1. For given $\varepsilon > 0$, we say that a graph $G$ on $n$ vertices is $\varepsilon$-far from satisfying a property $\mathcal{P}$ if one needs to add or to delete at least $\varepsilon n^2$ edges to turn it into a graph satisfying $\mathcal{P}$.

For given $\varepsilon > 0$, using the triangle removal lemma to design a (simple) randomized algorithm to test if a graph $G$ is $\varepsilon$-far from being triangle-free with confidence level 0.999 (i.e. we are 99.9% sure that the output is correct).

2. (a) Assume that $X_1, X_2, X_3$ are disjoint vertex sets in $G$ with $x_1, x_2, x_3$ vertices respectively. Assume furthermore that the graphs $(X_i, X_j), 1 \leq i < j \leq 3$ are all $\frac{1}{2}\varepsilon^2$-regular with density $d_{ij} \geq 2\varepsilon$ respectively. Let $N$ denote the number of copies of $K^2_3$ in $G$ (the complete tripartite graph with two vertices on each part). Show that

$$N \geq c(x_1 x_2 x_3)^2,$$

where $c$ is a constant depending on $\varepsilon$, and $x_1, x_2, x_3$ are sufficiently large.

(b) Deduce from here the Erdős-Stone theorem for $H = K^2_3$.

3. Let $(a_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ doubly stochastic matrix (i.e. $a_{ij} \geq 0$ and $\sum_{1 \leq k \leq n} a_{ik} = \sum_{1 \leq k \leq n} a_{kj} = 1$ for all $i, j$). Show that $A$ is in the convex hull of the $n \times n$ permutation matrices (i.e. $\exists \lambda_1, \ldots, \lambda_m \geq 0, \lambda_1 + \cdots + \lambda_m = 1$ such that $A = \lambda_1 P_1 + \cdots + \lambda_m P_m$).

4. (a) Prove that if $\mathcal{F}$ is a family of subsets of $[n]$ such that for any $A \neq B \in \mathcal{F}$ neither $A \subset B$ nor $B \subset A$, then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

(b) Use this to deduce the following: assume that $a_i \in \mathbb{R}, 1 \leq i \leq n$ with absolute value at least 1. Let $I \subset \mathbb{R}$ be any open interval of length 1. Then among the $2^n$ sums $\sum_{i=1}^{n} \epsilon_i a_i$, $\epsilon_i \in \{-1, 1\}$, there are at most $\binom{n}{\lfloor n/2 \rfloor}$ sums belong to the interval $I$.

5. Can you construct a directed graph $G = (V, E)$ together with a capacity function on its edge such that the augmenting path algorithm (for finding a maximum flow) might not terminate?