

INCLUSION-EXCLUSION PRINCIPLE

Several parts of this section are drawn from [1] and [2, 3.7].

1. PRINCIPLE OF INCLUSION AND EXCLUSION

Suppose that you have two sets A, B . The size of the union is certainly at most $|A| + |B|$. This way, however, we are counting twice all elements in $A \cap B$, the intersection of the two sets. To correct for this, we subtract $|A \cap B|$ to obtain the following formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In general, the formula gets more complicated because we have to take into account intersections of multiple sets. The following formula is what we call the principle of inclusion and exclusion.

Lemma 1.1. *For any collection of finite sets A_1, \dots, A_n we have*

$$|\cup_{1 \leq i \leq n} A_i| = \sum_{k=1}^n (-1)^{k-1} \sum_{I \subset \binom{[n]}{k}} |\cap_{i \in I} A_i|.$$

In other words, we add up the sizes of the sets, subtract intersections of pairs, add intersection of triples, etc. There are many ways to prove this identity, such as by induction on the number of sets (i.e. on n), or by counting (see [2, 3.7]). We present here a proof using the notion of a characteristic function.

Proof. Assume that $A_1, \dots, A_n \subset X$. For each set A_i , define the characteristic function $f_i(x)$ where $f_i(x) = 1$ if $x \in A_i$ and $f_i(x) = 0$ otherwise. We consider the following formula:

$$F(x) = \prod_{1 \leq i \leq n} (1 - f_i(x)).$$

Observe that this is the characteristic function of the complement of $\cup_{1 \leq i \leq n} A_i$: it is 1 if and only if x is not in any of the sets A_i . Hence,

$$\sum_{x \in X} F(x) = |X \setminus \cup_{1 \leq i \leq n} A_i|.$$

Now we write $F(x)$ differently, by expanding the product into 2^n terms:

$$F(x) = \prod_{1 \leq i \leq n} (1 - f_i(x)) = \sum_{I \subset [n]} (-1)^{|I|} \prod_{i \in I} f_i(x).$$

Observe that $\prod_{i \in I} f_i(x)$ is the characteristic function of $\cap_{i \in I} A_i$. Therefore, we get

$$\sum_{x \in X} F(x) = \sum_{I \subset [n]} (-1)^{|I|} \sum_{x \in X} \prod_{i \in I} f_i(x) = \sum_{I \subset [n]} (-1)^{|I|} |\cap_{i \in I} A_i|.$$

By comparing the estimates, we see that

$$|X \setminus \cup_{1 \leq i \leq n} A_i| = |X| - |\cup_{1 \leq i \leq n} A_i| = \sum_{I \subset [n]} (-1)^{|I|} |\cap_{i \in I} A_i|.$$

The first term in the sum here is $|X|$ by convention. Therefore, the lemma follows. \square

2. APPLICATION: THE EULER FUNCTION $\varphi(n)$

For a natural number n , the value of $\varphi(n)$ is defined as the number of natural numbers m from 1 to n that are relatively prime to n ; formally

$$\varphi(n) = |\{m \in \{1, 2, \dots, n\} : \gcd(n, m) = 1\}|.$$

Thus for instance if p is a prime, then $\varphi(p) = p - 1$, or more generally, if $n = p^k$, then $\varphi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$. The following result gives an exact formula for any n .

Theorem 2.1. *For any n of the form $p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ in its prime factorization, we have*

$$\varphi(n) = n(1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_r}).$$

Proof. Let us denote by $A_i = \{m \in \{1, 2, \dots, n\} : p_i | m\}$ the set of all multiples of p_i from 1 to n . We have

$$\varphi(n) = n - |A_1 \cup A_2 \cap \dots \cup A_r|.$$

The inclusion-exclusion principle commands that we find the sizes of the intersections of the sets A_i . For example, the intersection $A_1 \cap A_2$ contains the numbers divisible by both p_1 and p_2 , which are exactly the multiples of $p_1 p_2$, and hence $|A_1 \cap A_2| = n/(p_1 p_2)$. The same argument gives

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = n/(p_{i_1} p_{i_2} \dots p_{i_k}).$$

Thus, by the inclusion-exclusion principle,

$$\varphi(n) = n - \sum_{k=1}^r (-1)^{k-1} \sum_{I \subset \binom{[r]}{k}} |\cap_{i \in I} A_i| = n - \sum_{k=1}^r (-1)^{k-1} \sum_{I \subset \binom{[r]}{k}} n \prod_{i \in I} (1 - \frac{1}{p_i}) = n(1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_r}).$$

□

3. THE NUMBER OF DERANGEMENTS- THE HATCHECK PROBLEM

Consider the following problem. A sequence of n theatergoers want to pick up their hats on the way out. However, the deranged attendant does not know which hat belongs to whom and hands them out in a random order. What is the probability that nobody gets their own hat? More formally, we have a random permutation $\pi : [n] \rightarrow [n]$ and we are asking what is the probability that $\forall i; \pi(i) \neq i$. (Such permutations are called derangements.)

Theorem 3.1. *The probability that a random permutation $\pi : [n] \rightarrow [n]$ is a derangement is $\sum_{0 \leq k \leq n} (-1)^k / k!$, which tends to $1/e$ as $n \rightarrow \infty$.*

Proof. Let X be the set of all $n!$ permutations, and let A_i denote the set of permutations that fix element i , i.e.

$$A_i = \{\pi \in X : \pi(i) = i\}.$$

By simple counting, there are $(n - 1)!$ permutations in A_i , since by fixing i , we still have $n - 1$ elements to permute. Similarly, $\cap_{i \in I} A_i$ consists of the permutations where all elements of I are fixed, hence the number of such permutations is $(n - |I|)!$. By inclusion-exclusion, the number of permutations with some fixed point is

$$|\cup_{i \in [n]} A_i| = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} |\cup_{i \in I} A_i| = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (n - k)! = \sum_{k=1}^n (-1)^{k-1} n! / k!.$$

Hence, the probability that a random permutation has some fixed point is $\sum_{k=1}^n (-1)^{k-1} / k!$. By taking the complement, the probability that there is no fixed point is $1 - \sum_{k=1}^n (-1)^{k+1} / k! = \sum_{k=0}^n (-1)^k / k!$. In the limit, this tends to the Taylor expansion of e^{-1} . □

4. THE NUMBER OF SURJECTIONS

Next, consider the following situation. There are m teachers and n children, $m \geq n$. Each teacher gives one (random) child a cookie. What is the probability that all n children get at least one cookie?

Theorem 4.1. *The probability that all n children get cookies is $\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (1 - k/n)^m$.*

Proof. We formalize this problem as follows. A function $f : [m] \rightarrow [n]$ is called a surjection if it covers all elements of $[n]$. There are n^m functions total; we are interested in how many of these are surjections. We denote by A_i the set of functions that leave element i uncovered, i.e.

$$A_i = \{f : [m] \rightarrow [n] : \forall j, f(j) \neq i\}.$$

The number of such functions is $(n-1)^m$, since we have $n-1$ choices for each of $f(1), f(2), \dots, f(m)$. Similarly, because we have $|I|$ forbidden choices for each function value. By inclusion-exclusion, we get that the number of functions which are not surjections is

$$|\cup_{1 \leq i \leq m} A_i| = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} \binom{n}{|I|} (n - |I|)^m.$$

By taking the complement, the number of surjections is

$$n^m - |\cup_{1 \leq i \leq m} A_i| = \sum_{0 \leq k \leq n-1} (-1)^k \binom{n}{k} (n - k)^m.$$

Dividing by n^m , we get the desired probability. □

REFERENCES

- [1] J. Fox's note on Combinatorics at MIT
- [2] J. Matousek and J. Nešetřil, Invitation to Discrete Mathematics, Second edition.