1st Midterm

Q1. Consider the surface $z = f(x, y) = 4 - x^2 - 2y^2$ and the point $P(1, 1, 1)$.

(a) (**2pts**) Verify that P belongs to the surface.

Short answer: $4 - 1^2 - 2 * 1^2 = 1$.

+2

(b) (**8 pts**) Find the gradient of f at any point (x, y).

Short answer: $\nabla f =< -2x, -4y >$.

+4 +4

(c) (**5 pts**) Let C' be the path of the **steepest descent** on the surface beginning at P; and let C be the **projection** of C' on the xy-plane.
At a point (x, y) of C, what is the slope of the tangent line?

Short answer: The tangent line has the same direction as of the gradient vector, thus the slope is

$$\frac{-4y}{-2x} = \frac{2y}{x}.$$

+5

(d) (**5 pts**) (**No need to review**) What is the equation for C? (Hint: if $y'(x) = 2y/x$, then $y = ax^2$ for some a to be determined.)

Short answer: As the slope is $y'(x) = \frac{2y}{x}$, we have $y = ax^2$ for some constant a. As this line passes through $P(1, 1)$, we have

$$1 = a * 1^2 \text{ which yields } a = 1.$$

Thus

$$y = x^2.$$

Fall 2014, The Ohio State University **Calculus 2173**

September 29th, 2014 . First Midterm Exam, version A 55 minutes

Name: ____________________ Student ID: ____________________

Instructions.

1. This is a closed book exam, no calculator, NO CHEATING.
2. Detailed work is required for full credit.

Question	Points	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
bonus	5	
TOTAL	100	

Q2. Consider the following function on the given set R.

$$f(x,y) = \sqrt{x^2 + y^2 - 6y + 9}; \quad R = \{(x,y) : x^2 + y^2 \le 16\}.$$

Short answer: First we look in the interior of R: $x^2 + y^2 < 16$. To find critical points, set

$$f_x = 0, 2x = 0 \text{ yielding} x = 0$$ +3

$$f_y = 0, 2y - 6 = 0 \text{ yielding } y = 3.$$ +3

The point $(0,3)$ is clearly inside R, and the value of f at this point is

$$f^2(0,3) = 0.$$

Now we look on the boundary of R: $x^2 + y^2 = 16$. One can use either Lagrange multipliers of trigonometric substitution. But the fastest way is as follows: as $x^2 + y^2 = 1$, the range for y is

$$-1 \le y \le 1.$$ +2

On the other hand,

$$f^2(x,y) = x^2 + y^2 - 6y + 9 = 1 - 6y + 9 = 10 - 6y.$$ +6

This is a linear function in y, thus it achieves the absolute max at $y = -1$, corresponding to $f^2(0,-1) = 16$, and absolute minimum at $y = 1$, corresponding to $f^2(0,1) = 4$. +6

Compare the extreme values on the boundary, and in the interior of R, one concludes

(a) (**10 pts**) What is the absolute maximum of f on R: $f = 16$ at $(0,-1)$. +10

(b) (**10 pts**) What is the absolute minimum of f on R: $f = 4$ at $(0,1)$ +10

Q3. Find the volume of the solid body determined by the following surfaces and regions.

(a) (**10 pts**) Below the surface $z = 2e^{-y}$ and above the region $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$.

$$I = \int_0^1 \int_0^2 2e^{-y} dy dx = \int_0^1 \cancel{2()} dx$$

$$\int_0^2 2e^{-y} dy = 2(-e^{-y})\Big|_0^2 = 2(-e^{-2}+1)$$

(+5)

$$I = 2\int_0^1 (1-e^{-2})\, dx =$$

(+5)

$$= 2(1-e^{-2}).$$

(b) (**10 pts**) Below the surface $z = 24x^5 e^{x^3 y}$ and above the region $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$.

$$\int_0^2 \int_0^1 24x^5 e^{x^3 y} \, dx\, dy = \iint dy\, dx$$

$$\int_0^2 24x^5 e^{x^3 y} dy = 24x^5 \frac{e^{x^3 y}}{x^3}\Big|_0^2$$

$$= 24x^2 (e^{2x^3} - 1)$$

$$\int_0^1 24x^2 (e^{2x^3} - 1)\, dx = \underbrace{\int_0^1 24x^2 e^{2x^3}}_{A} - \underbrace{\int_0^1 24x^2 dx}_{B}$$

(+5) (+5)

$$A = \int_0^2 4e^u du = 4e^u\Big|_0^2 = 4(e^2-1)$$

$u = 2x^3$

$du = 6x^2 dx$

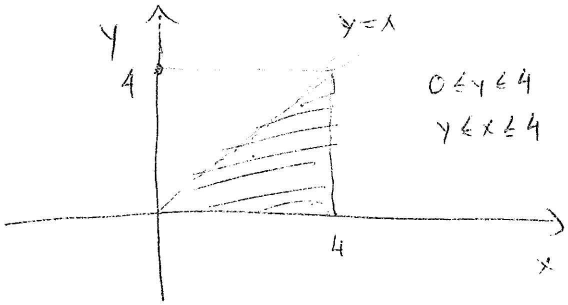
$$B = 24\frac{x^3}{3}\Big|_0^1 = 8$$

$$A+B = 4(e^2-1)+8. = 4(e^2+1)$$

Q4. Consider the following integral

$$I = \int_0^4 \int_y^4 x^2 e^{xy}\, dx dy.$$

(a) (**6 pts**) Sketch the region R of integration.



$0 \le y \le 4$

$y \le x \le 4$

+6

(b) (**4pts**) Rewrite the integral by changing the order of integration.

$$I = \int_0^4 \int_0^x x^2 e^{xy} \, dy \, dx$$

+4

(c) (**10 pts**) Evaluate the (new) integral.

$$\int_0^x x^2 e^{xy}\, dy = x^2 \cdot \frac{e^{xy}}{x} \Big|_0^x = x \cdot (e^{x^2} - 1)$$

$$\int_0^4 x \cdot e^{x^2} - x \, dx = \int_0^4 x \cdot e^{x^2} dx - \int_0^4 x \, dx$$

A B

+6 $A = \frac{1}{2} \int_0^{16} e^u du = \frac{1}{2} (e^{16} - 1).$

$u = x^2$

$du = 2x dx$

$A - B = \frac{1}{2}(e^{16} - 1) - 8$

+4 $B = \frac{x^2}{2} \Big|_0^4 = 8$

(Possible continuation of **Q3**)

(Possible continuation of **Q4**)

Q5. Complete the following steps in the procedure of Lagrange multipliers method for the following f (if they exist) subject to the given constraint

$$f(x, y) = xy + 5x + 5y, \text{ subject to } x^2y^2 = 9.$$

$g(x,y) = x^2y^2 - 9$

(a) (**10 pts**) Set up the equation system involving x, y and λ.

+4

$f_x = \lambda g_x$

$f_y = \lambda g_y$

$g = 0$

$$\begin{cases} y+5 = 2\lambda \cdot x \cdot y^2 & (1) \\ x+5 = 2\lambda \cdot x^2 y & (2) \\ x^2y^2 = 9 & (3). \end{cases}$$

+2

+2

+2

(b) (**8 pts**) Solve for x, y.

(1) − (2)

$y - x = 2\lambda xy(y - x)$

$\Rightarrow (y-x)(1 - 2\lambda xy) = 0$

$\Rightarrow$ either $\underbrace{y = x}_{\text{Case 1.}}$, but $x^2y^2 = 9 \Rightarrow$ $x = \sqrt{3}$ or $-\sqrt{3}$, $y = \sqrt{3}$ or $-\sqrt{3}$.

$\underbrace{1 - 2\lambda xy}_{\text{Case 2}} = 0 \Rightarrow 2\lambda xy = 1$. Substitute back into (1), we have

$y+5 = 2\lambda \cdot x \cdot y \cdot y =$

$y+5 = y$

$5 = 0$ CONTRADICTION

Thus Case 1 is the only case, $(x,y) = (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$

$f(x,y) = \underbrace{3 + 10\cdot\sqrt{3}}_{\text{abs. max}}$ $\quad$ $3 - 10\sqrt{3}$ abs. min

Bonus. (5pts) Consider the following integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy.$$

Evaluate I using polar coordinate.

$x = r\cos\theta \qquad 0 \le r \le \infty$

$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta.$$

$$\int_0^{\infty} e^{-r^2} r\,dr \underset{\substack{u=r^2 \\ du=2r\,dr}}{=} \frac{1}{2}\int_0^{\infty} e^{-u}\,du$$

$$= \frac{1}{2}\left. -e^{-u}\right|_0^{\infty}$$

$$= \frac{1}{2}.$$

$$\int_0^{2\pi} \frac{1}{2}\,d\theta = \boxed{\pi}.$$

(c) (**4pts**) Find the maximum and minimum of f.