Our main reference is [1, Section 13]. Two kinds of numbers that come up in many combinatorial problems are the so-called Stirling numbers of the first and second kind.

1. Stirling number of the first kind

Let \( c(n, k) \) denote the number of permutations \( \pi \in S_n \) with exactly \( k \) cycles. (This number is called a signless Stirling number of the first kind.) Furthermore define \( c(0, 0) = 1 \) and \( c(n, 0) = 0 \). It is also clear that for all \( n \geq 1 \)

\[
c(n, n) = 1.
\]

The Stirling numbers of the first kind \( s(n, k) \) are defined by

\[
s(n, k) := (-1)^{n-k}c(n, k).
\]

**Theorem 1.1.** The numbers \( c(n, k) \) satisfy the recurrence relation

\[
c(n, k) = (n - 1)c(n - 1, k) + c(n - 1, k - 1).
\]

**Proof.** If \( \pi \) is a permutation in \( S_{n-1} \) with \( k \) cycles, then there are \( n - 1 \) positions where we can insert the integer \( n \) to produce a permutation \( \pi' \in S_n \) with \( k \) cycles. We can also adjoin \( (n) \) as a cycle to any permutation in \( S_{n-1} \) with \( k - 1 \) cycles. This accounts for the two terms on the right-hand side of the formula. \( \square \)

**Theorem 1.2.** For \( n \geq 0 \) we have

\[
\sum_{k=0}^{n} c(n, k)x^k = x(x + 1) \ldots (x + n - 1) := (x)^n
\]

and

\[
\sum_{k=0}^{n} s(n, k)x^k = x(x - 1) \ldots (x - n + 1) := (x)_n.
\]

\( (x)^n \) and \( (x)_n \) are usually called rising and falling factorials respectively.
Proof. We establish the first identity. Write

\[ F_n(x) = x(x+1) \ldots (x+n-1) = \sum_{k=0}^{n} b(n,k)x^k. \]

We define by convention that \( b(0,0) = 1 \).

**Fact 1.3.** For \( n \geq 1 \), we have

\[ b(n,0) = 0 \quad \text{and} \quad b(n,n) = 1. \]

Proof. For the first identity, we just note that \( F_n(x) \) does not have free term (i.e. the coefficient of \( x^0 \) is 0). For the second identity, clearly the coefficient of \( x^n \) in the expansion of \( F_n(x) \) is 1. \( \square \)

By this fact, we have learned that

\[ b(0,0) = c(0,0); \quad b(n,0) = c(n,0) \quad \text{and} \quad b(n,n) = c(n,n). \]

Next, since

\[ F_n(x) = (x+n-1)F_{n-1}(x) \]
\[ = \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k, \]
we see that the numbers \( b(n,k) \) satisfy the same recurrence relation as the \( c(n,k) \), namely

\[ b(n,k) = (n-1)b(n-1,k) + b(n-1,k-1). \]

Since the numbers are equal if \( k = 0 \) or \( n = k \), they are equal for all \( n \geq k \geq 0 \).

For the second identity of the theorem, we just need to replace \( x \) by \( -x \) into the first identity, and then use the definition of \( s(n,k) \). \( \square \)

We next deduce some nice consequences.

**Corollary 1.4.** The number of permutations that have an even number of cycles is equal to the number of permutations that have an odd number of cycles.

Proof. Plug \( x = -1 \) into the first equation of Theorem 1.2. \( \square \)

**Corollary 1.5.** The total number of cycles in all permutations in \( S_n \) is equal to

\[ n!(1 + \frac{1}{2} + \cdots + \frac{1}{n}). \]
Proof. Recall that

\[ F_n(x) = x(x + 1) \ldots (x + n - 1) = \sum_{k=0}^{n} c(n, k)x^k. \]

Taking derivative,

\[ F'_n(x) = \sum_{k=1}^{n} kc(n, k)x^{k-1}. \]

Thus, when \( x = 1 \), \( F'_n(1) = \sum_{k=1}^{n} kc(n, k) \). This is the total number of cycles in all permutations.

On the other hand, as \( F_n(x) = x(x + 1) \ldots (x + n - 1) \), its derivative is

\[ F'_n(x) = [x(x + 1) \ldots (x + n - 1)]' = x(x + 1) \ldots (x + n - 1)[\frac{1}{x + \ldots + 1}]. \]

So

\[ F'_n(1) = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n}), \]

completing the proof.

\[ \square \]

2. Stirling number of the second kind

We now define the Stirling numbers of the second kind: denote by \( P(n, k) \) the set of all partitions of an \( n \)-set into \( k \) nonempty subsets (blocks). Then

\[ S(n, k) := |P(n, k)|. \]

Define by convention that \( S(0, 0) = 1 \). Similarly to \( c(n, k) \), we have the following recurrence relation.

**Theorem 2.1.** The Stirling numbers of the second kind satisfy the relation

\[ S(n, k) = kS(n - 1, k) + S(n - 1, k - 1). \]
Proof. The proof is nearly the same as for the first section. A partition of the set \{1, 2, \ldots, n-1\} can be made into a partition of \{1, 2, \ldots, n\} by adjoining \(n\) to one of the blocks or by increasing the number of blocks by one by making \(\{n\}\) a block. \(\square\)

We wonder if there is a ” formula ” for \(S(n, k)\).

**Theorem 2.2.** We have

\[ S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} j^n. \]

**Proof.** We remark that the number of surjective mappings from an \(n\)-set to a \(k\)-set is \(k!S(n, k)\) (a block of the partition is the inverse image of an element of the \(k\)-set). Thus, by our last application in the lecture on the Inclusion-Exclusion principle (counting the number of surjections map from \([n]\) to \([k]\)),

\[ S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} j^n. \]

\(\square\)

Now we state an analog of Theorem 1.2 for \(S(n, k)\).

**Theorem 2.3.** For \(n \geq 0\) we have

\[ x^n = \sum_{k=0}^{n} S(n, k)(x)_k. \]

**Proof.** Now let \(x\) be an integer. There are \(x^n\) mappings from the \(n\)-set \(N = [n]\) to the \(x\)-set \([x]\). For any \(k\)-subset \(Y\) of \([x]\), there are \(k!S(n, k)\) surjections from \(N\) to \(Y\). So we find

\[ x^n = \sum_{k=0}^{n} \binom{x}{k} k!S(n, k) = \sum_{k=0}^{n} S(n, k)(x)_k. \]

\(\square\)

Finally, using the recurrence formula for \(S(n, k)\), one can calculate its exponential generating function.

**Theorem 2.4.**

\[ \sum_{n \geq k} S(n, k)x^n/n! = \frac{1}{k!}(e^x - 1)^k (k \geq 0) \]

**Proof.** Let \(F_k(x)\) denote the sum on the left-hand side. By the recurrence formula, we have

\[ F'_k(x) = kF_k(x) + F_{k-1}(x). \]
The result now follows by induction. Since \( S(n,1) = 1 \), the assertion is true for \( k = 1 \) and the induction hypothesis yields a differential equation for \( F_k \), which with the condition \( S(k,k) = 1 \) has the right-hand side of the assertion as unique solution.

This point will be discussed in more detail later (on generating functions of various sequences.)

3. Conclusion: Relation between Stirling numbers

we conclude the note by showing the following inversion-type relation between \( S(n,k) \) and \( s(n,k) \).

**Theorem 3.1.** We have

\[
\sum_{k=m}^{n} S(n,k) s(k,m) = \delta_{mn}.
\]

**Proof.** This follows from the second equation of Theorem 1.2 and from Theorem 2.3, noting that \( \{1,x,\ldots,x^n\} \) and \( \{(x)_0,(x)_1,\ldots,(x)_n\} \) can be seen as two bases of the finite vector space (over \( R \)) of polynomials in \( x \) with degree at most \( n \). □

**References**