INTRODUCTION TO GRAPH THEORY, CONT'D

This lecture is based on [1, Chapter 4]

1. Connected graphs, a simple theorem

Definition 1.1 (2-connectivity)**.** A graph G is called 2-connected if it has at least 3 vertices, and by deleting any single vertex we obtain a connected graph.

Theorem 1.2. *A graph G is 2-connected if and only if there exists, for any two vertices of G, a cycle in G containing these two vertices.*

Proof. The given condition is sufficient since if two vertices v, v' lie on a common cycle then there exist two paths connecting them having no common vertices except for the end vertices, and so v and v' can never fall into distinct components by removing a single vertex.

We now prove the reverse implication. The existence of a common cycle for v, v' will be established by induction on $d_G(v, v')$, the distance of the vertices v and v'. First let $d_G(v, v') = 1$. By 2-connectivity of G, the graph $G - e$ is connected. Therefore there exists a path from v to v' in the graph $G - e$, and this path together with the edge e forms the required cycle containing both v and v'.

Next, suppose that any pair of vertices at distance less than k lies on a common cycle, for some $k \geq 2$. Consider two vertices $v, v' \in V$ at distance k. Let $P = (v = v_0, e_1, v_1, \ldots, e_k, v_k = v')$ be a shortest path from v to v'. Since $d_G(v, v_{k-1}) = k - 1$, a cycle exists containing both v and v_{k-1}. This cycle consists of two paths, P_1 and P_2, from v to v_{k-1}. Now consider the graph $G - v_{k-1}$. It is connected, and hence it has a path P' from v to v'. This path thus doesnt contain v_{k-1}. Let us look at the last vertex on the path P' (when going from v to v') belonging to one of the paths P_1, P_2, and denote this vertex by w.

Without loss of generality, suppose that w is a vertex of P_1. Then the desired cycle containing v and v' is formed by the path P_2, by the portion of the path P_1 between v and w, and by the portion of the path P' between w and v'. □

2. Further characterization of 2-connected graphs

Definition 2.1 (Some graph operations)**.** Let $G = (V, E)$ be a graph. We define various new graphs created from G:

- (Edge deletion)

$$G - e = (V, E\ \{e\}),$$

where $e \in E$ is an edge of G;

- (Adding a new edge)
$$G + e = (V, E \cup \{e\}),$$
where e is a pair of vertices that is not an edge of G;
- (Vertex deletion)
$$G - v = (V \setminus \{v\}, \{e \in E : v \notin e\}),$$
where $v \in V$ (we delete the vertex v and all edges having v as an endpoint);
- (Edge subdivision)
$$G\%e = (V \cup \{z\}, ((E \setminus \{\{x, y\}\}) \cup \{\{x, z\}, \{z, y\}\})),$$
where $e = \{x, y\} \in E$ is an edge, and $z \notin V$ is a new vertex (we draw a new vertex z on the edge $\{x, y\}$).

We say that a graph G' is a subdivision of the graph G if G' is isomorphic to a graph created from G by successive operations of edge subdivision.

Remark 2.2. *A graph G is 2-connected if and only if any subdivision of G is 2-connected.*

Proof. It is enough to show that, for any edge $e \in E(G)$, G is 2-connected if and only if $G\%e$ is 2-connected. If $v \in V(G)$ is a vertex of G, it is easy to see that $G - v$ is connected if and only if $(G\%e) - v$ is connected. Therefore, if $G\%e$ is 2-connected then so is G.

For the reverse implication, it remains to show that for a 2-connected G, the graph $(G\%e) - z$ is connected, where z is the newly added vertex. This follows from the fact that $G - e$ is connected for a 2-connected G. □

We now state our main result.

Theorem 2.3 (Characterization of 2-connected graphs)**.** *A graph G is 2-connected if and only if it can be created from a triangle (i.e. from K_3) by a sequence of edge subdivisions and edge additions.*

Proof. Every graph that can be produced from K_3 by the above mentioned operations is obviously 2-connected. So, we need to prove that we can construct each 2-connected graph.

Actually, we show the possibility of creating any 2-connected graph by a somewhat different construction. We start with a cycle G_0, and if a graph G_{i-1} has already been built, a graph G_i arises by adding a path P_i connecting two vertices of the graph G_{i-1}. The path P_i only shares its end vertices with G_{i-1}, while all edges and all inner vertices are new.

Since adding a path can be simulated by an edge addition and edge subdivisions, it suffices to show that every 2-connected graph G can be produced by a repeated ear addition.

Let us pick a cycle G_0 in the graph G arbitrarily. Suppose that graphs $G_j = (E_j, V_j)$ for $j \leq i$ have already been defined, with properties as described above. If $G_i = G$ the proof is over, so let us assume that $E_i \neq E(G)$. Since G is connected, there exists an edge $e \in E(G) \setminus E_i$ such that $e \cap V_i \neq \emptyset$.

If both vertices of e lie in V_i then we put $G_{i+1} = G_i + e$. In the other case, let $e = \{v, v?\}$, where $v \in V_i, v' \notin V_i$. Consider the graph $G - v$. This is connected (since G is 2-connected), and therefore a path P exists connecting the vertex v' to some vertex $v'' \in V_i$, where v'' is the only vertex of the path P belonging to V_i. To this end, take the shortest path joining v' to V_i in the graph $G - v$. Then we can define the graph G_{i+1} by adding the edge e and the path P to G_i, i.e. $V_{i+1} = V_i \cup V(P), E_{i+1} = E_i \cup \{e\} \cup E(P)$. □

3. A simple extremal problem

How many edges can a graph with n vertices have if we know that it doesnt contain a triangle? We are interested in the maximum possible number of edges of such a graph. Let us denote this number by $T(n)$.

Theorem 3.1. *For every natural number n we have*

$$T(n) = \lfloor \frac{n^2}{4} \rfloor.$$

Proof. First we establish the (considerably easier) inequality $T(n) \geq \lfloor n^2/4 \rfloor$. For this it suffices to find suitable triangle free graphs. For disjoint sets X and Y , let $K_{X,Y}$ denote the graph with vertex set $X \cup Y$ and edge set $\{\{x, y\} : x \in X, y \in Y\}$. The graph $K_{X,Y}$ is called a complete bipartite graph and if we set $a = |X|$ and $b = |Y|$. A complete bipartite graph contains no triangle, and the graph $K_{a,b}$ has ab edges. For our proof it suffices to find values a, b so that $a + b = n$ and $ab = \lfloor n^2/4 \rfloor$. We can easily check that $a = \lfloor n/2 \rfloor$ and $b = n - a$ will do.

Now we prove the harder inequality $T(n) \leq \lfloor n^2/4 \rfloor$. Since $T(n)$ is integral, it suffices to show $T(n) \leq n^2/4$. We will establish this statement by a somewhat unusual induction on n. We already know that the statement holds for $n \leq 4$. In the inductive step we prove the following implication:

$$T(n) \leq n^2/4 \to T(n + 2) \leq (n + 2)^2/4.$$

So let $G = (V, E)$ be any graph with $n + 2$ vertices containing no triangle. We aim at showing $|E| \leq (n + 2)^2/4$. Let us choose an edge $e_0 = \{x, y\} \in E$ arbitrarily, let us set $V' = V \backslash \{x, y\}$, and let $G' = (V', E')$ be the subgraph of G induced by the set V'. The graph G' is triangle-free, and so by the inductive hypothesis we have

$$|E'| \leq n^2/4.$$

Let E_x be the set of edges of G that are incident with the vertex x.We define the set E_y analogously for y. We thus have $E = E' \cup (E_x \cup E_y) \cup \{e_0\}$ and $|E| = |E'| + |E_x \cup E_y| + 1$. A key step in the proof is the inequality $|E_x \cup E_y| \leq n$, which follows from the fact that no edge of E_x has a common vertex with any edge of E_y.

Altogether we thus get

$$|E| \leq n^2/4 + n + 1 = (n+2)^2/4.$$

□

Next we introduce the following notion: we say that a graph $G = (V, E)$ with n vertices is extremal if it contains no triangle and has $\lfloor n^2/4 \rfloor$ edges. We already know that $K_{\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor}$ is extremal.

Theorem 3.2. *For every n each extremal graph is isomorphic to the graph $K_{a,b}$ with $a = \lfloor n/2 \rfloor$ and $b = n - \lfloor n/2 \rfloor$.*

Proof. We proceed as in the proof of Theorem 3.1. The statement again holds for $n = 1, 2, 3$, since we can easily consider all extremal graphs. In the inductive step we assume the uniqueness of the extremal graph with n vertices, and our goal is to show uniqueness for $n+2$ vertices. So let $G = (V, E)$ be a triangle-free graph with $n+2$ vertices and $\lfloor (n+2)^2/4 \rfloor$ edges. We choose an edge $e_0 = \{x, y\}$ arbitrarily and we consider the graph $G' = (V', E')$ with $V' = V \backslash \{x, y\}$ and $E' = E \cap \binom{V'}{2}$. Since $|E| = \lfloor n^2/4 \rfloor + n + 1$ and since $|E'| \leq n^2/4$ and $|E_x \cup E_y| \leq n$, we obtain that $|E'|$ must be even equal to $\lfloor n^2/4 \rfloor$. So G' is extremal and thus isomorphic to $K_{a,b}$, with $a = \lfloor n/2 \rfloor$ and $b = n - \lfloor n/2 \rfloor$.

Hence $G' = K_{X,Y}$ for a suitable partition of V' into two sets X and Y, $|X| = a, |Y| = b$. We note that x cannot be connected to both X and Y , and similarly for y. But since we also have $|E_x| + |E_y| = n$, we see that one of the following two possibilities has to occur: either x is connected to all vertices of X and y is connected to all vertices of Y , or $x$is connected to all of Y and y is connected to all of X. These two possibilities may look different but actually they yield isomorphic graphs. □

We conclude this section with a short presentation of another proof of Theorem 3.1.

Theorem 3.3. *Let $G = (V, E)$ be a triangle-free graph. Then there exists a partition of V into two subsets X and Y such that for every vertex $x \in V$ we have $d_G(x) \leq d_{K_{X,Y}}(x)$.*

The deduction of Theorem 3.1 is left as an exercise.

Proof of Theorem 3.3. Let $G = (V, E)$ be a triangle-free graph. We pick a vertex $x_0 \in V$ whose degree in G is maximum. We set $Y = \{y : \{x_0, y\} \in E\}$ and $X = V \backslash Y$. Clearly $x_0 \in X$, and for every $x \in X$ we have $d_{K_{X,Y}}(x) = |Y| = d_G(x_0) \geq d_G(x)$, and hence the inequality in the theorem holds for all $x \in X$. Next, we note that no two vertices of Y are adjacent, because all vertices of Y are neighbors of x_0 and G contains no triangles. So all neighbors of each $y \in Y$ lie in X and we have $d_G(y) \leq |X| \leq d_{K_{X,Y}}(y)$. □

References

[1] J. Matousek and J. Nesetril, Invitation to Discrete Mathematics, Second edition.