Partial solutions to Homework #11, Math 2173

Reference: all of these questions are drawn from Chapters 16.3, 16.4 of the e-book.

1. (30pts, Method of undetermined coefficients with polynomials/exponential/trigonometric functions) Find a particular solutions of the following differential equations

(a)

$$y'' + 3y' - 18y = 2t^4 - t^2$$

Answer: The trial solution is

$$y_p = At^4 + Bt^3 + Ct^2 + Dt + E.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B, C, D, E. One obtains

$$y_p = -\frac{1}{9}t^4 - \frac{2}{27}t^3 - \frac{1}{18}t^2 - \frac{7}{162}t - \frac{13}{972}.$$

(b)

$$y'' + 5y = 6t^3 - t^2$$

Answer: The trial solution is

$$y_p = At^3 + Bt^2 + Ct + D.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B, C, D. One obtains

$$y_p = \frac{6}{5}t^3 - \frac{1}{5}t^2 - \frac{36}{25}t + \frac{2}{25}.$$

(c)

$$y'' - 4y' - 32y = 6e^{-3t}$$

Answer: The trial solution is

$$y_p = Ae^{-3t}.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A. One obtains

$$y_p = -\frac{6}{11}e^{-3t}.$$

(d)

$$y'' + 2y' - 8y = e^{-t} - 2e^t$$

Answer: The trial solution is

$$y_p = Ae^t + Be^{-t}.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = -\frac{1}{9}e^t + \frac{2}{5}e^{-t}.$$

(e)

$$y'' - y' - 6y = \sin t + 3\cos t$$

Answer: The trial solution is

$$y_p = A\sin t + B\cos t.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = -\frac{1}{5}\sin t - \frac{2}{5}\cos t.$$

(f)

$$y'' - 3y' - 4y = 2\cos(2t) - 3\sin(2t).$$

Answer: The trial solution is

$$y_p = A\sin 2t + B\cos 2t.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = \frac{3}{25}\sin 2t - \frac{17}{50}\cos 2t.$$

2. (30pts, Method of undetermined coefficients with combined functions) Find a particular solutions of the following differential equations

(a)

$$y'' - 4y = 2e^t - 1$$

Answer: The trial solution is

$$y_p = Ae^t + B.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = -\frac{2}{3}e^t + \frac{1}{4}.$$

(b)

$$y'' + y = \cos(2t) + t^3$$

Answer: The trial solution is

$$y_p = A\sin 2t + B\cos 2t + Ct^3 + Dt^2 + Et + F.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = -\frac{1}{3}\cos 2t + t^3 - 6t.$$

(c)

$$y'' + y' - 12y = 2te^{-t}$$

Answer: The trial solution is

$$y_p = (At + B)e^{-t}.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = (-\frac{1}{6}t + \frac{1}{72})e^{-t}.$$

(d)

$$y'' + 4y = e^{-t} \cos t$$

Answer: The trial solution is

$$y_p = e^{-t}(A \sin t + B \cos t).$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = -\frac{1}{10}e^{-t} \sin t + \frac{1}{5}e^{-t} \cos t.$$

(e)

$$y'' - y' - 2y = t(\cos t - \sin t)$$

Answer: The trial solution is

$$y_p = At \sin t + Bt \cos t + C \sin t + D \cos t.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B, C, D. One obtains

$$y_p = \frac{1}{5}t \sin t - \frac{2}{5}t \cos t + \frac{13}{50} \sin t - \frac{9}{50} \cos t.$$

(f)

$$y'' + 16y = t \cos(2t)$$

Answer: The trial solution is again

$$y_p = At \sin 2t + Bt \cos 2t + C \sin 2t + D \cos 2t.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B, C, D. One obtains

$$y_p = \frac{1}{36} \sin 2t + \frac{1}{12}t \cos 2t.$$

3. (30pts, Method of undetermined coefficients with special cases) Find a particular solutions and/or initial value problems of the following differential equations

(a)

$$y'' - y = 3e^t$$

Answer: The general solution is

$$c_1e^t + c_2e^{-t} + y_p.$$

Because e^t also appears in the non-homogeneous part on the right hand side, our choice of trial for y_p is

$$y_p = Ate^t.$$

We then calculate y'_p, y''_p and substitute into the original solution to solve for A. One obtains

$$y_p = \frac{3}{2}te^t.$$

(b)

$$y'' + y = 3\cos t$$

Answer: The general solution is

$$c_1 \sin t + c_2 \cos t + y_p.$$

Because $\cos t$ also appears in the non-homogeneous part on the right hand side, our choice of trial for y_p is

$$y_p = At\sin t + Bt\cos t.$$

We then calculate y'_p, y''_p and substitute into the original solution to solve for A, B. One obtains

$$y_p = \frac{3}{2}t\sin t.$$

(c)

$$y'' + 5y' + 6y = 2e^{-2t}$$

Answer: The general solution is

$$c_1e^{-2t} + c_2e^{-3t} + y_p.$$

Because e^{-2t} appears in the non-homogeneous part on the right hand side, our choice of trial for y_p is

$$y_p = Ate^{-2t}.$$

We then calculate y'_p, y''_p and substitute into the original solution to solve for A. One obtains

$$y_p = 2te^{-2t}.$$

(d) (longer calculations)

$$y'' + 2y' + 2y = 5e^{-2t}\cos t$$

Answer: The trial solution is

$$y_p = e^{-2t}(B\sin t + C\cos t).$$

In other words, $y_p = Be^{-2t}\sin t + Ce^{-2t}\cos t$.

$$\begin{aligned} y_p' &= -2Be^{-2t}\sin t + Be^{-2t}\cos t - 2Ce^{-2t}\cos t - Ce^{-2t}\sin t \\ &= (-2B - C)e^{-2t}\sin t + (B - 2C)e^{-2t}\cos t. \\ y_p'' &= (4B + 2C)e^{-2t}\sin t + (-2B - C)e^{-2t}\cos t + (-2B + 4C)e^{-2t}\cos t + (-B + 2C)e^{-2t}\sin t \\ &= [(4B + 2C) + (-B + 2C)]e^{-2t}\sin t + [(-2B - C) + (-2B + 4C)]e^{-2t}\cos t \\ &= (3B + 4C)e^{-2t}\sin t + (-4B + 3C)e^{-2t}\cos t. \end{aligned}$$

Thus

$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= (3B + 4C)e^{-2t}\sin t + (3C - 4B)e^{-2t}\cos t + 2[(-2B - C)e^{-2t}\sin t + (-2C + B)e^{-2t}\cos t] + \\ &\quad + 2[e^{-2t}(B\sin t + C\cos t)] \\ &= [(3B + 4C) + 2(-2B - C) + 2B]e^{-2t}\sin t + [(3C - 4B) + 2(-2C + B) + 2C]e^{-2t}\cos t \\ &= (B + 2C)e^{-2t}\sin t + (C - 2B)e^{-2t}\cos t. \end{aligned}$$

We thus deduce that $B + 2C = 0$ and $C - 2B = 5$. Plug the value of C into the first equation, $B + 2C = B + 4B + 10 = 5B + 10 = 0$, and so $B = -2$, and $C = 1$.

$$y_p = -2e^{-2t}\sin t + e^{-2t}\cos t.$$

(e)

$$y'' + 3y' + 2y = 2e^{-3t}, y(0) = 0, y'(0) = 1$$

Answer: The general solution is

$$c_1 e^{-t} + c_2 e^{-2t} + y_p.$$

Our choice of trial for y_p is

$$y_p = Ae^{-3t}.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A. One obtains

$$y_p = e^{-3t}.$$

Now we use the initial conditions to solve for c_1, c_2, one obtains

$$y = 2e^{-t} - 3e^{-2t} + e^{-3t}.$$

(f)

$$y'' + 9y = 6\cos(3t), y(0) = 0, y'(0) = 0.$$

Answer: The general solution is

$$y = c_1 \sin 3t + c_2 \cos 3t + y_p.$$

To find a particular solution y_p, one set

$$y_p = At\sin 3t + Bt\cos 3t.$$

We then calculate y_p', y_p'' and substitute into the original solution to solve for A, B. One obtains

$$y_p = t \sin 3t.$$

Thus the general solution is

$$y = c_1 \sin 3t + c_2 \cos 3t + t \sin 3t.$$

Now we use the initial conditions to solve for c_1, c_2, one obtains

$$y = t \sin 3t.$$

4. (10pts, Free undamped oscillations) A 0.5-kg block hangs on a spring and stretches the spring 0.49 m. The block is pulled down 0.6 m and released with a downward velocity of 0.25 m/s. Assuming no resistance or external forcing, find the position of the block at all times after it is released. Graph and describe the motion of the block, and give its period of oscillation.

Answer: At equilibrium, $mg = ky$, thus

$$k = \frac{mg}{y} = \frac{0.5 * 9.8}{0.49} = 10.$$

We then calculate ω_0,

$$\omega_0 = \sqrt{k/m} = \sqrt{10/0.5} = \sqrt{20}.$$

The oscillation equation is

$$y'' + \omega_0^2 y = 0,$$

which has general solution

$$y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.$$

Now we use the initial conditions to solve for c_1, c_2, one obtains

$$y = \frac{.25}{\sqrt{20}} \sin(\sqrt{20}t) + 0.6 \cos(\sqrt{20}t).$$

Remark: the phase-amplitude form is $A \sin(\omega_0 t + \phi)$, where A, ϕ can be computed.

The period of this oscillation is

$$period = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{20}} (seconds).$$

5. (15pts, Forced undamped oscillations) A $\frac{4}{3}$-kg block hangs on a spring with spring constant k = 12.0 N/m. The support of the spring vibrates and produces an external force of $F_{ext} = 9\cos\omega t$. Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: $\omega = 3$ and $\omega = 0.5$. In which case does the solution have resonance?

Answer: Case 1: $\omega = 3$.

We first calculate ω_0,

$$\omega_0 = \sqrt{k/m} = \sqrt{12/(4/3)} = 3.$$

The oscillation equation is

$$y'' + \omega_0^2 y = 9\cos\omega t.$$

As $\omega_0 = \omega = 3$, this is the resonance case. A particular solution to this equation is

$$y_p = \frac{F_0}{2m\omega_0} t \sin\omega_0 t = \frac{4.5}{3} t\sin 3t = 1.5t\sin 3t.$$

Remark 1: $\frac{F_0}{m} = 9$ here, and you don't have to calculate F_0.

With the initial condition $y(0) = y'(0) = 0$, the actual solution equals y_p as above, thus

$$y = y_p = 1.5t\sin 3t.$$

Remark 2.: Notice that the formula

$$y = \frac{F_0}{2m\omega_0} t\sin\omega_0 t$$

works only for the initial conditions $y(0) = y'(0) = 0$.

Case 2: $\omega = 0.5$.

The oscillation equation is

$$y'' + \omega_0^2 y = 9\cos 0.5t.$$

As $\omega_0 \neq \omega$, this is the beat case. A particular solution to this equation is

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{9}{9 - 0.25} \cos 0.5t.$$

The general solution is

$$c_1 \sin 3t + c_2 \cos 3t + \frac{9}{9 - 0.25} \cos 0.5t.$$

With the initial condition $y(0) = y'(0) = 0$, one can infer that $c_1 = 0, c_2 = -\frac{9}{9-0.25}$. Thus

$$y = \frac{9}{9 - 0.25}(\cos 0.5t - \cos 3t) = \frac{9}{9 - 0.25} \sin \frac{3 - 0.5}{2} \sin \frac{3 + 0.5}{2} = 2.06 \sin 1.25t \times \sin 1.75t.$$

Remark 3.: Notice that the formula

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}$$

works only for the initial conditions $y(0) = y'(0) = 0$.

6. (15pts, Free damped oscillations) Three 10-kg blocks hang on springs with spring constant k. Friction in each system is modeled with a damping coefficient of $c = 40kg/s$. Solve an initial value problem with the following values of k and initial conditions $y(0) = 2, y'(0) = 0$. Graph each solution and state whether the problem exhibits underdamping, overdamping, or critical damping.

a. k=30N/m

b. k=40N/m

c. k=62.5N/m.

Answer: a. The oscillation equation is

$$y'' + 4y' + 3y'' = 0.$$

This is the overdamping case, which has general solution

$$y = c_1 e^{-t} + c_2 e^{-3t}.$$

Now we use the initial conditions to solve for c_1, c_2, one obtains

$$y = 3e^{-t} - e^{-3t}.$$

b. The oscillation equation is

$$y'' + 4y' + 4y'' = 0.$$

This is the critical damping case, which has general solution

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Now we use the initial conditions $y(0) = 2, y'(0) = 0$ to solve for c_1, c_2, one obtains

$$y = 2e^{-2t} + 4te^{-2t}.$$

c. The oscillation equation is

$$y'' + 4y' + 6.25y'' = 0.$$

This is the under damping case, which has general solution

$$y = c_1 e^{-2t} \sin 3t/2 + c_2 e^{-2t} \cos 3t/2.$$

Now we use the initial conditions $y(0) = 2, y'(0) = 0$ to solve for c_1, c_2, one obtains

$$y = \frac{8}{3} e^{-2t} \sin 3t/2 + 2e^{-2t} \cos 3t/2.$$

7. (15pts, pendulum equation) A pendulum consisting of a bob of mass m swinging on a massless rod of length l can be modeled as an oscillator. Let $\theta(t)$ be the angular displacement of the pendulum t seconds after it is released (measured in radians). Assuming that the only force acting on the bob is the gravitational force, we write Newton's second law in the direction of motion (perpendicular to the rod). Notice that the distance along the arc of the swing is $s(t) = l\theta(t)$, so the velocity of the bob is $s'(t) = l\theta'(t)$ and the acceleration is $s''(t) = l\theta''(t)$.

a. Considering only the component of the force in the direction of motion, explain why Newton's second law is

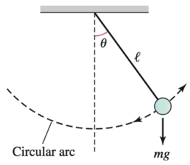
$$ml\theta''(t) = -mg \sin\theta,$$

where $g = 9.8m/s^2$ is the acceleration due to gravity.

b. Write this equation as $\theta'' + \omega_0^2 \sin\theta = 0$, where $\omega_0^2 = g/l$.

c. Notice that this equation is nonlinear. It can be linearized by assuming that the angular displacements are small and using the approximation $\sin\theta \approx \theta$. Show that the resulting linear pendulum equation is $\theta'' + \omega_0^2\theta = 0$.

d. Express the period of the pendulum in terms of g, l. If the length of the pendulum is increased by a factor of 2, by what factor does the period change?



Answer: a. By Newton's second law of motion $ma = F_{gravitational force projected onto the direction of motion}$. As $a = l\theta''$, one has

$$ml\theta''(t) = -mg\sin\theta(t)$$

b. One can rewrite the above as

$$\theta'' = -\frac{g}{l}\sin\theta, \text{ or } \theta'' + \omega_0^2\sin\theta = 0,$$

where

$$\omega^2 = \frac{g}{l}.$$

c. As θ is small, $\sin\theta \approx \theta$, and thus we can think the above equation as

$$\theta'' + \omega_0^2\theta = 0.$$

d. We know that any solution to the above has the form $A\sin(\omega_0 t + \phi)$. The frequency is $\omega_0 = \sqrt{\frac{g}{l}}$. The period can be computed by

$$period = \frac{2\pi}{frequency} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = \frac{2\pi\sqrt{l}}{\sqrt{g}}(seconds).$$

Thus if we double the length, then the period increases (of course), and by a factor of $\sqrt{2}$.