

LECTURE 4-5: DOUBLE COUNTING

The main part of this lecture is from [1, Chapter 7]. An elementary but extremely powerful trick is double counting. This is a discrete version of the Fubini's theorem in Analysis. As the name suggests, the trick involves counting a certain quantity in two different ways and comparing the results.

$$\sum_{i=1}^n \sum_{j=1}^m a_{ij} = \sum_{j=1}^m \sum_{i=1}^n a_{ij}.$$

In this lecture we will present three (related) applications of this simple fact.

0.1. **warming-up.** The following observation is due to Leonard Euler (1736).

Theorem 0.2 (The handshake lemma). *For any graph G , the sum of degrees over all vertices is even.*

Proof. For a vertex v and edge e , let $i(v, e) = 1$ if $v \in e$ and 0 otherwise. We count all the incidences between vertices and edges in two ways:

- $\sum_{v \in V, e \in E} i(v, e) = \sum_{e \in E, v \in V} i(v, e) = \sum_{v \in V} d(v)$ because $d(v)$ is exactly the number of edges incident with v .
- $\sum_{v \in V, e \in E} i(v, e) = \sum_{e \in E} \sum_{v \in V} i(v, e) = 2|E|$

because every edge is incident with exactly two vertices. Thus we have proved that the sum of all degrees is exactly twice the number of edges. $\square$

It follows from the Handshake Lemma that

Corollary 0.3. *If we know that a graph G has at least one vertex of odd degree, then it must have at least two such vertices.*

0.4. **Sperner's lemma in the plane.** Let us draw a big triangle in the plane with vertices A_1, A_2, A_3 . We divide it arbitrarily into a finite number of smaller triangles. No triangle may have a vertex inside an edge of any other small triangle, so that if we consider the resulting picture as a drawing of a planar graph, all inner faces are triangular. (See examples in lecture.)

Let us assign the labels 1, 2, 3 to the vertices of the big and the small triangles, under the following rules:

- The vertex A_i gets the label $i, i = 1, 2, 3$,
- All vertices lying on the edge A_iA_j of the big triangle may be assigned only the label i or j .
- Otherwise the assignment is completely arbitrary.

Theorem 0.5 (Planar Sperner's lemma). *In the situation described above, a small triangle always exists whose vertices are assigned all the three labels 1, 2, 3.*

Proof. We define an auxiliary graph G as follows: its vertices are the faces of our triangulation, i.e. all small triangles plus the outer face. The vertex for the outer face is denoted by v . Two vertices, i.e. two faces of the original drawing, are joined by an edge in G if they are neighboring faces and the endpoints of their common edge have **labels 1 and 2**. (This also concerns the outer face vertex v : it is connected to all small triangles adjacent to the circumference of the big triangle by a side labeled 12.) A small triangle can be connected to some of its neighbors in this graph G only if one of its vertices is labeled by 1 and another by 2. We observe the followings:

- If the remaining vertex is labeled 1 or 2, the considered small triangle is adjacent to exactly two of its neighbors.
- If the remaining vertex has label 3, then the considered triangle is adjacent to exactly one neighbor, and this is the only case where the degree of a small triangle in the graph G is odd.

We now show that the vertex v (the outer face) has an odd degree in G . Then, by the above corollary of the handshake lemma, there exists at least one other vertex of odd degree in G , and this is the desired small triangle labeled 1, 2, 3.

The edges of the graph G incident to v can obviously only cross the side A_1A_2 of the big triangle. By the rules of the labeling, this side only contains vertices labeled by 1 or 2. Let us write down the sequence of these labels, starting at A_1 and ending at A_2 . The number of neighbors of v is just the number of alterations between 1 and 2 in this sequence (the number of times a segment of 1s ends and a segment of 2s starts or the other way round). The key observation follows.

Lemma 0.6. *The number of such alterations must be odd.*

Proof. Because the sequence begins with 1 and ends with 2. □

As such, v has odd degree in G , the proof of Sperner's lemma is thus complete. □

0.7. Brouwer's theorem in the plane. Let's now give a nice application of Sperner's lemma as introduced in the first lecture. Let Δ denote a triangle in the plane. For simplicity, let us take the triangle with vertices $A_1 = (1, 0)$, $A_2 = (0, 1)$, and $A_3 = (0, 0)$. A function $f : \Delta \rightarrow \Delta$ is called continuous if:

for every $a \in \Delta$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $b \in \Delta$ is a point at distance at most δ from a , then the distance of $f(a)$ and $f(b)$ is at most ε .

Briefly, f maps close points to close points.

Theorem 0.8 (Planar Brouwer's fixed point theorem). *Every continuous function $f : \Delta \rightarrow \Delta$ has a fixed point.*

Proof. We define three auxiliary real-valued functions $\beta_1, \beta_2, \beta_3$ on the triangle Δ . For a point $a \in \Delta$ with coordinates (x, y) , we set

$$\beta_1(a) = x, \beta_2(a) = y, \beta_3(a) = 1 - x - y.$$

The key properties of these functions are:

$$\beta_i(a) \geq 0 \text{ and } \beta_1(a) + \beta_2(a) + \beta_3(a) = 1, \forall a \in \Delta.$$

We define sets $M_1, M_2, M_3 \subset \Delta$ as follows:

$$M_i = \{a \in \Delta : \beta_i(a) \geq \beta_i(f(a))\}, i = 1, 2, 3.$$

Thus, M_i consists of the points that are not moved farther apart from the side opposite to A_i by the function f .

Lemma 0.9. *Assume that there exists $p \in M_1 \cap M_2 \cap M_3$, then this is a fixed point of the function f .*

Proof. Because we have $\beta_i(p) \geq \beta_i(f(p))$ for all $i = 1, 2, 3$, and since $\sum_i \beta_i(p) = \sum_i \beta_i(f(p)) = 1$ we get $\beta_i(p) = \beta_i(f(p))$ for all i , which implies

$$p = f(p).$$

□

Our goal now is to find a point in the intersection $M_1 \cap M_2 \cap M_3$. Consider a sequence of successively refining triangulations of the triangle Δ by the midpoints. (See picture in lecture.)

In each of these triangulations, we label all vertices of the triangles by 1, 2, or 3. We require that a vertex labeled i belongs to the set M_i , and, moreover, that the assignment satisfies the **rules of Sperner's lemma**. We have to make sure this can always be arranged.

- The vertex A_1 has the largest possible distance from its opposite side; hence this distance cannot increase under f . Therefore $A_1 \in M_1$ and we can label A_1 by 1; similarly for A_2 and A_3 .

- A point a lying on the side A_1A_2 has $\beta_1(a) + \beta_2(a) = 1$, which implies that $f(a)$ cannot satisfy both $\beta_1(f(a)) > \beta_1(a)$ and $\beta_2(f(a)) > \beta_2(a)$. Thus $a \in M_1 \cup M_2$, and so we can label all vertices on the side A_1A_2 only by 1 and 2. A similar argument works for the other sides.
- Finally, each point of Δ belongs to at least one of the sets M_i since it cannot be moved farther from all three sides at once.

Now the planar Sperner's theorem implies that each of the successively refining triangulations has a triangle labeled 1, 2, 3. Let us denote the vertices of some such triangle in the j -th triangulation by $a_{j,1}, a_{j,2}$, and $a_{j,3}$ in such a way that $a_{j,i}$ belongs to M_i for $i = 1, 2, 3$.

Consider the infinite sequence of points $(a_{1,1}, a_{2,1}, a_{3,1}, \dots)$. By compactness (as the triangle is bounded and closed in $\mathbf{R}^2$), we can choose an infinite convergent subsequence from it. So suppose that we have chosen a convergent subsequence $(a_{j_1,1}, a_{j_2,1}, a_{j_3,1}, \dots, j_1 < j_2 < j_3 < \dots)$, and let us denote its limit point by p .

Claim 0.10. *We have $p \in M_1$.*

Proof. Indeed, by the definition of M_1 , we have $\beta_1(a_{j_k,1}) \geq \beta_1(f(a_{j_k,1}))$ for all j_k , and taking a limit on both sides yields $\beta_1(p) \geq \beta_1(f(p))$, because taking a limit preserves non-strict inequalities between continuous functions. $\square$

Since the diameter of triangles in the successive triangulations tends to 0, the sequences of the other vertices, i.e. $(a_{j_1,2}, a_{j_2,2}, a_{j_3,2}, \dots)$ and $(a_{j_1,3}, a_{j_2,3}, a_{j_3,3}, \dots)$, also converge to the point p . This implies that $p \in M_2$ and $p \in M_3$ as well by the same argument as in the claim above. Thus p is the desired fixed point of the function f . $\square$

REFERENCES

- [1] J. Matousek and J. Nešetřil, Invitation to Discrete Mathematics, second edition.