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1) Method of undetermined coefficients. Find a particular solutions of following

a. $y'' + 3y' - 18y = 2t^4 - t^2$

$$y_p = At^4 + Bt^3 + Ct^2 + Dt + E$$

$$y_p' = 4At^3 + 3Bt^2 + 2Ct + D$$

$$y_p'' = 12At^2 + 6Bt + 2C$$

$$2t^4 - t^2 = 12At^2 + 6Bt + 2C + 3(4At^3 + 3Bt^2 + 2Ct + D) - 18(At^4 + Bt^3 + Ct^2 + Dt + E)$$

$$= 12At^2 + 6Bt + 2C + 12At^3 + 9Bt^2 + 6Ct + 3D - 18At^4 - 18Bt^3 - 18Ct^2 - 18Dt - 18E$$

$$= [-18At^4] + [12At^3 - 18Bt^3] + [12At^2 + 9Bt^2 - 18Ct^2] + [6Bt + 6Ct - 18Dt] + [2C + 3D - 18E]$$

$$-18A = 2 \qquad A = \frac{-1}{9}$$

$$12A - 18B = 0$$

$$12A = 18B$$

$$B = \frac{12\left(\frac{-1}{9}\right)}{18} = \frac{-2}{27}$$

$$12A + 9B - 18C = -1$$

$$18C = 12A + 9B + 1$$

$$C = \frac{12\left(\frac{-1}{9}\right) + 9\left(\frac{-2}{27}\right) + 1}{18} = \frac{-1}{18}$$

$$6B + 6C - 18D = 0$$

$$18D = 6B + 6C$$

$$D = \frac{6\left(\frac{-2}{27}\right) + 6\left(\frac{-1}{18}\right)}{18} = \frac{-7}{162}$$

$$2C + 3D - 18E = 0$$

$$18E = 2C + 3D$$

$$E = \frac{2\left(\frac{-1}{18}\right) + 3\left(\frac{-7}{162}\right)}{18} = \frac{-13}{972}$$

$$\boxed{y_p = -\frac{1}{9}t^4 - \frac{2}{27}t^3 - \frac{1}{18}t^2 - \frac{7}{162}t - \frac{13}{972}}$$

1) b. $y'' + 5y = 6t^3 - t^2$

$y_p = At^3 + Bt^2 + Ct + D$

$y_p' = 3At^2 + 2Bt + C$

$y_p'' = 6At + 2B$

$$6t^3 - t^2 = 6At + 2B + 5(At^3 + Bt^2 + Ct + D)$$
$$= 6At + 2B + 5At^3 + 5Bt^2 + 5Ct + 5D$$
$$= [5At^3] + [5Bt^2] + [6At + 5Ct] + [2B + 5D]$$

$5A = 6$ $5B = -1$ $6A + 5C = 0$

$A = \frac{6}{5}$ $B = \frac{-1}{5}$ $5C = -6A$

$C = \frac{-6\left(\frac{6}{5}\right)}{5} = \frac{-36}{25}$

$2B + 5D = 0$

$5D = -2B$

$D = \frac{-2\left(-\frac{1}{5}\right)}{5} = \frac{2}{25}$

$$\boxed{y_p = \frac{6}{5}t^3 - \frac{1}{5}t^2 - \frac{36}{25}t + \frac{2}{25}}$$

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1) c. $y'' - 4y' - 32y = 6e^{-3t}$

$y_p = Ae^{-3t}$

$y_p' = -3Ae^{-3t}$

$y_p'' = 9Ae^{-3t}$

$$6e^{-3t} = 9Ae^{-3t} - 4(-3Ae^{-3t}) - 32(Ae^{-3t})$$
$$= 9Ae^{-3t} + 12Ae^{-3t} - 32Ae^{-3t}$$

$6 = 9A + 12A - 32A$

$A = \frac{-6}{11}$

$$\boxed{y_p = \frac{-6}{11}e^{-3t}}$$

[redacted]

1) a. $y'' + 2y' - 8y = e^{-t} - 2e^{t}$

$y_p = Ae^{-t} + Be^{t}$

$y_p' = -Ae^{-t} + Be^{t}$

$y_p'' = Ae^{-t} + Be^{t}$

$e^{-t} - 2e^{t} = Ae^{-t} + Be^{t} - 2Ae^{-t} + 2Be^{t} - 8Ae^{-t} - 8Be^{t}$

$= (A - 2A - 8A)e^{-t} + (B + 2B - 8B)e^{t}$

$-9A = 1 \qquad -5B = -2$

$A = \frac{-1}{9} \qquad B = \frac{2}{5}$

$$\boxed{y_p = -\frac{1}{9}e^{-t} + \frac{2}{5}e^{t}}$$

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1) e. $y'' - y' - 6y = \sin t + 3\cos t$

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$$\sin t + 3\cos t = -A\cos t - B\sin t - (-A\sin t + B\cos t) - 6(A\cos t + B\sin t)$$
$$= -A\cos t - B\sin t + A\sin t - B\cos t - 6A\cos t - 6B\sin t$$
$$= (A - 7B)\sin t + (-7A - B)\cos t$$

$A - 7B = 1$

$A = 1 + 7B$

$-7A - B = 3$

$-7(1+7B) - B = 3$

$-7 - 50B = 3$

$50B = -10$

$B = -\frac{1}{5}$

$A = 1 + 7(-\frac{1}{5})$

$= -\frac{2}{5}$

$$\boxed{y_p = -\frac{2}{5}\cos t - \frac{1}{5}\sin t}$$

1) f. $y'' - 3y' - 4y = 2\cos 2t - 3\sin 2t$

$$y_p = A\cos 2t + B\sin 2t$$
$$y_p' = -2A\sin 2t + 2B\cos 2t$$
$$y_p'' = -4A\cos 2t - 4B\sin 2t$$

$$2\cos 2t - 3\sin 2t = -4A\cos 2t - 4B\sin 2t - 3(-2A\sin 2t + 2B\cos 2t) - 4(A\cos 2t + B\sin 2t)$$
$$= -4A\cos 2t - 4B\sin 2t + 6A\sin 2t - 6B\cos 2t - 4A\cos 2t - 4B\sin 2t$$
$$(-8A - 6B)\cos 2t + (6A - 8B)\sin 2t$$

$$-8A - 6B = 2$$
$$A = \frac{2 + 6B}{-8}$$

$$6A - 8B = -3$$
$$6\left(\frac{2+6B}{-8}\right) - 8B = -3$$
$$-\frac{3}{2} - \frac{9}{2}B - 8B = -3$$
$$-\frac{25}{2}B = -\frac{3}{2}$$
$$B = \frac{3}{25}$$

$$A = \frac{2 + 6\left(\frac{3}{25}\right)}{-8}$$
$$= -\frac{17}{50}$$

$$\boxed{y_p = -\frac{17}{50}\cos 2t + \frac{3}{25}\sin 2t}$$

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2. Method of undetermined coefficients with combined functions

Find a particular solution of the following differential equations

a. $y'' - 4y = 2e^t - 1$

$y_p = Ae^t + B$

$y_p' = Ae^t$

$y_p'' = Ae^t$

$2e^t - 1 = Ae^t - 4(Ae^t + B)$

$= -3Ae^t - 4B$

$-3A = 2$ $\quad\quad$ $-4B = -1$

$A = -\frac{2}{3}$ $\quad\quad$ $B = \frac{1}{4}$

$$\boxed{y_p = -\frac{2}{3}e^t + \frac{1}{4}}$$

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2) b. $y'' + y = \cos 2t + t^3$

$$y_p = A\cos 2t + B\sin 2t + Ct^3 + Dt^2 + Et + F$$
$$y_p' = -2A\sin 2t + 2B\cos 2t + 3Ct^2 + 2Dt + E$$
$$y_p'' = -4A\cos 2t - 4B\sin 2t + 6Ct + 2D$$

$$\cos 2t + t^3 = -4A\cos 2t - 4B\sin 2t + 6Ct + 2D + A\cos 2t + B\sin 2t + Ct^3 + Dt^2 + Et + F$$
$$-3A\cos 2t - 3B\sin 2t + Ct^3 + Dt^2 + [6Ct + Et] + [2D + F]$$

$-3A = 1$ $\quad -3B = 0$ $\quad C = 1$ $\quad D = 0$ $\quad 6C + E = 0$

$A = \frac{-1}{3}$ $\quad B = 0$ $\quad E = -6$

$2D + F = 0$

$F = 0$

$$\boxed{y_p = -\frac{1}{3}\cos 2t + t^3 - 6t}$$

2) c. $y'' + y' - 12y = 2te^{-t}$

$y_p = (At + B)e^{-t} = Ate^{-t} + Be^{-t}$

$y_p' = A(e^{-t} + (-e^{-t})t) - Be^{-t} = Ae^{-t} - Ate^{-t} - Be^{-t}$

$y_p'' = -Ae^{-t} - A(e^{-t} + (-e^{-t})t) + Be^{-t}$

$= -2Ae^{-t} + Ate^{-t} + Be^{-t}$

$2te^{-t} = -2Ae^{-t} + Ate^{-t} + Be^{-t} + Ae^{-t} - Ate^{-t} - 12(Ate^{-t} + Be^{-t}) - Be^{-t}$

$= e^{-t}(-2A + At + B + A - At - 12At - 12B - B)$

$2t = [-A - 12B] + [-12At]$

$-12A = 2$ $\qquad$ $-A - 12B = 0$

$A = -\frac{1}{6}$ $\qquad$ $-(-\frac{1}{6}) = 12B$

$B = \frac{1}{72}$

$$\boxed{y_p = \left(-\frac{1}{6}t + \frac{1}{72}\right)e^{-t}}$$

$r^2+4=0 \quad r=\pm 2i$

gen homogeneous soln:

$y = c_1\cos 2t + c_2 \sin 2t$

2) d. $y'' + 4y = e^{-t}\cos t$

$$y_p = e^{-t}(A\cos t + B\sin t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

$$y_p' = A((-\sin t)(e^{-t}) + (-e^{-t})(\cos t)) + B((\cos t)(e^{-t}) + (-e^{-t})(\sin t))$$

$$= -Ae^{-t}\sin t - Ae^{-t}\cos t + Be^{-t}\cos t - Be^{-t}\sin t$$

$$y_p'' = -A(-e^{-t}\sin t + e^{-t}\cos t) - A(-e^{-t}(\cos t) + e^{-t}(-\sin t))$$
$$+ B(-e^{-t}\cos t + e^{-t}(-\sin t)) - B(-e^{-t}\sin t + e^{-t}\cos t)$$

$$= Ae^{-t}\sin t - Ae^{-t}\cos t + Ae^{-t}\cos t + Ae^{-t}\sin t$$
$$-Be^{-t}\cos t - Be^{-t}\sin t + Be^{-t}\sin t - Be^{-t}\cos t$$

$$= 2Ae^{-t}\sin t - 2Be^{-t}\cos t$$

$$e^{-t}\cos t = 2Ae^{-t}\sin t - 2Be^{-t}\cos t + 4Ae^{-t}\cos t + 4Be^{-t}\sin t$$

$$\cos t = 2A\sin t - 2B\cos t + 4A\cos t + 4B\sin t$$

$$= (2A+4B)\sin t + (4A-2B)\cos t$$

$$4A - 2B = 1 \qquad 2A + 4B = 0$$

$$A = \frac{2B+1}{4} \longrightarrow 2\left(\frac{2B+1}{4}\right) + 4B = 0$$

$$B + \frac{1}{2} + 4B = 0$$

$$5B = -\frac{1}{2} \qquad B = -\frac{1}{10}$$

$$A = \frac{1}{5}$$

$$\boxed{y_p = \frac{1}{5}e^{-t}\cos t - \frac{1}{10}e^{-t}\sin t}$$

[redacted]

$r^2 - r - 2 \quad (r-2)(r+1)$

$r = 2, -1$

homogeneous soln $= c_1 e^{-r_1 t} + c_2 e^{r_2 t}$

2) e. $y'' - y' - 2y = t(\cos t - \sin t)$ simplifies to $t\cos t - t\sin t$

$y_p = (At + B)\cos t + (Ct + D)\sin t$

$= At\cos t + B\cos t + Ct\sin t + D\sin t$

$y_p' = A(\cos t + t(-\sin t)) - B\sin t + C(\sin t + t(\cos t)) + D\cos t$

$= A\cos t - At\sin t - B\sin t + C\sin t + Ct\cos t + D\cos t$

$y_p'' = -A\sin t - A(\sin t + t\cos t) - B\cos t + C\cos t + C(\cos t + t(-\sin t)) - D\sin t$

$= -A\sin t - A\sin t - At\cos t - B\cos t + C\cos t + C\cos t - Ct\sin t - D\sin t$

$= -2A\sin t - At\cos t - B\cos t + 2C\cos t - Ct\sin t - D\sin t$

$t(\cos t - \sin t) = -2A\sin t - At\cos t - B\cos t + 2C\cos t - Ct\sin t - D\sin t - A\cos t + At\sin t + B\sin t - C\sin t$

$- Ct\cos t - D\cos t - 2At\cos t - 2B\cos t - 2Ct\sin t - 2D\sin t$

$(-At - B + 2C - A - Ct - D - 2At - 2B)\cos t + (-2A - Ct - D + At + B - C - 2Ct - 2D)\sin t$

$[(-3A - C)t + (-3B + 2C - A - D)]\cos t + [(A - 3C)t + (-2A - 3D + B - C)]\sin t$

$= \downarrow \; t\cos t \quad + \quad \downarrow \; -t\sin t$

$-3A - C = 1$ $\qquad$ $A - 3C = -1$

$C = -3A - 1 \longrightarrow A - 3(-3A - 1) = -1$

$A + 9A + 3 = -1$

$10A = -4$ $\qquad$ $A = -\frac{2}{5}$

$C = \frac{1}{5}$

$-3B + 2C - A - D = 0$ $\qquad$ $-2A - 3D + B - C = 0$

$-3B + 2\left(\frac{1}{5}\right) - \left(-\frac{2}{5}\right) - D = 0$ $\qquad$ $3\left(-2\left(-\frac{2}{5}\right) - 3D + B - \left(\frac{1}{5}\right) = 0\right)$

sum equations cancel B

$2\left(\frac{1}{5}\right) + \frac{2}{5} - D - 6\left(-\frac{2}{5}\right) - 9D - 3\left(\frac{1}{5}\right) = 0$

$10D = \frac{13}{5}$ $\qquad$ $D = \frac{13}{50}$

$B = 2A + 3D + C = 2\left(-\frac{2}{5}\right) + 3\left(\frac{13}{50}\right) + \frac{1}{5} = \frac{9}{50}$ $\qquad$ $B = \frac{9}{50}$

$$\boxed{y_p = \left(-\frac{2}{5}t + \frac{9}{50}\right)\cos t + \left(\frac{1}{5}t + \frac{13}{50}\right)\sin t}$$

$r^2 + 16 \qquad r = \pm 4i$

due to no y' term, the homogeneous soln is $c_1 \cos 2t$ (can't repeat!) in y_p

2) f. $\quad y'' + 16y = t\cos 2t \quad \ldots$

$(At+B)\sin 2t + (Ct+D)\cos 2t$

$$y_p = A\sin 2t + B\underline{t}\cos 2t$$

$$y_p' = 2A\cos 2t + B(\cos 2t + t(-2\sin 2t))$$

$$= 2A\cos 2t + B\cos 2t - 2Bt\sin 2t$$

$$y_p'' = -4A\sin 2t - 2B\sin 2t - 2B(\sin 2t + t(2\cos 2t))$$

$$= -4A\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t$$

$$= -4A\sin 2t - 4B\sin 2t - 4Bt\cos 2t$$

$$t\cos 2t = -4A\sin 2t - 4B\sin 2t - 4Bt\cos t + 16A\sin 2t + 16Bt\cos 2t$$

$$(-4A - 4B + 16A)\sin 2t + (-4B + 16B)t\cos 2t$$

$$(12A - 4B)\sin 2t + (12B)t\cos 2t$$

$$12A - 4B = 0 \qquad\qquad 12B = 1$$

$$A = \frac{4}{12}B = \frac{B}{3} \qquad\qquad B = \frac{1}{12}$$

$$A = \frac{1}{36}$$

$$\boxed{y_p = \frac{1}{36}\sin 2t + \frac{1}{12}t\cos 2t}$$

3) Method of undetermined coefficients with special cases

Find a particular solution and/or initial value problems of following:

a. $y'' - y = 3e^t$ $r^2 - 1 = 0$

$(r+1)(r-1)$ $r = \pm 1$

homogeneous sol'n $= c_1 e^{r_1 t} + c_2 e^{r_2 t}$

y_p needs extra linear term t

$y_p = A\underline{t}e^t$

$y_p' = A(e^t + te^t)$

$= Ae^t + Ate^t$

$y_p'' = Ae^t + A(e^t + te^t) = Ae^t + Ae^t + Ate^t$

$= 2Ae^t + Ate^t$

$3e^t = 2Ae^t + \cancel{Ate^t} - \cancel{Ate^t}$

$3 = 2A$

$A = \frac{3}{2}$

$$\boxed{y_p = \frac{3}{2}te^t}$$

3) b. $y'' + y = 3\cos t$ $\quad r^2 + 1 = 0 \quad r = \pm i$

homogeneous soln' $= C_1 \sin t + C_2 \cos t$

y_p needs extra term t

$y_p = t(A\cos t + B\sin t)$

$= At\cos t + Bt\sin t$

$y_p' = A(\cos t + t(-\sin t)) + B(\sin t + t\cos t)$

$= A\cos t - At\sin t + B\sin t + Bt\cos t$

$y_p'' = -A\sin t - A(\sin t + t\cos t) + B\cos t + B(\cos t + t(-\sin t))$

$= -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t$

$= -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$

$3\cos t = -2A\sin t + 2B\cos t - At\cos t - Bt\sin t + At\cos t + Bt\sin t$

$= -2A\sin t + 2B\cos t$

$-2A = 0 \qquad 2B = 3$

$A = 0 \qquad B = \frac{3}{2}$

$$\boxed{y_p = \frac{3}{2}t\sin t}$$

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3) c. $y'' + 5y' + 6y = 2e^{-2t}$

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) \quad r = -3, -2$

homogeneous soln' $= C_1 e^{-3t} + C_2 e^{-2t}$

y_p needs extra linear term t

$y_p = Ate^{-2t}$

$y_p' = A(e^{-2t} + (-2e^{-2t})t)$

$= Ae^{-2t} - 2Ate^{-2t}$

$y_p'' = -2Ae^{-2t} - 2A(e^{-2t} + (-2e^{-2t})t) = -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}$

$= -4Ae^{-2t} + 4Ate^{-2t}$

$2e^{-2t} = -4Ae^{-2t} + 4Ate^{-2t} + 5(Ae^{-2t} - 2Ate^{-2t}) + 6Ate^{-2t}$

$= -4Ae^{-2t} + 4Ate^{-2t} + 5Ae^{-2t} - 10Ate^{-2t} + 6Ate^{-2t}$

$2 = -4A + \cancel{4At} + 5A - \cancel{10At} + \cancel{6At}$

$A = 2$

$\boxed{y_p = 2te^{-2t}}$

3) d. $y'' + 2y' + 2y = 5e^{-2t}\cos t$ $\qquad r^2 + 2r + 2 = 0$

$y_p = Ae^{-2t}\cos t + Be^{-2t}\sin t$ $\qquad r = \frac{-2 \pm \sqrt{8-4}\, i}{2} = -1 \pm i$

$y'_p = A(-2e^{-2t}\cos t + e^{-2t}(-\sin t))$
$\quad + B(-2e^{-2t}\sin t + e^{-2t}(\cos t))$

homogeneous sol'n: $C_1 e^{-t}\sin t + C_2 e^{-t}\cos t$

$= -2Ae^{-2t}\cos t - Ae^{-2t}\sin t - 2Be^{-2t}\sin t + Be^{-2t}\cos t$

y_p does not need t term

$y''_p = -2A(-2e^{-2t}\cos t + e^{-2t}(-\sin t)) - A(-2e^{-2t}\sin t + e^{-2t}(\cos t))$
$\quad - 2B(-2e^{-2t}\sin t + e^{-2t}(\cos t)) + B(-2e^{-2t}\cos t + e^{-2t}(-\sin t))$

$= 4Ae^{-2t}\cos t + 2Ae^{-2t}\sin t + 2Ae^{-2t}\sin t - Ae^{-2t}\cos t$
$\quad + 4Be^{-2t}\sin t - 2Be^{-2t}\cos t - 2Be^{-2t}\cos t - Be^{-2t}\sin t$

$= 3Ae^{-2t}\cos t + 4Ae^{-2t}\sin t + 3Be^{-2t}\sin t - 4Be^{-2t}\cos t$

$5e^{-2t}\cos t = 3Ae^{-2t}\cos t + 4Ae^{-2t}\sin t + 3Be^{-2t}\sin t - 4Be^{-2t}\cos t - 4Ae^{-2t}\cos t$
$\quad - 2Ae^{-2t}\sin t - 4Be^{-2t}\sin t + 2Be^{-2t}\cos t + 2Ae^{-2t}\cos t + 2Be^{-2t}\sin t$

$5\cos t = 3A\cos t + 4A\sin t + 3B\sin t - 4B\cos t - 4A\cos t - 2A\sin t - 4B\sin t$
$\quad + 2B\cos t + 2A\cos t + 2B\sin t$

$= [3A - 4B - 4A + 2B + 2A]\cos t + [4A + 3B - 2A - 4B + 2B]\sin t$

$(A - 2B)\cos t + (2A + B)\sin t$

$A - 2B = 5 \qquad 2A + B = 0$

$A - 2(-2A) = 5 \longleftarrow B = -2A$

$5A = 5 \quad A = 1 \quad B = -2$

$$\boxed{y_p = e^{-2t}\cos t - 2e^{-2t}\sin t}$$

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3) e. $y'' + 3y' + 2y = 2e^{-3t}$, $y(0) = 0$, $y'(0) = 1$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \quad r = -2, -1$

$y = c_1 e^{-2t} + c_2 e^{-t} + y_p$

$y_p = Ae^{-3t}$

$y_p' = -3Ae^{-3t}$

$y_p'' = 9Ae^{-3t}$

$2e^{-3t} = 9Ae^{-3t} + 3(-3Ae^{-3t}) + 2(Ae^{-3t})$

$= 9Ae^{-3t} - 9Ae^{-3t} + 2Ae^{-3t}$

$2Ae^{-3t}$

$2A = 2 \qquad A = 1$

$y_p = e^{-3t}$

$y = c_1 e^{-2t} + c_2 e^{-t} + e^{-3t}$

$0 = c_1 + c_2 + 1 \qquad c_2 = -1 - c_1$

$y' = -2c_1 e^{-2t} - c_2 e^{-t} - 3e^{-3t}$

$1 = -2c_1 - c_2 - 3$

$4 = -2c_1 - c_2$

$4 = -2c_1 - (-1 - c_1) = 1 - c_1 \qquad c_1 = -3$

$c_2 = 2$

$$\boxed{y = -3e^{-2t} + 2e^{-t} + e^{-3t}}$$

3) f. $y'' + 9y = 6\cos 3t \quad , \quad y(0)=0,\ y'(0)=0$

$r^2 + 9 = 0$

$r = \pm 3i$

$y = c_1 \sin 3t + c_2 \cos 3t + y_p$

$y_p = (A\sin 3t + B\cos 3t)\,t = At\sin 3t + Bt\cos 3t$

$y'_p = A(\sin 3t + 3t\cos 3t) + B(\cos 3t - 3t\sin 3t)$

$= A\sin 3t + 3At\cos 3t + B\cos 3t - 3Bt\sin 3t$

$y''_p = 3A\cos 3t + 3A(\cos 3t + t(-3\sin 3t)) - 3B\sin 3t - 3B(\sin 3t + 3t\cos 3t)$

$= 6A\cos 3t - 9At\sin 3t - 6B\sin 3t - 9Bt\cos 3t$

$6\cos 3t = 6A\cos 3t - 9Bt\cos 3t - 9At\sin 3t - 6B\sin 3t + 9At\sin 3t + 9Bt\cos 3t$

$[6A - 9Bt + 9Bt]\cos 3t + [-9At - 6B + 9At]\sin 3t$

$6A\cos 3t \qquad -6B\sin 3t$

$6A = 6 \qquad A = 1 \qquad -6B = 0 \qquad B = 0$

$y_p = t\sin 3t$

$y = c_1\sin 3t + c_2\cos 3t + t\sin 3t$

$0 = 0 + c_2 + 0 \qquad c_2 = 0$

$y' = 3c_1\cos 3t - 3c_2\sin 3t + (\sin 3t + 3t\cos 3t)$

$0 = 3c_1 - 0 + 0 + 0 \qquad c_1 = 0$

$\boxed{y = t\cdot\sin 3t}$

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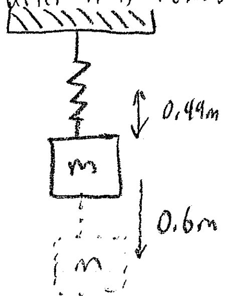
(Free Undamped Oscillations)

4) No resistance or external force, find position of block at all times after it is released. Graph and describe motion of block, and give its period.



$m = 0.5kg$ block stretches spring $0.49m$

$x = 0.49m$

$b = 0$

$f(t) = 0$

$mg = kx$

$k = 10$

Then block is pulled down $0.6m$ and released with velocity of $0.25 m/s$.

$w_0 = \sqrt{\frac{k}{m}}$

$y(0) = 0.6$

$y'(0) = 0.25$

$w_0 = \sqrt{\frac{10}{0.5}} = \sqrt{20}\ s^{-1}$

$\boxed{\text{period of oscillation} = \frac{2\pi}{\sqrt{20}} \text{ seconds}}$

$y'' + w_0^2 y = 0$

$y'' + 20y = 0$

$r^2 + 20 = 0$

$r = \pm i\sqrt{20}$

$y = c_1 \sin\sqrt{20}t + c_2 \cos\sqrt{20}t$

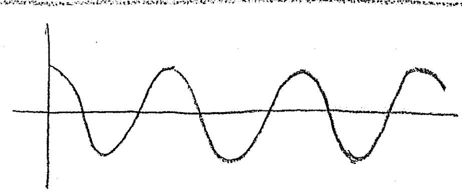
$0.6 = 0 + c_2$ $\quad c_2 = 0.6$

$y' = \sqrt{20}\, c_1 \cos(\sqrt{20}t) - \sqrt{20}\, c_2 \sin(\sqrt{20}t)$

$0.25 = \sqrt{20}\, c_1 - 0$ $\quad c_1 = \frac{0.25}{\sqrt{20}}$

$\boxed{\text{Position at all time: } y = \frac{0.25}{\sqrt{20}} \sin(\sqrt{20}t) + 0.6\cos(\sqrt{20}t)}$

Approximate Graph:

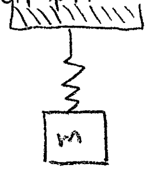


...

The motion is periodic and the sinusoidal wave pattern has a constant amplitude.

(Forced Undamped Oscillations) $b=0$

5) Assume no initial displacement and velocity of block, and no damping. Graph the solution in two cases: $w=3$ and $w=0.5$. In which case does the solution have resonance?



$m = \frac{4}{3}kg$

$k = 12.0 N/m$

$F_{ext} = 9\cos wt$ $\qquad y(0)=0 \quad y'(0)=0$

a. $w = 3$ $\qquad y'' + w_0^2 y = 9\cos wt$

$w_0^2 = \frac{k}{m} = \frac{12}{4/3} = 9 \quad w_0 = 3 = w$ $\boxed{\text{resonance case}}$

$y'' + 9y = 9\cos 3t$ $\qquad F_0 = 9$ wrong here; F_0/m = 9

$r^2 + 9 = 0 \qquad r = \pm 3i \qquad y_p = \frac{F_0}{2mw_0} t\sin w_0 t$

$y = c_1 \sin 3t + c_2 \cos 3t + \frac{9}{8} t\sin 3t$

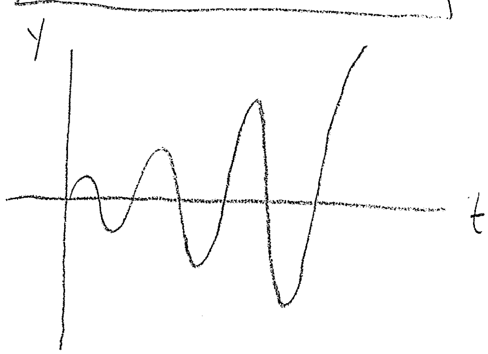
$0 = c_2$

$y'' = 3c_1\cos 3t - 3c_2 \sin 3t + \frac{9}{8}(\sin 3t + 3t\cos 3t)$

$0 = 3c_1 - 0 + 0 + 0$

wrong coefficient here 1.5 t sin 3t

$$\boxed{y = \frac{9}{8} t\sin 3t}$$



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5) b. $\omega = 0.5$ $\quad y'' + \omega_0^2 y = 9\cos\omega t$ $\quad$ non-resonance case
beats case

$\omega_0^2 = \frac{12}{4/3} = 9 \quad \omega_0 = 3 \neq 0.5$

$y'' + 9y = 9\cos 0.5t$ $\qquad F_0 = 9$ $\qquad y_p = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$

So again F_0/m =9

$r^2 + 9 = 0 \qquad r = \pm 3i$

$y = C_1\sin 3t + C_2\cos 3t + \frac{27}{35}\cos 0.5t$

$0 = 0 + C_2 + \frac{27}{35} \qquad C_2 = -\frac{27}{35}$

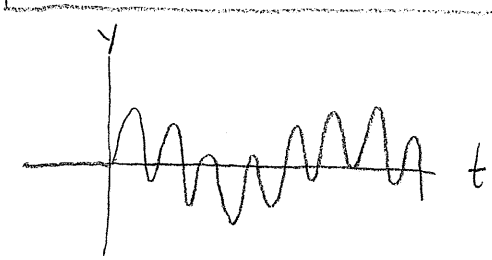
$y' = 3C_1\cos 3t - 3C_2\sin 3t - \frac{27}{70}\sin 0.5t$

$0 = 3C_1 - 0 - 0$

$C_1 = 0$

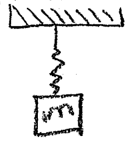
$$\boxed{y = -\frac{27}{35}\cos 3t + \frac{27}{35}\cos 0.5t}$$

should be
9/8.75 *(cos .5 t-cos 3t)



(Free Damped Oscillations) $f = 0$

6) Three 10kg blocks hang on springs with spring constants k. Friction is modeled with damping coefficient $c = 40\,kg/s$



Solve an initial value problem with the following values of k and initial values $y(0) = 2$, $y'(0) = 0$. Graph each solution and state whether the problem exhibits underdamping, overdamping, or critical damping.

$m = 10$

a. $k = 30\,N/m$ $\qquad y'' + by' + \omega_0^2 y = 0$

$$b = \frac{c}{m} = \frac{40}{10} = 4 \qquad \omega_0^2 = \frac{k}{m} = \frac{30}{10} = 3$$

$$y'' + 4y' + 3y = 0$$

$$r^2 + 4r + 3 = 0 \qquad (r+3)(r+1) = 0 \quad r = -3, -1$$

$$y = c_1 e^{-3t} + c_2 e^{-t}$$

$$2 = c_1 + c_2$$

$$y' = -3c_1 e^{-3t} - c_2 e^{-t}$$

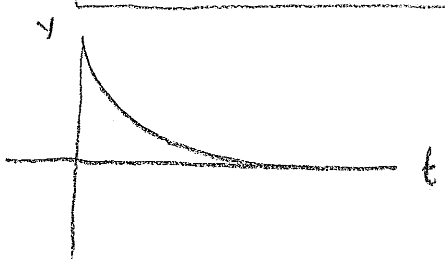
$$0 = -3c_1 - c_2$$

$$c_2 = -3c_1$$

$$2 = c_1 + (-3c_1) = -2c_1$$

$$c_1 = -1 \qquad c_2 = 3$$

$$\boxed{y = -e^{-3t} + 3e^{-t}}$$



overdamped

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6) b. $k = 40\,N/m$ $\quad y'' + by' + w_0^2 y = 0 \quad y(0) = 2,\ y'(0) = 0$

$b = 4 \quad w_0^2 = \frac{40}{10} = 4$

$y'' + 4y' + 4 = 0$

$r^2 + 4r + 4 = 0 \quad (r+2)^2 = 0 \quad r = -2$ repeated

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

$2 = c_1 + c_2$

$y' = -2c_1 e^{-2t} + c_2\left(e^{-2t} + t(-2e^{-2t})\right)$

$= -2c_1 + c_2 e^{-2t} - 2c_2 t e^{-2t}$

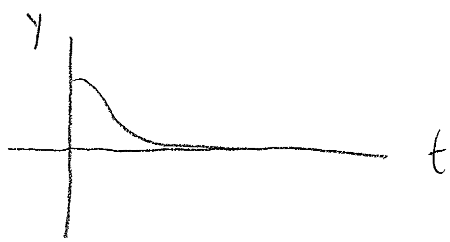
$0 = -2c_1 + c_2 - 0$

$c_1 = 2 - c_2$

$0 = -2(2 - c_2) + c_2 = -4 + 2c_2 + c_2$

$3c_2 = 4 \quad c_2 = \frac{4}{3} \quad c_1 = \frac{2}{3}$

$$\boxed{y = \frac{2}{3}e^{-2t} + \frac{4}{3}te^{-2t}}$$



critically damped

6) c. $k = 62.5\, N/m$ $\quad y'' + by' + w_0^2 y = 0 \quad y(0) = 2,\ y'(0) = 0$

$b = 4 \qquad w_0^2 = \frac{62.5}{10} = 6.25$

$y'' + 4y' + 6.25 = 0$

$$r = \frac{-4 \pm \sqrt{4(6.25) - 4^2}\, i}{2} = -2 \pm \frac{3}{2} i$$

$$y = c_1 e^{-2t} \sin \tfrac{3}{2} t + c_2 e^{-2t} \cos \tfrac{3}{2} t$$

$$2 = 0 + c_2 \qquad c_2 = 2$$

$$y' = c_1\left(-2e^{-2t} \sin \tfrac{3}{2}t + e^{-2t}\left(\tfrac{3}{2}\cos\tfrac{3}{2}t\right)\right)$$
$$+ c_2\left(-2e^{-2t} \cos\tfrac{3}{2}t + e^{-2t}\left(-\tfrac{3}{2}\sin\tfrac{3}{2}t\right)\right)$$

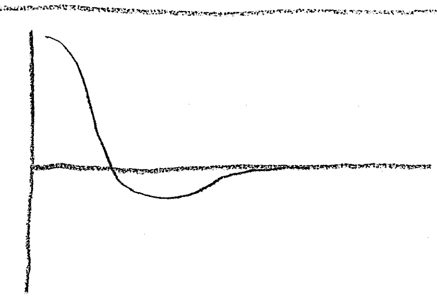
$$y' = -2c_1 e^{-2t} \sin\tfrac{3}{2}t + \tfrac{3}{2}c_1 e^{-2t} \cos\tfrac{3}{2}t - 2c_2 e^{-2t}\cos\tfrac{3}{2}t - \tfrac{3}{2}c_2 e^{-2t} \sin\tfrac{3}{2}t$$

$$0 = 0 + \tfrac{3}{2}c_1 - 2c_2 - 0$$

$$0 = \tfrac{3}{2}c_1 - 2(2)$$

$$4 = \tfrac{3}{2}c_1 \qquad c_1 = \frac{8}{3}$$

$$\boxed{y = \frac{8}{3} e^{-2t} \sin\tfrac{3}{2}t + 2e^{-2t}\cos\tfrac{3}{2}t}$$

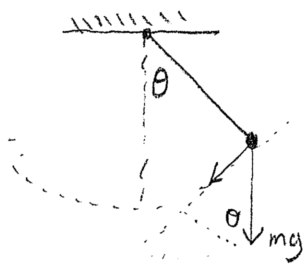


Underdamped

7) Pendulum Equation:

A pendulum consisting of a bob of mass m swinging on a massless rod of length ℓ. Let $\theta(t)$ be angular displacement of the pendulum t seconds after it is released (radians). Assuming only force is gravity, write Newton's 2nd law in direction of motion ($\perp$ to rod). Distance along arc is $s(t) = \ell\theta(t)$, so velocity is $s'(t) = \ell\theta'(t)$ and acceleration is $s''(t) = \ell\theta''(t)$

a. Explain why Newton's 2nd Law is $m\ell\theta''(t) = -mg\sin\theta$ where $g = 9.8\ m/s^2$ is the acceleration due to gravity.



since $F = ma$ and $s''(t)$ of the distance of arc = acceleration = $\ell\theta''(t)$

$a = \ell\theta''(t)$ which is equal to the a on the other side of the eqn $g\sin\theta$, the component of acceleration of the force done by gravity.

Gravity points in negative direction against θ.

b. Write this equation as $\theta'' + w_0^2\sin\theta = 0$ where $w_0^2 = \frac{g}{\ell}$

$$m\ell\theta'' + mg\sin\theta = 0$$
$$\ell\theta'' + g\sin\theta$$
$$\theta'' + \frac{g}{\ell}\sin\theta = 0$$
$$\theta'' + w_0^2\sin\theta = 0$$

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7) c. Equation is nonlinear. It can be linearized by assuming the angular displacements are small and using the approximation sin theta $\approx \theta$. Show that the resulting linear pendulum equation is $\theta'' + \omega_0^2\theta = 0$

When $\theta \approx 0$ then $\sin\theta \approx 0$

so $\theta \approx \sin\theta$

Substitute θ for $\sin\theta$ in equation from part b.

$\boxed{\theta'' + w_0^2(\theta) = 0}$, the linear pendulum equation.

d. Express the period of the pendulum in terms of g, l. If the length of the pendulum is increased by a factor of 2, by what factor does the period change?

$$\text{period} = \frac{2\pi}{w_0} \qquad w_0^2 = \frac{g}{l} \qquad w_0 = \sqrt{\frac{g}{l}}$$

$$\boxed{\text{period} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}}$$

$\boxed{\text{period} \propto \sqrt{l}\text{, so for } 2l\text{, factor of period changes by } \sqrt{2}}$