could a true approximator care for an entire subject lacking limits, and, oh no, capital Oh's, small oh's, and similar relationships? It will come as a pleasant surprise to the Reader that there are a combined eleven limits here, including a few lim infs and lim sups. On top of that, it has a proof that I immediately understood; see Corollary 2.46 on p. 101. It goes as follows. If the set A is uncountable and B is countable, then $A \setminus B$ is uncountable as well. That's elementary, isn't it?

Many problems here start out as absolutely elementary and progress toward sophistication and depth. For instance, let's take the misnamed Pell equation[23]

$$x^2 = 1 + y^2$$

whose natural integer solutions are all easy to find, but then moving to the next round

$$x^2 = 1 + 2y^2$$

the triviality turns into depth and substance. Although the history of this equation depends on who you ask and what you read, there is no question

[21]What about starting with the Borwein brothers' *Pi and the AGM*; Google allows one to peek inside for free.

[22]Professor Khrushcheff prefers writing *Markoff*.

[23]Google *euler brouncker pell*; `tinyurl.com/2bx7hbo` is fun to read as well.

that it was studied before Brouncker. Nevertheless, Brouncker found solutions, see (2.30) on p. 84, and, according to Sergey's and others' speculation based on Brouncker's letter to Wallis, he might have used continued fractions to arrive at the potential conclusion that the solutions are related to the continued fraction expansion on $\sqrt{2}$. One needs to be very careful how the historical facts are stated since no one really knows them for a fact, so to speak.

Here is a nice problem from the exercise section: a natural integer is a sum of two integer squares if and only if it is a finite product of multipliers each of which is either 2, or the square of an integer, or else a prime number of the form $4n+1$. According to Sergey, this generalization of Fermat's theorem on the sums of two squares belongs to A. Girard.[24]

HOMEWORK. Find all the limits (not lim infs and lim sups) in Chapter 2.

Ch. 3. Continued fractions: analysis

HOMEWORK. Prove Viète's product formula from 1593

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \dots ,$$

but please do **not** see (3.9) in *OPCF*.

HINT. Either cheat by looking up, say, `tinyurl.com/mpmon9` (or your calculus lecture notes) or don't be ashamed to admit defeat, since, as Sergey tells us, it took 170 years until Euler found the easy, now standard proof. Of course, it is quite possible that you will prove it on your own, in which case, hats off to you.

Did you know that the standard calculus book proof of Wallis' product formula, anno 1656, for $\frac{2}{\pi}$, see (3.10) in *OPCF*, also goes back to Euler in 1768 who improved upon Wallis's original arguments?

Most of this chapter deals with the work of Brouncker and Wallis. Ramanujan's formula for Brouncker's function $b(s)$ in terms of Γ-functions is the icing on the cake. Here $b(s)$ ain't no bee-ess; it solves the functional equation

$$b(s)b(s+2) = (s+1)^2 .$$

[24] Professor Jhrushchev spells it as *Jirard*; see `tinyurl.com/68skyro`

HOMEWORK. Prove

$$\pi \equiv \frac{1^2}{6} \mathop{+}\frac{3^2}{6} \mathop{+}\frac{5^2}{6} \mathop{+}\frac{7^2}{6} \mathop{+}\frac{9^2}{6} \mathop{+} \dots \qquad (\mathrm{mod}\ 3)\ .$$

HINT. See Exercise 3.11 on p. 155.

Ch. 4. Continued fractions: Euler

I heard it through the grapevine that poor Euler who couldn't even afford a single pair of ϵ–δ has never committed a single crime that the post-Cauchy generation could have held against him; e.g., we all know now that $1 - 1 + 1 - 1 + 1 \cdots$ is indeed equal to $0.4999\ldots$.[25] So it is safe to assume that Euler's work in CFs is impeccable.

Euler, in 1744, solved the very basic problem of writing sums (and, thereby, series as well) as continued fractions the following way

$$\sum_{k=0}^{n} c_k = \frac{c_0}{1} \mathop{-} \frac{c_1/c_0}{1 + c_1/c_0} \mathop{-} \frac{c_2/c_1}{1 + c_2/c_1} \mathop{-} \dots \mathop{-} \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1}}\ , \qquad \text{all } c_k \in \mathbb{C} \setminus \{0\}\ ,$$

see Theorem 4.2 on p. 159, or, defining (ρ_k) by[26]

$$c_k = \rho_0 \rho_1 \ldots \rho_k\ , \qquad k = 0\,, 1\,, \ldots\ ,$$

this becomes the *beautiful formula*[27]

$$\sum_{k=0}^{n} \rho_0 \rho_1 \ldots \rho_k = \frac{\rho_0}{1} \mathop{-} \frac{\rho_1}{1 + \rho_1} \mathop{-} \dots \mathop{-} \frac{\rho_n}{1 + \rho_n}\ .$$

The latter is called Euler's continued fraction for $\sum c_k$.[28]

This chapter is quite long and it is not at all limited to works of Euler alone. It even discusses Euler's work on Ricatti's equation and its solution using CFs.

Let me just give a couple of extra tributes to Euler by mentioning the well-known and easily googlable

$$e = 2 + \frac{1}{1} \mathop{+} \frac{1}{2} \mathop{+} \frac{1}{1} \mathop{+} \frac{1}{1} \mathop{+} \frac{1}{4} \mathop{+} \frac{1}{1} \mathop{+} \frac{1}{1} \mathop{+} \frac{1}{6} \mathop{+} \frac{1}{1} \mathop{+} \dots\ ,$$

[25]\dots + period = ...

[26]Professor Khrushchev Khrushchev also defines $\rho_0 = c_0$.

[27]With Sergey's own words.

[28]Professor Khrushchev forgot to point out that this also holds if, G·d forbid, one or more of the ρ_k's vanish.

and, say,

$$\sqrt[3]{e} = 1 + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{1} + \frac{1}{14} + \frac{1}{1} + \frac{1}{1} + \frac{1}{20} + \dots,$$

where one needs to keep adding 6's to those non-ones; see, e.g., Mathematica's `ContinuedFraction[E^(1/3), gazillion]`.

The chapter concludes with 54 torturous exercises.

HOMEWORK. Find out who found it first, and then who proved it first that

$$\sin x = \frac{x}{1} + \frac{x^2}{2 \times 3 - x^2} + \frac{2 \times 3x^2}{4 \times 5 - x^2} + \frac{4 \times 5x^2}{6 \times 7 - x^2} + \dots.$$

I must confess, I was unable to solve this excruciatingly complicated problem.

HOMEWORK. Prove the above.

HINT. There must be a reason that Exercise 4.5 on p. 215 has no hint.

Appetite whetted? Let's move on.

Ch. 5. Continued fractions: Euler's influence

PUZZLE. Is it possible to write a book on CFs without mentioning chain sequences?

ANSWER. Yes, e.g., *OPCF*.

PUZZLE. Is it possible to write a book on CFs without having chain sequences in them?

ANSWER. Maybe, but it's not gonna be *OPCF*. The careful Reader will find those pesky chain sequences right here in this chapter, and only here, and only twice; see the RHS of (5.28) and the second displayed formula afterward.

What are chain sequences? Well, look them up in Wikipedia[29] or ask Ted Chihara.[30]

One application of chain sequences is *Worpitsky's* test: if $|c_k| \leq 1/4$ for all $k \in \mathbb{N}$, then $\mathbf{K}_{k\geq 1}(c_k/1)$ converges absolutely, that is, if (f_n) denotes the

[29] `tinyurl.com/3owlfk3`

[30] Ted did more than any other soul to study chain sequences.

sequence of the nth convergents, then $\sum |f_k - f_{k-1}|$ converges; see Corollary 5.14 on p. 237.

Then Sergey talks about Pringsheim's theorem.[31]. Then a bunch of more theorems about convergence of CFs.

I found Exercises 5.7–5.9 on pp. 245–246 fascinating. Are there any simpler solutions? I wonder if Sergey realized that Exercise 6.1 on p. 293 is a special case of Exercises 5.9 on p. 246 (Euler $\subset$ Ramanujan).

Ch. 6. P-fractions

"P" stands for polynomials, so P-fractions are CFs where the terms are polynomials. Well, almost. They are CFs of the form

$$\mathbf{b}_0(z) + \mathop{\mathbf{K}}_{k=1}^{\infty} \left(\frac{1}{\mathbf{b}_k(z)} \right) ,$$

where $\mathbf{b}_k$'s are non-constant polynomials.[32]

This is a rich area with large intersection with standard approximation theory, special functions, and orthogonal polynomials.

Do you remember the short introductory section on the relationship between CFs and OPs in Szegős's book on OPs? So you know that (first and second kind) OPs are, in fact, denominators and numerators of convergents of P-fractions with a special kind of $\mathbf{b}_k$'s that are of exact degree 1.

Let $\mathbb{C}\left([1/z]\right)$ consist of formal *left-terminating* Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{c_k}{z^k} , \qquad c_k \in \mathbb{C} \ \& \ \inf\{k \in \mathbb{Z} : c_k \neq 0\} > -\infty ,$$

which can be equipped with the non-Archimedean[33] norm

$$\|f\| = \exp(\deg(f)) , \qquad \deg(f) = -\inf\{k \in \mathbb{Z} : c_k \neq 0\} .$$

[31] Until the modern era, Pringsheim might have been the richest of all mathematicians, perhaps even beating the next one by orders of magnitude; see, e.g., Wikipedia at `tinyurl.com/43tk6ow` Look at his family room at `tinyurl.com/3qe2xav`. No wonder that Thomas Mann married his daughter, who wouldn't?

[32] Sergey requires that the degree of $\mathbf{b}_k$ be at least 1, but that leads one to a slippery road. Call me a nitpicker but I believe it is legitimate to call a polynomial of exact degree, say, 5, of degree 7 as well.

[33] Sergey spells it as *nonarchimedean*.

Here is a lovely familiar looking best-approximatorial result in $\mathbb{C}\,([1/z])$; see Theorem 6.3 on p. 252. Given an infinite continued P-fraction as above with convergents P_n/Q_n,[34] let P and Q be polynomials with $0 < \deg(Q) < \deg(Q_n)$ and $P/Q \neq P_{n-1}/Q_{n-1}$. Then

$$\left\| \frac{P}{Q} - \frac{P_{n-1}}{Q_{n-1}} \right\| > \left\| \frac{P_{n-1}}{Q_{n-1}} - \frac{P_n}{Q_n} \right\| .$$

PUZZLE. If traditionally $\mathbb{P}$ denotes polynomials, then what is $\mathbb{P}_+$ and $\mathbb{P}_\varepsilon$?

ANSWER. $\mathbb{P}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$ and $\mathbb{P}_\varepsilon = \{z \in \mathbb{C} : |\arg(z)| < \pi/2 - \varepsilon\}$.

Edward Burr Van Vleck proved in 1901 the marvelous theorem, see Theorem 6.52 on p. 286, that if there is $\varepsilon \in (0, \pi/2)$ such that $b_k \in \mathbb{P}_\varepsilon$ for $k \geq 1$, then

(a) the n-th convergent f_n of $\mathop{\mathbf{K}}\limits_{k=1}^{\infty}(1/b_k)$ is in $\mathbb{P}_\varepsilon$;

(b) both limits $\lim_{n\to\infty} f_{2n}$ and $\lim_{n\to\infty} f_{2n+1}$ exist and are finite;

(c) the continued fraction converges if $\sum_k |b_k| = +\infty$;

(d) if $\mathop{\mathbf{K}}\limits_{k=1}^{\infty}(1/b_k)$ converges to f, then $f \in \mathbb{P}_+$.

PUZZLE. Where was the very first office space that I occupied in the US?

ANSWER. In Van Vleck Hall.

HOMEWORK. Why do I dislike that Sergey actually wrote *... the n-th convergent f_n of $\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(1/b_n)$...*?

HOMEWORK.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{2}{1} \underset{+}{} \frac{3}{2} \underset{+}{} \frac{4}{3} \underset{+}{} \frac{5}{4} \underset{+}{} \frac{6}{5} \underset{+}{} \frac{7}{6} \underset{+}{} \cdots .$$

Ch. 7. Orthogonal Polynomials (aka OPs)

[34] P on the top and Q on the bottom make sense from the alphabetical point of view. However, it differs from the traditonally accepted way of writing convergents, at least within the OPs community.

If you expect a yet-another-run-of-the-mill-introduction to OPs, then you are going to be hugely disappointed. When was the last time you saw a book on OPs that studied zeros of OPs using Sturm sequences? I bet Sergey chose this approach because he remembered the algebra classes by Zenon Ivanovich Borevich[35] back in 1967, although, as I recall, the words OPs were never explicitly pronounced.[36]

Sergey succeeds to give an overview of OPs all the way from orthogonal matrices (Euler) and quadrature formulas (Newton, Cotes, Gauss, Jacobi) to Abelian integrals and equilibrium measures (Franz Peherstorfer and Robert Steinbauer[37]).

Of course, this is not a substitute for an elementary introduction. Still it serves well for those of us who had already way-too-many what's-their-names' orthogonalization and what's-his-name's moment problem.

Here is a nice problem from the exercise section: If f and g are polynomials with real zeros, $\deg(g) \le \deg(f)$, and between two zeros of g there is at least one zero of f, then $f'g - fg'$ doesn't vanish on $\mathbb{R}$.

HOMEWORK. Whose theorem is this and how is it related to OPs?

Ch. 8. Orthogonal Polynomials on the Unit Circle (aka OPUCs)

OPUC, OPUC, OPUC everywhere. I googled OPUC variations on May 20, 2011, and I ended up with the following scientific survey: *opuc saff* had 12,600 hits (but mostly because of the common word *staff*), *opuc lopez* had 5,440 hits (thanks to Ms. Jennifer Lopez), *opuc simon* had 3,220 hits (including many other unrelated Simons), *opuc nevai* had 1,080 hits, *opuc rakhmanov* had 832 hits (so many ways to spell Jenya/Zhenya's family name), *opuc totik* had 700 hits, *opuc assche* had 462 hits, *opuc lubinsky* had 377 hits, and *opuc khrushchev* had a miserly 95 hits only (including Nikita). Thus, it is a matter of scientific fact, Sergey is the least qualified to discuss OPUC. Right? Wrong!

This chapter beautifully complements Barry Simon's book titled OPUC from the point of view of CFs, Schur functions, Hardy spaces, and many other considerations that historically have not been viewed as standard tools

[35]Google *vershik3-15-1990_transcript*; for limited biographical information, see `tinyurl.com/6xy27r7`, and, if you have access to SpringerLink, then use `tinyurl.com/6bzq3hl`.

[36]It turns out, I am wrong; Sergey simply followed Chebyshev's line of reasoning.

[37]Professor khrushchev spells it as *steinbauer*.

for OPUC at all. In short, it is not at all business as usual.

The discussion moves from general to specific, from Szegő (the log thingie is integrable), to Erdős (the a.c. component of the measure is positive a.e.), to Rakhmanov (the OPUC^2 weak-$\star$ converge to Lebesgue), to Nevai (hurrah, that's me; those infamous recursion coefficients[38] go to 0). to singular measures, to ratio asymptotics, to López (this requires more words; see Definition 8.1.115 on p. 391), to periodic measures,[39] etc.

Highlights? Too many. Any expert in OPUC worth his salt will be unable to stop drooling when seeing this rich collection of sexy results. Unfortunately, many of them require quite a few definitions and notations.

For instance,[40] one of my favorites is Sergey's proof of Rakhmanov theorem that the Verblunsky coefficients $a_n \to 0$ in the Erdős class. Sergey deduces this from the inequalities

$$|a_n| = |f_n(0)| = \left|\int_T f_n \, dm\right| \leqslant \left(\int_T |f_n|^2 \, dm\right)^{1/2} .$$

Here (f_n) are the Schur functions obtained from f via the Schur algorithm, where f is the Schur function associated with the measure (Herglotz & Co.); see Theorem 8.59 on p. 358. It is easy to see that $|f| < 1$ a.e. on the unit circle $\iff$ the measure's a.c. component is positive a.e., see, e.g., formula (8.7) on p. 325. Then Sergey uses his own marvelous theorem that if f is a Schur function then $|f| < 1$ a.e. on the unit circle if and only if the Schur functions (f_n) satisfy

$$\lim_{n\to\infty} \int_{\mathbb{T}} |f_n|^2 \, dm = 0 ,$$

see Theorem 8.58 on p. 356, to finish the proof.

I really don't know what other results to quote. Too many results too little time.

HOMEWORK. Find an example for Exercise 8.23 on p. 424 where strict inequality $<$ holds in the displayed $\leq$ formula.

NOTE. Did you fail? Don't despair. Actually, although Sergey's claim is technically correct, the truth of the matter is that the OPUC (φ_n) associated

[38] Reflection/Schur/Szegő/Verblunsky/etc.

[39] Professor Absentminded forgot to define them. Anyway it's not the measure that's periodic but the Schur parameters associated with it; see the introduction in Sergey's 2009 JAT paper, vol. 159, pp. 243–289.

[40] If you are not an OPUC person, please skip this paragraph.

with a measure σ satisfy

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^2 = \frac{1}{\sigma(\{z\})}$$

for every z on the unit circle. Prove this.[41]

Let me point out, and also let me register this as a formal complaint against Sergey, that practically every time my name is mentioned in this chapter, it should be replaced by either MNT[42] or by a suitable subset or permutation thereof.

Appendix; Continued fractions, observations; L. Euler (1739)

This is Sergey's English translation of a Russian translation of the Latin translation of Euler's work. Well, it might have been written in Latin in the first place. I don't know what was Euler's working language although it is highly unlikely that he conversed in Latin with his Swiss/German/Russian colleagues. I think it needs to be translated to German and then Professor Euler could be consulted if anything was lost in the process.

For me this was a first ever peek at Euler's original work and as such it was immensely fascinating. Let me warn the Reader that it is not easy to read it.

Interestingly, Euler explicitly talks about *convergence* although it is not clear to me at all what he means by it. The same applies to induction; he concludes the validity of a formula by evaluating it for $n = 1, 2, \ldots, 6$. He denotes the the natural logarithm by the letter ℓ as in $\ell 2$, or rather $l2$.

One thing that strikes me is that Euler was using neither sums ($\sum$), nor products ($\prod$), nor factorials (!), nor indices (a_k). This makes his formulas look rather awkward; e.g., he takes the series

$$A + \frac{B}{P} - \frac{BD}{PQ} + \frac{BDF}{QR} - \cdots ,$$

and transforms it to

$$A + \frac{B}{2P} + \frac{BE}{2Q} - \frac{BDG}{2PR} + \frac{BDFI}{2QS} - \frac{BDFHL}{2RT} + \cdots .$$

[41] I asked four very distinguished friends of mine if this is true and they all agreed; this is a rigorous enough proof for me.

[42] Máté, Nevai, Totik.

Are you following me? Good, because I am already hopelessly lost.

Another example is

$$\int y^{m-1}\, dy\, (1-y^{2r})^{\kappa}(1-y^{r})^{n}$$

If I understand it correctly, this means

$$\int y^{m-1}\, d\left(y\,(1-y^{2r})^{\kappa}(1-y^{r})^{n}\right)$$

as opposed to

$$\left(\int y^{m-1}\, dy\right)\,(1-y^{2r})^{\kappa}(1-y^{r})^{n}$$

Thanks to Sergey's footnotes though, there is a way to decrypt Euler's formulas.

I was curious to see if he used e for e but, couldn't find it.[43] Somewhat puzzlingly, I found more than once expressions such as xx or aa instead of the otherwise regularly used power notation.

Of course, keep in mind, that these are just Euler's *observations* and not a polished work ready for the critical eye of a dedicated 18th century copy editor.

References

It's not as extensive as it could be but then MathSciNet, Wikipedia, and Google take care of most of our needs anyway. I am grateful to Sergey for spelling his own name in a consistent manner[44] and for not spelling Chebyshev as Tschebyscheff. However, please, why Markoff? Also, Chebotarëv looks kind of unusual.

The Cambridge University Press reference style is of the form *Khrushchev (1993)* that can lead innocent bystanders to unexpected conclusions, e.g., that Euler was still alive in 1813; see Exercise 6.1 on p. 293, which, by the way, has an extra left-parenthesis as in *(Euler*.

Index

[43] According to Wikipedia, Euler introduced the notations e, and $f(x)$, and, maybe, even i.

[44] This itself is a remarkable feat.

I wish Sergey had hired Ms. Cherie Galvez, Barry Simon's business manager, of the OPUC fame (google *barry simon colloquium 54*), because *OPCF* badly needs a well organized index of notation. For some mysterious reason it has no author index. In addition, a much more detailed general index wouldn't hurt either.

EXAMPLE. Look at Lemma 8.22 on p. 338. Since *OPCF* is more or less an *encyclopedia*, I should be able to figure out what the lemma says without reading the entire book. Is this Kolmogorov's theorem with the anti-Szegő condition? Well, it took me quite a while to figure it out that the mysterious $N(\sigma)$ is defined near the top of p. 336, and that it is just the cardinality of the set of points of increases of the measure σ. Elementary, my dear Khrushchev (`tinyurl.com/q67mu`).

MISCELLANEOUS OBSERVATIONS

Each chapter concludes with an excellent set of exercises, eight sets altogether, many of which even have accompanying hints. As it frequently happens in mathematical monographs, and this book is no exception, many of the exercises are, in fact, serious and deep results. As a consequence of a typesetting bug, these exercises are not mentioned in the table of contents or anywhere else in the book, not even in the preface.

On the other hand, the book could have contained more and more detailed historical information on recent results on OPUC many of which appear in a book the first time in *OPCF*.

I believe there is no formal definition of the (standard) symbol $\mathop{\mathbf{K}}\limits_{k=1}^{\infty}$ used in denoting continued fractions in a linearly (horizontally) readable form. The closest we have is (1.12) on p. 11 where Sergey casually writes

$$b_0 + \frac{a_1}{b_1} {}_{+} \frac{a_2}{b_2} {}_{+\cdots} = b_0 + \mathop{\mathbf{K}}_{k=1}^{\infty} \left(\frac{a_k}{b_k} \right) \,,$$

end then the index refers to this formula as **continued fraction**.

On p. ix, Sergey credits Yakov Lazarevich Geronimus, a guy from Kharkov, with translating Szegő's OPs into Russian, whereas it was Viktor Solomonovich Videnskiĭ,[45] a guy from Leningrad and, possibly, Sergey's former professor, who did it.

[45] In 2008, at the age of 84, he still published a research paper.

In Section 1.4, on p. 36, Sergey discusses Jean Bernoulli sequences. What? Jean Bernoulli? Oh yes, Jean (Johann) is one of the many Bernoullis, including his brother Jacob. That explains it. But then why Andrey-less Markoff conditions in the very same section, see §27, on p. 44, and many more times elsewhere; sometimes even in the very same sentence? Didn't Andrey have a brother too (Vladimir)? Yes, he did; see Exercise 7.3 on p. 320.

In §43, on p. 84, Sergey places the city of Toulouse into Italy.[46]

In the second line of formula (2.37) on p. 87, x_1 should be y_1. I wonder why Sergey doesn't tell the Reader that $(x_1)_{n\geq 1}$ and $(y_1))_{n\geq 1}$ are the 1st and 2nd kind Chebyshev polynomials, resp.

On p. 128, line 6-, there is a missing space character as in *Theorem 3.6gives*.

I truly appreciate that Sergey defines weak-$\star$ topology and weak-$\star$ convergence, see p. 322, since ϵ-differences exist in various definitions. On the other hand, I wonder if it's a glitch that he refers to them as $\star$-*weak* (*topology*|*convergence*).[47] Of course, since Sergey provides the definition, it is absolutely fair and legitimate to use any terminology, even if it is a slight permutation of the standard one.

When I teach calculus or real analysis, I always complain to my students that calculus books (and astronomers/navigators wanting to annoy us, mathematicians; see, e.g., `tinyurl.com/kr3lzk`) invented those weird trigonometric functions that are reciprocals of good guys, such as secant, cosecant, and even perhaps cotangent. I tell them that I've never seen a mathematician worth his salt to ever use these functions just as no professional mathematician will ever read -3 as *negative three* (we say *minus three*, right?). So it was a big surprise for me to see Smirnov's inequality (Theorem 8.8 on p. 326) with the secant function in it. I feel guilty now that all these years I lied to my students. Speaking of Vladimir Ivanovich Smirnov, let me put it on the record that, unlike so many other Soviet mathematicians, he was a real gentleman and a wonderful human being, and, just like so many other Soviet mathematicians, he was a great mathematician as well, albeit somewhat unknown in the Western Hemisphere despite his phenomenally successful `A Course in Higher Mathematics`;[48]

[46] To lose Toulouse or not to lose Toulouse, that is the question.
[47] This is a REGEXP.
[48] The first two volumes was written jointly with Jacob David (aka Yakov Davydovich)

see, e.g., `tinyurl.com/mgyqgs`.

Unimodularity is simultaneously both defined and assumed to be known in Lemma 8.12 on p. 329. Writing *…unimodular, that is,* $|f| = 1, \ldots$ instead of *…unimodular,* $|f| = 1, \ldots$ would have made the exposition more fluid.[49]

Now comes Blaschke, I mean, *blaschke*; see three times on p. 329, and approximately 22 times altogether in *OPCF*. Many years ago, as my own personal protest against nazis and their cronies, I stopped capitalizing the names of nazis such as bieberbach, blaschke, and teichmüller, and I wish Sergey had continued this tradition, especially, having been born in Leningrad, he should have excellent reasons for a lifetime disgust toward all nazis. It is also tragically ironic that in one line we read about blaschke products, and in the next one about Schur functions; perpetrator vs. victim.[50] Speaking of Schur and nazis, nevanlinna[51] appears three times in *OPCF* too; twice on p. 360, right above Schur.

Sergey left out a $\lim_n$ in front of the integral in Corollary 8.50 on p. 352.

Somehow, $1 < \theta \leq 2$ ended up as $0 < \theta \leq 1$ on p. xiii, right after formula (2).[52]

In Exercise 8.23 on p. 424, Sergey talks about Parseval's inequality. Google yields under 1K hits for it, whereas *Bessel's inequality* gets close

Tamarkin whose two unforgivable sins (a Jew and a defector) resulted in having his name disassociated from the books. Being familiar with the Soviet system (google *great purge* or *great terror*), I don't blame Smirnov for agreeing to this. No one knows who actually wrote or co-wrote the other books in this collection; see the amazing story of Tamarkin in Russian (but `translate.google.com` does a respectable job) at `tinyurl.com/nmvfyy` by N. S. Ermolaeva

[49]If the Reader thinks that I am nitpicking, I am guilty as charged. To minimize my guilt, I won't even point out that Sergey knows the spelling of Szegő but overlooks Erdős (Erdös).

[50]Issai Schur left Germany for Palestine in 1939, broken in mind and body, having the final humiliation of being forced to find a sponsor to pay the *Reichs flight tax* to allow him to leave Germany. Without sufficient funds to live in Palestine he was forced to sell his beloved academic books to the Institute for Advanced Study in Princeton. For more details, see MacTutor at `tinyurl.com/kksynh`. In favor of blaschke, it appears that he was not as bad as bieberbach who was personally involved in persecuting Schur; see the previous web reference.

[51]It is well known and well documented that nevanlinna was a nazi. Just google *nevanlinna nazi* although, surprise surprise, our Finnish colleagues have an excellent explanation that this is only an optical illusion.

[52]Both the misprint and the correction was contributed by Sergey.

to 16K hits so, if I may say so, let's stick to Bessel.[53]

Regarding the aesthetic quality of *OPCF*, Cambridge University Press did an outstanding job, and the people involved in the production process should be both commended and thanked for this; the binding, the slightly ivory colored paper, and the typesetting are all superb. In addition, apart from some minor and mostly unavoidable issues, the copy-editing is also flawless; I suspect that we must thank Sergey for the latter although the more I read *OPCF* the more I think that the anonymous copy-editor must have played a non-trivial role in minimizing Sergey's occasional Runglish.[54]

PRE-FINALLY...

I am almost finished and I haven't been able to make up my mind yet what's the best way to utilize *OPCF*. Should one read it from α to ω or else just find and digest the appropriate chapter or section or result. Since, as pointed out above, the indexing system is less than perfect, Sergey himself couldn't have meant the latter.[55]

As a practical advice, especially if you are filthy rich,[56] I recommend keeping a copy of *OPCF* at every location where you perform CF.

FINALLY...

Wrapping up this review, let me make the following statement, on the record, that I promise to never ever regret although, as history shows, many, if not most predictions fail to materialize, especially those related to the future. Sergey, the OPs guy, is of the same caliber as some of the greatest OPs persons who have ever lived such as (gentle Reader, please insert your choices here, e.g., Stieltjes, Bernstein, Szegő, or Geronimus[57]) and his *OPCF* is yet another proof that his work is destined for immortality as long as OPs remain a viable mathematical subject. Paraphrasing John Wallis,[58]

PRAISE BE TO SERGEY

[53] Well, at least Parseval beats Bessel when it comes to *equality*. I was surprised to see that, as of May 21, 2011, even *Bessel's equality* scored 300+ hits.

[54] Lack of articles in Russian make it difficult for Russians to master English language.

[55] I have a confession to make but my NDA pevents me from confessing it so I can't tell you how I use the book.

[56] This can be substituted by being a dear friend of Sergey.

[57] I deliberately avoid mentioning OP people who are still kicking.

[58] Wallis actually refers to a well-known and hugely successfully well-publicized fairy-tale-concept, see, e.g., `tinyurl.com/m27pp4` at amazon.com.

P.S. I strongly urge the Reader to check out the web references mentioned above; for a click friendly pdf of this review see `mw.nevai.us/AT/3.pdf`.

P.P.S. I thank the anonymous referee for supplying me with information that I thought only Sergey was privy to.