2nd Midterm

Q1.

(a) (**10 pts**) A solid box D is bounded by the planes $x = 0, x = 2, y = 0, y = 2$ and $z = 1, z = 4$. The density of the box is x^2y/z. Find the total mass of the box.

$$\int\int_D\int \frac{x^2y}{z} dV = ?$$

$$\iiint_D \frac{x^2y}{z}\, dV = \int_0^2\int_0^2\int_1^4 \frac{x^2y}{z}\, dz\, dy\, dx \quad (+1)$$

$$x^2y \ln z \Big|_1^4 \quad (+3)$$

$$\int_0^2 x^2y \ln 4 \, dy$$

$$\ln 4\, x^2 \frac{y^2}{2}\Big|_0^2 = \int \ln 4 \cdot 2 \cdot x^2 dx \quad (+3)$$

$$2\ln 4 \int_0^2 x^2 dx = 2 \ln 4 \cdot \frac{x^3}{3}\Big|_0^2 \quad (+3)$$

$$= \frac{16 \ln 4}{3}.$$

(b) (**10 pts**) Evaluate the following integral

$$\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y-z}dxdydz.$$

$$e^{y-z} \quad e^{x}\Big|_0^{\ln 2}$$

$$e^{y-z}(2-1) = e^{y-z}$$

(+3)

$$\int_0^{\ln 3} e^{y-z}dy = e^{-z} \quad e^{y}\Big|_0^{\ln 3} = e^{-z}(3-1)$$

$$= 2e^{-z}$$

(+3)

$$2\int_0^{\ln 4} e^{-z}dz = 2\left(-e^{-z}\Big|_0^{\ln 4}\right) =$$

$$= 2\left(-\frac{1}{4}-(-1)\right) =$$

$$= 2\left(\frac{3}{4}\right) = \frac{3}{2}.$$

(+4)

(c) **(10 pts)** Evaluate the following integral

$$\int_0^4\int_{x^{1/2}}^2\int_0^2 \frac{z}{y^3+1}dzdydx.$$

(Possible continuation of **Q1**)

$$\int_0^2 \frac{z}{y^3+1}\, dz = \frac{1}{y^3+1} \frac{z^2}{2}\Big|_0^2 = \frac{2}{y^3+1}$$ (+4)

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{2}{y^3+1}\, dy\, dx =$$

$$= \int_0^2 \int_0^{y^2} \frac{2}{y^3+1}\, dx\, dy$$ (+4)

$\sqrt{x} = y$

$x = y^2$

$$= \int_0^2 \frac{2y^2}{y^3+1}\, dy$$

$u = y^3+1$

$du = 3y^2\, dy = \frac{3}{2} \cdot 2y^2 \cdot dy$ (+2)

$2y^2 dy = \frac{2}{3}\, du$

$$\frac{2}{3} \int \frac{1}{u}\, du$$

$$\frac{2}{3} \left(\ln(8) - \ln 1 \right) =$$

$$\frac{2}{3} \ln 8 .$$

Q2. **(15 pts)** Let R be the parallelogram bounded by the lines $y = -x+2; y = -x+4; y = 2x$ and $y = 2x+2$. Evaluate

$$\int\int_R 9x\,dA.$$

(Hint: $u = x+y, v = -2x+y$.)

$u = x+y$

$v = -2x+y$ $\quad u - v = 3x \quad x = \frac{u-v}{3}$ $\quad$ +2

$2u+v = 3y$

$y = \frac{2u+v}{3}$ $\quad$ +2

$|J(u,v)| = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$ $\quad$ +2

$9x = 3(u-v)$ $\quad$ +2

$\frac{1}{3}\int_2^4\int_0^2 3(u-v)\,dv\,du$ $\quad$ +2

$6u - 3\int_0^2 v\,dv$

$3\frac{v^2}{2}\Big|_0^2 = 6$ $\quad$ +3

$\int_2^4 6u - 6\,du$

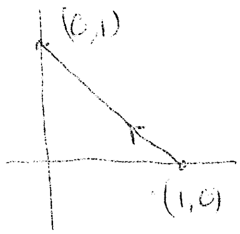
$6\frac{u^2}{2} - 6u\Big|_2^4 = 6\cdot(8-2) - 6(4-2)$ $\quad$ +2

$= 36 - 12 = \boxed{24}\cdot\frac{1}{3}$

$\boxed{8}$

Q3.

(a) **(15 pts)** Let C_1 be the line segment from $A(1,0)$ to $B(0,1)$. Evaluate the line integral of $\mathbf{F} = \langle x, -y \rangle$ on C_1.



$r(t) = \langle 1,0 \rangle + t\,\langle -1,1 \rangle \qquad 0 \le t \le 1$

$= \langle 1-t, t \rangle$

$= \langle x(t), y(t) \rangle$

$\int_a^b f x' + g y' \, dt =$

$= \int_0^1 (1-t)\cdot(-1) + (-t)\cdot(1) \, dt$

$t - 1 \quad -t$

-1

$\int_0^1 -1 \, dt = \boxed{-1}$

(b) **(15 pts)** Let C_2 be the counter-clockwise path around the circle with radius 1 centered at $(2,4)$. Calculate the **flux** of the field $\mathbf{F} = \langle x, y \rangle$ across the plane curve C_2.

$(\cos t, \sin t) \qquad 0 \le t \le 2\pi$

+5

$(2 + \cos t, \; 4 + \sin t)$

$x(t) \qquad y(t)$

(Possible continuation of **Q3**)

(+2)

$$\int_a^b f\, y'(t) - g\, x'(t)\, dt$$

$$\int_0^{2\pi} (2+\cos t)\cdot \cos t - (4+\sin t)\cdot(-\sin t)$$

$$2\cos t + \cos^2 t + 4\sin t + \sin^2 t$$

(+5)

$$\int_0^{2\pi} 2\cos t + 4\sin t + 1 \quad dt$$

$$2\ (\sin t)\Big|_0^{2\pi} + 4\ (-\cos t)\Big|_0^{2\pi} + 2\pi$$

$$0 \qquad\qquad 0 \qquad\qquad 2\pi$$

(+3)

$$\boxed{2\pi}$$

Q4.

above the cone $z = \sqrt{x^2+y^2}$

below the paraboloid $z = 12 - x^2 - y^2$.

cylindrical / polar coordinates

(r, θ, z) $\quad x = r\cos\theta, \ y = r\sin\theta$

$x^2+y^2 = r^2, \quad r \geq 0, \quad \theta \in [0, 2\pi]$

$\sqrt{x^2+y^2} \leq 12 - x^2 - y^2$

+2 $\quad r \leq 12 - r^2$

$0 \leq r \leq 3$

+5 $\quad$ Volume $= \displaystyle\int_0^{2\pi}\int_0^3\int_r^{12-r^2} r \cdot dz \, dr \, d\theta =$

$$= \int_0^{2\pi}\int_0^3 (12 - r^2 - r) \, r \, dr \, d\theta =$$

+3 $\quad = \displaystyle\int_0^{2\pi} 12 \cdot \frac{r^2}{2}\Big|_0^3 - \frac{r^4}{3}\Big|_0^3 - \frac{r^3}{3}\Big|_0^3 \, d\theta$

$$= \frac{99\pi}{2}.$$

Q5. Find the gradient fields F of the following functions.

(a) (6 pts)

$$f(x,y,z) = (\frac{y+z}{x+y})^4.$$

$$\mathbf{F} = \nabla f = ?$$

+2 $\quad f_x = \dfrac{-(y+z)^4 \cdot (x+y)^3 \cdot 4}{(x+y)^8} = -\dfrac{(y+z)^4}{(x+y)^5}$

+2 $\quad f_y = \dfrac{4(y+z)^3 (x+y)^4 - 4(y+z)^4 (x+y)^3}{(x+y)^8} = \dfrac{4(y+z)^3 (x+y-(y+z))}{(x+y)^5}$

$= \dfrac{4(y+z)^3 (x-z)}{(x+y)^5}$

+2 $\quad f_z = \dfrac{4(y+z)^3 (x+y)^4 - 4(y+z)^4 \cdot 0}{(x+y)^8} = \dfrac{4(y+z)^3}{(x+y)^4}$

(b) (4 pts)

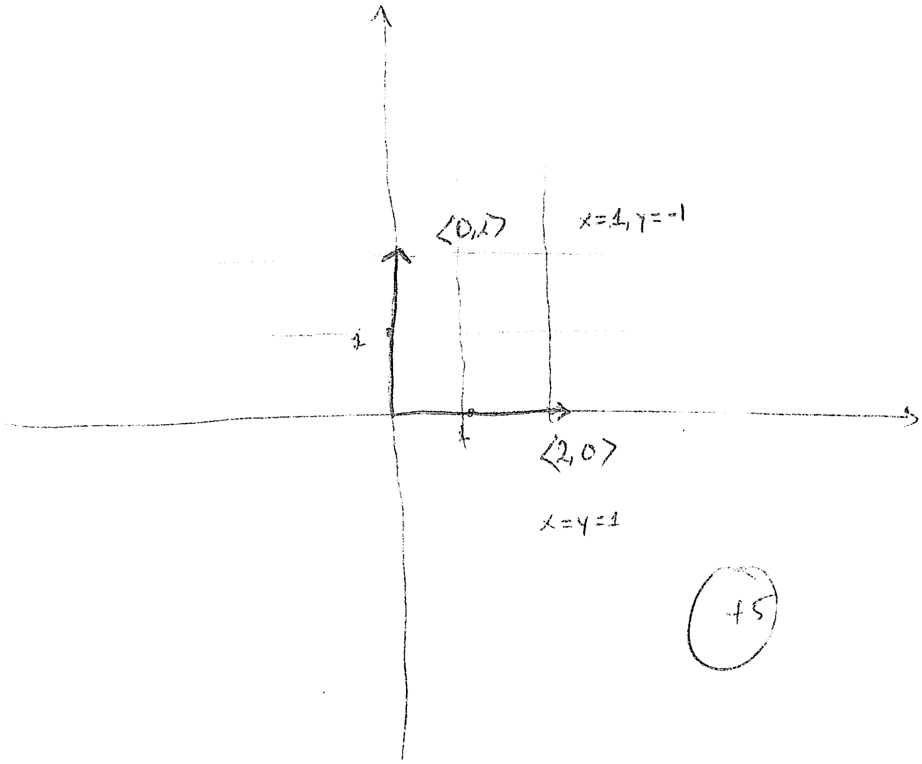
$$\varphi(x,y) = (x^2 + 2xy - y^2)/2.$$

$$\mathbf{F} = \nabla\varphi = ?$$

$= \langle \frac{2x+2y}{2}, \frac{2x-2y}{2} \rangle = \langle x+y, x-y \rangle.$

+2 +2

(c) (**5 pts**) For **F** from (b), draw (any) two representative vectors (not at the origin) of the vector field on the xy-plane.



+5

(d) (**Bonus 5 pts**) For **F** from (b), show that the vector field is orthogonal to the equipotential curve at all points (x, y).

This is due to the fact that $F = \langle \varphi_x, \varphi_y \rangle$

while the tangent line to the equipotential curve at (x,y) has slope $-\dfrac{\varphi_x}{\varphi_y}$ +4

(Recall that the tangent line has eqn:

$\varphi_x \cdot (x - x_0) + \varphi_y (y - y_0) = 0$)

Thus the tangent line is parall to $\langle \varphi_y, -\varphi_x \rangle$

One checks that F is orthogonal to this direction because

$\langle \varphi_x, \varphi_y \rangle \cdot \langle \varphi_y, -\varphi_x \rangle = \varphi_x \cdot \varphi_y - \varphi_y \varphi_x = 0$. +1