Lecture 2-3: The Pigeon-Hole Principle

Theorem 0.1. If \( n \) objects are placed in \( k \) boxes, \( k < n \), then at least one box contains \( \lceil \frac{n}{k} \rceil \) objects or more.

In what follows we will deduce some corollaries of this simple principle.

Theorem 0.2 (Two equal degrees). In any graph, there are two vertices of equal degree.

Proof. For any graph on \( n \) vertices, the degrees are between 0 and \( n - 1 \). Therefore, the only way all degrees could be different is that there is exactly one vertex of each possible degree. In particular, there is a vertex \( v \) of degree 0 and a vertex \( w \) of degree \( n - 1 \). However, if there is an edge \((v, w)\), then \( v \) cannot have degree 0, and if there is no edge \((v, w)\) then \( w \) cannot have degree \( n - 1 \). This is a contradiction. \( \square \)

Let \([2n] = \{1, 2, \ldots, 2n\}\). Suppose you want to pick a subset \( S \subset [2n] \) so that no number in \( S \) divides another. How many numbers can you pick? Obviously, you can take \( S = \{n + 1, n + 2, \ldots, 2n\} \) and no number divides another. Can you pick more than \( n \) numbers? The answer is negative.

Theorem 0.3 (Subsets without divisors). For any subset \( S \subset [2n] \) of size \(|S| > n\), there are two numbers \( a, b \in S \) such that \( a | b \).

Proof. For each odd number \( a \in [2n] \), let \( D_a = \{2^k a : k \geq 0, 2^k a \leq 2n\} \). The number of these classes is \( n \) and every element \( b \in [2n] \) belongs to exactly one of them. Consider \( S \subset [2n] \) of size \(|S| > n\). By the pigeonhole principle, there is a class \( D_a \) that contains at least two elements of \( S \), say \( n_1 \) and \( n_2 \) with \( n_1 < n_2 \). Then clearly \( n_1 | n_2 \). \( \square \)

Theorem 0.4 (Dirichlet approximation, 1879). For any \( x \in \mathbb{R} \) and \( n \) positive integer, there is a rational number \( p/q, 1 \leq q \leq n \), such that

\[ |x - \frac{p}{q}| < \frac{1}{nq}. \]

Proof. Let \( \{x\} \) denote the fractional part of \( x \). Consider \( \{ax\}, a = 1, 2, \ldots, n + 1 \) and place these \( n + 1 \) numbers into \( n \) buckets \( [0, 1/n), [1/n, 2/n), \ldots, [(n-1)/n, 1) \). By the pigeonhole principle, there must be a bucket containing at least two numbers \( \{ax\} \geq \{a'x\} \). We set \( q = a - a' \) and we get

\[ \{qx\} = \{ax\} - \{a'x\} < 1/n. \]

Hence \( qx = p + \delta \) where \( p \) is an integer and \( \delta = \{qx\} < 1/n \). By definition, \( q = a - a' \leq n \), and
Finally, we give an application which is less immediate. Given an arbitrary (ordered) sequence of distinct real numbers, what is the largest monotone subsequence that we can always find? It is easy to construct sequences of \( mn \) numbers such that any increasing subsequence has length at most \( m \) and any decreasing subsequence has length at most \( n \). For instance the following ordered sequence

\[
mn - m + 1, \ldots, mn, mn - 2m + 1, \ldots, mn - m, mn - 3m + 1, \ldots, mn - 2m, \ldots
\]

\[
\ldots, mn - (n - 1)m + 1, \ldots, mn - (n - 1)m + m, 1, \ldots, m.
\]

We show that this is an extremal example.

**Theorem 0.5** (Monotone subsequences). For any sequence of \( mn + 1 \) distinct real numbers \( a_0, a_1, \ldots, a_{mn} \), there is an increasing subsequence of length \( m + 1 \) or a decreasing subsequence of length \( n + 1 \).

**Proof.** Let \( t_i \) denote the maximum length of an increasing subsequence starting with \( a_i \). If \( t_i > m \) for some \( i \), we are done. So assume \( t_i \in \{1, 2, \ldots, m\} \) for all \( i \); i.e. we have \( mn + 1 \) numbers in \( m \) buckets. By the pigeonhole principle, there must be a value \( s \in \{1, 2, \ldots, m\} \) such that \( t_i = s \) for at least \( n + 1 \) indices, \( i_0 < i_1 < \cdots < i_n \).

Now we claim that \( a_{i_0} > a_{i_1} > \cdots > a_{i_n} \), and hence done. Indeed, if there were a pair such that \( a_{i_j} < a_{i_{j+1}} \), we could extend the increasing subsequence starting at \( a_{i_{j+1}} \) by adding \( a_{i_j} \), to get an increasing subsequence of length \( s + 1 \). However, this contradicts \( t_{i_j} = s \). \( \square \)