LINEAR ALGEBRA METHOD IN COMBINATORICS

1. Warming-up examples

Theorem 1.1 (Oddtown theorem). *In a town of n citizens, no more than n clubs can be formed under the rules*

- *each club have an odd number of members*
- *each pair of clubs share an even number of members.*

Proof. It is enough to show that the incidence vectors v_i are linearly independent over $\mathbf{F}_2$. To prove this, one just observes that $\langle v_i, v_j \rangle = \delta_{ij}$ over $\mathbf{F}_2$. □

Theorem 1.2. *The same conclusion holds if we reverse the rules:*

- *each club have an even number of members*
- *each pair of clubs share an odd number of members.*

Let $a_1, \dots, a_m$ be points in $\mathbf{R}^n$, it is clear that if all the pairwise distances $d(a_i, a_j)$ are equal, then $m \le n+1$. Now assume that $d(a_i, a_j)$ can take two values, then how big m can be?

Theorem 1.3 (Two distance). *Let $m(n)$ be the largest number m can take, then one has*

$$n(n+1)/2 \le m(n) \le (n+1)(n+4)/2.$$

Proof. The lower bound can be obtained by considering $(0, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$. For the upper bound, assume that the distances take values δ_1 and δ_2. For each $1 \le i \le m$, we define the polynomials $f_i : \mathbf{R}^n \to \mathbf{R}$:

$$f_i(x) := (\|x - a_i\|_2^2 - \delta_1^2)(\|x - a_i\|_2^2 - \delta_2^2).$$

Notice that $f_i(a_i) \neq 0$ and $f_i(a_j) = 0$ for all $j \neq i$. Because of this, $f_i(x)$ are linearly independent over the linear space generated by $\{(\sum_{k=1}^n x_k^2)^2, (\sum_{k=1}^n x_k^2)x_j, x_i x_j, x_i, 1.\}$

□

Similarly, one also has

Theorem 1.4 (*s*-distance)**.**

$$\binom{n+1}{s} \leq m(n,s) \leq \binom{n+s+1}{s}.$$

Our next example is on the decomposition of K_n into complete bipartite graphs.

Theorem 1.5. *If the edge set of the complete graph on n vertices is the disjoint union of the edge sets of m complete bipartite graph, then $m \geq n-1$.*

Proof. For each complete bipartite graph (X_k, Y_k), we assign an $n \times n$ matrix A_k in which $a_{ij} = 1$ if and only if $i \in X_k$ and $j \in Y_k$. It is clear that $S = \sum A_k$ has the property that $S + S^T = J - I$. We next claim that $r(S) \geq n-1$. Indeed, otherwise there exists $x = (x_1, \dots, x_n)$ with $x_1 + \cdots + x_n = 0$ and $Sx = 0$. Thus $S^T x = -x$, and so $0 = x^T S^T x = -\|x\|^2$. □

Here is another example using more spectral properties.

Theorem 1.6. *Assume that G is r regular graph with r^2+1 vertices and with girth 5, then $r \in \{1, 3, 5, 7, 57\}$.*

Proof. One first observe that for any vertex-pair (v_1, v_2), $|N(v_1) \cap N(v_2)| = 1$. This motivates us to consider the adjacency matrix A. It has the following properties:

- $A^2 = rI + \bar{A}$
- $I + A + \bar{A} = J$.

From here, it is not hard to show that either $r = 2$ or $s^4 - 2s^2 - 16(m_1 - m_2)s = 15$ with $r = (s^2+3)/4$. Thus $r \in \{1, 3, 7, 57\}$. □

2. Set systems with restricted intersections

Theorem 2.1 (Non-uniform Fisher inequality, Majumdar 1953)**.** *Let $C_1, \dots, C_m$ be distinct subsets of $[n]$ such that for every $i \neq j$, $|C_i \cap C_j| = \lambda$ for some $1 \leq \lambda \leq n$. Then $m \leq n$.*

Proof. If suffices to assume that $|C_i| > \lambda$ for all i. Let M of size $m \times n$ be the incidence matrix of our system, then

$$MM^T = \lambda J + C,$$

where C is a diagonal matrix of positive entries. It is easy to check that MM^T is positive definite, and thus M must have full rank. □

Remark 2.2. *Fisher's original result (1940) was for $\lambda = 1$ together with uniformity assuming on the size of C_i's. This uniformity condition was then relaxed by Erdos and de Bruijn (1948), and generalized by Bose a year later by a linear algebraic method argument.*

It is perhaps useful to summarize the algebraic techniques we have used so far:

Proposition 2.3 (Matrices).

- *If M is positive-definite, then it has full rank.*
- *If A, B has size $a \times b$ and $b \times a$ respectively, and if* $\operatorname{rank}(AB) = a$*, then* $a \leq b$.
- *If A, B are matrices of the same size, then* $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.
- *Spectral decomposition for normal matrices (symmetric matrices).*

Proposition 2.4 (Criteria for linear independence). *Let $f_i : \Omega \to \mathbf{F}, 1 \leq i \leq m$ be functions. Then they are linearly independent over $\mathbf{F}$ if one of the following holds for some $a_i \in \Omega, 1 \leq i \leq m$.*

- *(diagonal) $f_i(a_j) = 0$ if $i \neq j$ and $\neq 0$ if $i = j$,*
- *(triangular) $f_i(a_j) = 0$ if $i < j$ and $\neq 0$ if $i = j$.*

Theorem 2.5 (Frankl-Wilson 1981). *Let L be a set of s integers and $\mathcal{F}$ an L-intersecting family of subsets of $[n]$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

Proof. Arrange the set size in increasing order $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. Let v_i be the indicator function of A_i, define $f_i : \mathbf{R}^n \to \mathbf{R}$ as follows:

$$f_i(x) = \prod_{|l_k| < |A_i|} (x \cdot v_i - l_k).$$

Notice that f_i has degree at most s, and $f_i(v_i) \neq 0$ while $f_i(v_j) = 0$ for $j \leq i$. Deform f_i to multilinear $\tilde{f}_i$ such that $f_i = \tilde{f}_i$ over all v_i. As $\tilde{f}_i$ are linearly independent by the triangular criterion, $m \leq \sum_{i \leq s} \binom{n}{s}$.

□

By the same method, one can obtain the following modulo version.

Theorem 2.6. *Let p be a prime number, and L be a set of s integers. Assume that $\mathcal{F} = \{A_1, \ldots, A_m\}$ such that*

- $|A_i| \notin L(\mod p)$;
- $|A_i \cap A_j| \in L(\mod p)$.

Then

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

As a corollary, one deduces that

Corollary 2.7. *Let p be a prime and $\mathcal{F}$ a $(2p-1)$-uniform family of subsets of a set of $4p-1$ elements. If no two of $\mathcal{F}$ intersect in precisely $p-1$ elements, then*

$$|\mathcal{F}| \leq 2\binom{4p-1}{p-1}$$

Proof. Let $L := \{0, \dots, p-2\}$ and use $\binom{n}{s} + \cdots + \binom{n}{0} \leq 2\binom{n}{s}$. □

Corollary 2.8. *Let L be a set of s integers and $\mathcal{F}$ an L-intersecting k-uniform family of subsets of a set of n elements, where $s \leq k$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

The following celebrated result of Ray-Chaudhuri and Wilson shows that the bound can be improved to $\binom{n}{s}$.

Theorem 2.9 (Ray-Chaudhuri -Wilson 1975)**.** *Let L be a set of s integers and $\mathcal{F}$ an L-intersecting k-uniform family of subsets of a set of n elements, where $s \leq k$. Then*

$$|\mathcal{F}| \leq \binom{n}{s}.$$

Proof. (of Theorem 2.9) One shows that the f_i (from the proof of Theorem 2.5) together with $x_I(x_1+\cdots+x_n-k), |I| \leq s-1$ are independent. Indeed, a vanishing linear combination $\sum_{i=1}^m \lambda_i f_i + \sum_{|I|\leq s-1} \mu_I x_I(x_1+\cdots+x_n-k)$ would imply $\lambda_i = 0$ (by replacing v_i into the indentity). For $\sum_{|I|\leq s-1} \mu_I x_I(x_1+\cdots+x_n-k)$, one has $x_I(x_1+\cdots+x_n-k)(v_J) = 0$ if $|J| \leq |I|, J \neq I$, and $\neq 0$ if $J = I$. □

We now present an application of Corollary 2.7.

Theorem 2.10. *[Chromatic number of unit distance graph] For large n, the chromatic number of the unit distance graph on $\mathbf{R}^n$ is greater than 1.13^n.*

Notice that the upper bound is at most $n^{n/2}$.

Proof. (of Theorem 2.10) Without loss of generality, assume that $n = 4p-1$ and define a graph G on $\binom{[n]}{2p-1}$ by connecting A to B if $|A \cap B| = p-1$. Notice that $d(A,B) = \sqrt{2p}$ in this case. As any independent set of G has size at most $2\binom{4p-1}{p-1}$ according to Corollary 2.7. The chromatic number of G is at least $\binom{4p-1}{2p-1}/\binom{4p-1}{p-1} \geq 1.13^{4p-1}$. □

Theorem 2.11 (Kahn-Kalai's disproving of Borsuk's conjecture)**.** *Let $f(n)$ denote the minimum number such that every set of diameter 1 in $\mathbf{R}^d$ can be partitioned into $f(d)$ pieces of smaller diameter. Then $f(d) \geq 1.2^{\sqrt{d}}$.*

Proof. For each $A \in V(G)$ from the proof of Theorem 2.10, we define $\Phi(A) \subset 2^X$ with $X = \binom{[n]}{2}$ as follows:

$$\phi(A) = \{\{x, y\} : x \in A, y \in \bar{A}\}.$$

One checks that

(1) $\mathcal{F} = \{\phi(A) : A \in V(G)\}$ is $k(n-k)$ uniform, with $k = 2p - 1$.
(2) Assume that $|A \cap B| = r$, then $|\phi(A) \cap \phi(B)| = r(n - 2k + r) + (k - r)^2$, which is minimized when r is as close to $k - n/4 = p - 3/4$ as possible, i.e. when $r = p - 1$.

It follows from (2) that G and $\phi(G)$ is isomorphic, where $\phi(G)$ is the graph over $\phi(A), A \in V(G)$, and $\phi(A)$ is connected to $\phi(B)$ if their distance is the diameter $\mu(\mathcal{F})$ of $\mathcal{F}$.

By the proof of Theorem 2.10, with $d = \binom{n}{2}$,

$$\chi(\phi(G)) \geq 1.13^n = 1.13^{\sqrt{2d}}$$

□

References

[1] L. Babai and P. Frankl, Linear Algebra method in Combinatorics