COMBINATORIAL COUNTING: SPECIAL NUMBERS

Our main reference is [1, Section 13]. Two kinds of numbers that come up in many combinatorial prolems are the so-called Stirling numbers of the first and second kind.

1. Stirling number of the first kind

Let $c(n,k)$ denote the number of permutations $\pi \in S_n$ with exactly k cycles. (This number is called a signless Stirling number of the first kind.) Furthermore define $c(0,0) = 1$ and $c(n,0) = 0$. It is also clear that for all $n \geq 1$

$$c(n,n) = 1.$$

The Stirling numbers of the first kind $s(n,k)$ are defined by

$$s(n,k) := (-1)^{n-k} c(n,k).$$

Theorem 1.1. *The numbers $c(n,k)$ satisfy the recurrence relation*

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1).$$

Proof. If π is a permutation in S_{n-1} with k cycles, then there are $n-1$ positions where we can insert the integer n to produce a permutation $\pi' \in S_n$ with k cycles. We can also adjoin (n) as a cycle to any permutation in S_{n-1} with $k-1$ cycles. This accounts for the two terms on the right-hand side of the formula. $\square$

Theorem 1.2. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\dots(x+n-1) := (x)^n$$

and

$$\sum_{k=0}^{n} s(n,k)x^k = x(x-1)\dots(x-n+1) := (x)_n.$$

$(x)^n$ and $(x)_n$ are usually called *rising* and *falling* factorials respectively.

Proof. We establish the first identity. Write

$$F_n(x) = x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} b(n,k)x^k.$$

We define by convention that $b(0,0) = 1$.

Fact 1.3. *For $n \geq 1$, we have*

$$b(n,0) = 0 \text{ and } b(n,n) = 1.$$

Proof. For the first identity, we just note that $F_n(x)$ does not have free term (i.e. the coefficient of x^0 is 0). For the second identity, clearly the coefficient of x^n in the expansion of $F_n(x)$ is 1. □

By this fact, we have learned that

$$b(0,0) = c(0,0); b(n,0) = c(n,0) \text{ and } b(n,n) = c(n,n).$$

Next, since

$$F_n(x) = (x+n-1)F_{n-1}(x)$$
$$= \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k,$$

we see that the numbers $b(n,k)$ satisfy the same recurrence relation as the $c(n,k)$, namely

$$b(n,k) = (n-1)b(n-1,k) + b(n-1,k-1).$$

Since the numbers are equal if $k = 0$ or $n = k$, they are equal for all $n \geq k \geq 0$.

For the second identity of the theorem, we just need to replace x by $-x$ into the first identity, and then use the definition of $s(n,k)$. □

We next deduce some nice consequences.

Corollary 1.4. *The number of permutations that have an even number of cycles is equal to the number of permutations that have an odd number of cycles.*

Proof. Plug $x = -1$ into the first equation of Theorem 1.2. □

Corollary 1.5. *The total number of cycles in all permutations in S_n is equal to*

$$n!(1 + \frac{1}{2} + \dots + \frac{1}{n}).$$

Proof. Recall that

$$F_n(x) = x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} c(n,k)x^k.$$

Taking derivative,

$$F_n'(x) = \sum_{k=1}^{n} kc(n,k)x^{k-1}.$$

Thus, when $x = 1$, $F_n'(1) = \sum_{k=1}^{n} kc(n,k)$. This is the total number of cycles in all permutations.

On the other hand, as $F_n(x) = x(x+1)\dots(x+n-1)$, its derivative is

$$F_n'(x) = [x(x+1)\dots(x+n-1)]' = x(x+1)\dots(x+n-1)[\frac{1}{x} + \dots + \frac{1}{x+n-1}].$$

So

$$F_n'(1) = n!(1 + \frac{1}{2} + \dots + \frac{1}{n}),$$

completing the proof.

□

2. Stirling number of the second kind

We now define the Stirling numbers of the second kind: denote by $P(n,k)$ the set of all partitions of an n-set into k nonempty subsets (blocks). Then

$$S(n,k) := |P(n,k)|.$$

Define by convention that $S(0,0) = 1$. Similarly to $c(n,k)$, we have the following recurrence relation.

Theorem 2.1. *The Stirling numbers of the second kind satisfy the relation*

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

Proof. The proof is nearly the same as for the first section. A partition of the set $\{1, 2, \ldots, n-1\}$ can be made into a partition of $\{1, 2, \ldots, n\}$ by adjoining n to one of the blocks or by increasing the number of blocks by one by making $\{n\}$ a block. □

We wonder if there is a " formula " for $S(n, k)$.

Theorem 2.2. *We have*

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

Proof. We remark that the number of surjective mappings from an n-set to a k-set is $k!S(n, k)$ (a block of the partition is the inverse image of an element of the k-set). Thus, by our last application in the lecture on the Inclusion-Exclusion principle (counting the number of surjections map from $[n]$ to $[k]$),

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

□

Now we state an analog of Theorem 1.2 for $S(n, k)$.

Theorem 2.3. *For $n \geq 0$ we have*

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k.$$

Proof. Now let x be an integer. There are x^n mappings from the n-set $N = [n]$ to the x-set $[x]$. For any k-subset Y of $[x]$, there are $k!S(n, k)$ surjections from N to Y. So we find

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S(n,k) = \sum_{k=0}^{n} S(n,k)(x)_k.$$

□

Finally, using the recurrence formula for $S(n, k)$, one can calculate its exponential generating function.

Theorem 2.4.

$$\sum_{n\geq k} S(n,k)x^n/n! = \frac{1}{k!}(e^x - 1)^k (k \geq 0)$$

Proof. Let $F_k(x)$ denote the sum on the left-hand side. By the recurrence formula, we have

$$F_k'(x) = kF_k(x) + F_{k-1}(x).$$

The result now follows by induction. Since $S(n,1) = 1$, the assertion is true for $k = 1$ and the induction hypothesis yields a differential equation for F_k, which with the condition $S(k,k) = 1$ has the right-hand side of the assertion as unique solution. $\square$

This point will be discussed in more detail later (on generating functions of various sequences.)

3. Conclusion: Relation between Stirling numbers

we conclude the note by showing the following inversion-type relation between $S(n,k)$ and $s(n,k)$.

Theorem 3.1. *We have*

$$\sum_{k=m}^{n} S(n,k)s(k,m) = \delta_{mn}.$$

Proof. This follows from the second equation of Theorem 1.2 and from Theorem 2.3, noting that $\{1, x, \ldots, x^n\}$ and $\{(x)_0, (x)_1, \ldots, (x)_n\}$ can be seen as two bases of the finite vector space (over R) of polynomials in x with degree at most n. $\square$

References

[1] J.H. van Lint and R. M. Wilson, A Course in Combinatorics.
[2] J. Matousek and J. Nesetril, Invitation to Discrete Mathematics, Second edition.