EIGENVECTORS OF RANDOM MATRICES OF SYMMETRIC ENTRY DISTRIBUTIONS

SEAN MEEHAN AND HOI NGUYEN

Abstract. In this short note we study a non-degeneration property of eigenvectors of symmetric random matrices with entries of symmetric sub-gaussian distributions. Our result is asymptotically optimal under the sub-exponential regime.

1. introduction

Let $\mathbf{x}$ be a random vector uniformly distributed on the unit sphere S^{n-1}, where $n \to \infty$. It is well known that $\mathbf{x}$ can be represented as

$$\mathbf{v} := (\frac{\xi_1}{S}, \dots, \frac{\xi_n}{S})$$

where ξ_i are iid standard Gaussian and $S = \sqrt{\sum_{i=1}^n |\xi_i|^2}$. One then can deduce that for any deterministic vector $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{R}^n$ with $\sum_i f_i^2 = n$,

$$\mathbf{f}^T \mathbf{v} \xrightarrow{d} \mathbf{N}(0,1).$$

We also refer the reader to the survey [17] for further nice properties of $\mathbf{x}$.

Let M_n be a random symmetric matrix of size $n \times n$ of real-valued entries. When M_n is GOE, then by the rotation invariance, the individual eigenvectors of M_n have the same distribution as $\mathbf{x}$ above. One then can deduce various nice properties of these eigenvectors. Motivated by the *universality phenomenon*, it is natural to ask if these properties are universal.

Question 1.1. *Is it true that the eigenvectors of M_n are "asymptotically uniformly distributed" for more general random ensemble M_n?*

We assume for the moment that M_n has simple spectrum. Let $\lambda_1 < \dots < \lambda_n$ be the real eigenvalues of M_n, and $\mathbf{u}_1, \dots, \mathbf{u}_n$ be the corresponding unit eigenvectors (which are unique up to a sign). Among many nice results, the followings can be read from [26, Theorem 13] and [1, Theorem 1.2] regarding Question 1.1.

Theorem 1.2. *Let M_n be a random symmetric matrix where $m_{ij}, 1 \le i \le j \le n$ are iid copies of a random variable ξ. Let $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{R}^n$ be any deterministic vector with $\sum_i f_i^2 = n$.*

- [26] *Assume that ξ is symmetric, $\xi \overset{d}{=} -\xi$, and has moment matching up to the fourth order with $\mathbf{N}(0,1)$. Then for any $1 \le i \le n$,*

Key words and phrases. Random matrices, eigenvectors, delocalization.
The authors are supported by NSF grant DMS 1600782.

$$\mathbf{f}^T\mathbf{u}_i \xrightarrow{d} \mathbf{N}(0,1).$$

More precisely, there exists a positive constant c such that for any $x > 0$

$$\mathbf{P}(|\mathbf{f}^T\mathbf{u}_i| \le x) = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt + O(n^{-c}). \tag{1}$$

- [1] *Assume that ξ has mean zero, variance one, and having finite moment of all orders. Then* (1) *holds for any eigenvector $\mathbf{u}_i$ with $i \in [1, n^{1/4}] \cup [n^{1-\delta, n-n^{1-\delta}}] \cup [n - n^{1/4}, n]$, with possibly different c.*

We also refer the readers to [26] and [1] for further beautiful results such as the joint independence and gaussianity of the eigenvectors.

Note that the constants c above can be made explicit but are rather small in both results. Thus, assume that if we are interested in the tail bound estimates $|\mathbf{f}^T\mathbf{u}_i| \le x$, then the above results are less effective when $x \ll n^{-c}$. In fact, it was not even known whether asymptotically almost surely $\mathbf{f}^T\mathbf{u}_i \neq 0$. This question was raised by Dekel, Lee and Linial in [5] for $\mathbf{f} = (1, 0, \dots, 0)$ in connection to the notion of strong and weak nodal domains in random graph $G(n,p)$. This question has been confirmed in [14] in the following form.

Theorem 1.3. *Assume that F_n is a symmetric matrix with $\|F_n\|_2 \le n^\gamma$ for some constant $\gamma > 0$. Consider the matrix $M_n + F_n$ with the random symmetric matrix M_n of entries $m_{ij}, 1 \le i < j \le n$, being iid copies of a random variable ξ of mean zero, variance one, and bounded $(2+\varepsilon)$-moment for given $\varepsilon > 0$. Then for any A, there exists B depending on A and γ, ε such that*

$$\mathbf{P}\Big(\exists \text{ a unit eigenvector } \mathbf{u} = (u_1, \dots, u_n) \text{ of } M_n \text{ with } |u_i| \le n^{-B} \text{ for some } i\Big) = O(n^{-A}).$$

Although the above result holds for very general matrices, the approach does not seem to extend to the case that $\mathbf{f}$ has many non-zero entries, which is the main focus of our current note.

Condition 1.1. *Let c, K_1, K_2 be positive parameters.*

- *(assumption for $\mathbf{f}$) We assume that the following holds for all but cn indices $1 \le i \le n$*

$$n^{-c} \le |f_i| \le n^c.$$

- *(assumption for M_n) We assume that the entries of $m_{ij}, 1 \le i \le j \le n$, are iid copies of a random variable ξ of mean zero, variance one, and so that*
  - *For every $t > 0$,*

$$\mathbf{P}(|\xi| \ge t) \le K_1 \exp(-t^2/K_2), \tag{2}$$

  - *ξ is symmetric.*

For the rest of this note we will be conditioning on the following result.

Theorem 1.4. [27, 14] *With M_n as above, there exists a constant $c > 0$ such that with probability at least $1 - \exp(-n^c)$, M_n has simple spectrum.*

In the above setting, we are able to prove the following

Theorem 1.5 (Main result)**.** *Let M_n and $\mathbf{f}$ be as in Condition 1.1 for some positive constants K_1, K_2, and for some sufficiently small constant c. Conditioning on the event of Theorem 1.4, let $\lambda_1 < \cdots < \lambda_n$ be the eigenvalues of M_n and $\mathbf{u}_1, \dots, \mathbf{u}_n$ be the associated eigenvectors. Then the following holds for any $\delta \geq \exp(-n^c)$*

$$\mathbf{P}\Big(\sup_i |\langle \mathbf{u}_i, \mathbf{f}\rangle| \leq \delta\Big) \leq n^c \delta.$$

It seems that our result can also be extended to the case when m_{ij} and m_{ii} have different distributions, but we will not focus on this setting for simplicity. The current method does not extend to non-symmetric ξ, although we believe that our result should hold in this generality.

In what follows we connect our result to the study of controllability of matrices. Consider the discrete-time linear state-space system whose state equation is

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k).$$

In the above, A and B are $n \times n$ and $n \times r$ matrices, respectively, while each $\mathbf{u}(k)$ is an $r \times 1$ vector that we wish to solve for based on the state values $\mathbf{x}(k)$ of size $n \times 1$.

We say that our system is controllable if we can always find the control values $\mathbf{u}(n-1), \mathbf{u}(n-2), \cdots, \mathbf{u}(0)$ based on the state values $\mathbf{x}(\cdot)$. Note that

$$\begin{aligned}
\mathbf{x}(1) &= A\mathbf{x}(0) + B\mathbf{u}(0) \\
\mathbf{x}(2) &= A\mathbf{x}(1) + B\mathbf{u}(1) = A^2\mathbf{x}(0) + AB\mathbf{u}(0) + B\mathbf{u}(1) \\
&\ \vdots \\
\mathbf{x}(n) &= A^n\mathbf{x}(0) + A^{n-1}B\mathbf{u}(0) + A^{n-2}B\mathbf{u}(1) + \cdots + AB\mathbf{u}(n-2) + B\mathbf{u}(n-1).
\end{aligned}$$

That is

$$\mathbf{x}(n) - A^n\mathbf{x}(0) = (A^{n-1}B \quad A^{n-2}B \quad \cdots \quad AB \quad B)(\mathbf{u}^T(0) \quad \mathbf{u}^T(1) \quad \cdots \quad \mathbf{u}^T(n-1))^T.$$

From here it is easy to see that we can always find the control values $\mathbf{u}(\cdot)$ if and only if the left matrix has full rank. Restricting to the case where $r = 1$ and switching around columns to remain consistent with the literatures, this motivates the following definition of controllability.

Definition 1.6. Let A be an $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. We say that the pair $(A, \mathbf{b})$ is controllable if the $n \times n$ column matrix

$$(\mathbf{b} \quad A\mathbf{b} \quad \cdots \quad A^{n-1}\mathbf{b})$$

has full rank.

As it turns out, the notion of controllability is related to the existence of eigenvectors orthogonal to $\mathbf{b}$ via the Popov-Belevitch-Hautus test [15].

Theorem 1.7. *With A and $\mathbf{b}$ as above, $(A, \mathbf{b})$ is uncontrollable if and only if there exists an eigenvector $\mathbf{v}$ of A such that $\langle \mathbf{b}, \mathbf{v}\rangle = 0$.*

This is [16, Lemma 1], we insert it here for completeness.

Proof. (of Theorem 1.7) The backward direction follows almost immediately. Indeed, if we can find an eigenvalue-eigenvector pair $(\lambda, \mathbf{v})$ of A such that $\mathbf{v}^T\mathbf{b} = 0$, then for each k, we have $\mathbf{v}^T A^k \mathbf{b} = \lambda^k \mathbf{v}^T \mathbf{b} = 0$. Letting A' denote the controllability matrix in Definition 1.6, we have that $\mathbf{v}^T A' = 0$ and thus A' is uncontrollable.

For the forward direction, suppose that each eigenvector $\mathbf{v}$ satisfies $\mathbf{v}^T\mathbf{b} \neq 0$. Then each eigenspace of A has dimension one (if we can find an eigenspace of dimension at least 2, then considering the intersection of that eigenspace with the orthogonal complement of the subspace spanned by $\mathbf{b}$ leads us to an eigenvector $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{b} = 0$). Since A is symmetric, it thus follows that the eigenvalues are distinct so that A has simple spectrum. Now suppose that the spectrum of A is simple and assume that $(A, \mathbf{b})$ is uncontrollable, i.e. we can find a nonzero vector $\mathbf{a} = (a_0, \cdots, a_{n-1})$ such that $A'\mathbf{a} = 0$, where

$$A' = (\mathbf{b} \quad A\mathbf{b} \quad \cdots \quad A^{n-1}\mathbf{b})$$

is our controllability matrix. Further suppose that our eigenvalue-eigenvector pairs are denoted $(\lambda_i, \mathbf{v}_i)$ with $\lambda_1 < \cdots < \lambda_n$. We begin to use the spectral theorem to decompose each $A^k\mathbf{b}$ as

$$A^k\mathbf{b} = \sum_{j=1}^{n} (\lambda_j^k \mathbf{v}_j^T \mathbf{b}) \mathbf{v}_j.$$

Since $A'\mathbf{a} = 0$, we have that

$$0 = A'\mathbf{a} = \sum_{k=0}^{n-1} a_k A^k \mathbf{b} = \sum_{k=0}^{n-1} a_k \sum_{j=1}^{n} (\lambda_j^k \mathbf{v}_j^T \mathbf{b}) \mathbf{v}_j = \sum_{j=1}^{n} \mathbf{v}_j (\sum_{k=0}^{n-1} \mathbf{v}_j^T \mathbf{b} \lambda_j^k a_k).$$

Letting

$$\beta_j = \sum_{k=0}^{n-1} \mathbf{v}_j^T \mathbf{b} \lambda_j^k a_k,$$

we have that each $\beta_j = 0$ by linear independence of our eigenbasis. Write

$$\beta_j = \mathbf{v}_j^T \mathbf{b} \begin{pmatrix} 1 \\ \lambda_j^1 \\ \vdots \\ \lambda_j^{n-1} \end{pmatrix}^T \mathbf{a}.$$

Since each $\mathbf{v}_j^T \mathbf{b} \neq 0$ by assumption, it must then be the case that

$$\begin{pmatrix} 1 \\ \lambda_j^1 \\ \vdots \\ \lambda_j^{n-1} \end{pmatrix}^T \mathbf{a} = 0.$$

But this implies that the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}^T$$

is singular, and hence $\lambda_i = \lambda_j$ for some $i \neq j$, a contradiction. □

Recent developments in the area of matrix controllability have come from imposing randomness on the matrix A and imposing varying rigidity on the deterministic vector $\mathbf{b}$. For example, in [16] O'Rourke and Touri were able to prove the following conjecture of Godsil.

Conjecture 1.8. *Let $\mathbf{1}_n$ be the vector in $\mathbb{R}^n$ consisting of all 1's and A_n be the adjacency matrix of $G(n, 1/2)$. Then as n approaches infinity, $(A_n, \mathbf{1}_n)$ is controllable asymptotically almost surely.*

This has been verified recently by O'Rourke and Touri in stronger form. To state their result, we first introduce a notion called $(K, \delta)-$*delocalization*.

Definition 1.9. Let K, δ be positive parameters. We say that a unit vector $\mathbf{v} = (v_1, \cdots, v_n)$ is (K, δ)-delocalized if at least $n - \lfloor \delta n \rfloor$ coordinates k satisfy the following

(1) $v_k = \frac{p_k}{q_k}$, where $p_k, q_k \in \mathbb{Z}$,
(2) $|p_k|, |q_k| \leq K$,
(3) $p_k, q_k \neq 0$.

Thus if $\mathbf{v}$ is (K, δ)-delocalized then most of the entries of $\mathbf{v}$ are non-zero rational numbers of bounded height. Through this notion, the authors of [15, 16] were able to prove Godsil's conjecture by the following theorem.

Theorem 1.10. [15, Theorem 3.4] *Assume that M_n is a random symmetric matrix where the off-diagonal entries $m_{ij}, 1 \leq i < j \leq n$, are iid copies of ξ as in Theorem 1.5, while the diagonal entries are iid copies of a possibly different subgaussian random variable ζ. Fix $K \geq 1$ and $\alpha > 0$. Then there exist constants $C > 0$ and $\delta \in (0, 1)$ (depending on K, α, ξ, and ζ) such that the following holds. Let $\mathbf{b}$ be a vector in $\mathbb{R}^n$ which is $(K, \delta)-$delocalized. Then $(M_n, \mathbf{b})$ is controllable with probability at least $1 - Cn^{-\alpha}$.*

Our result, Theorem 1.5, can be seen as a near optimal generalization of Theorem 1.10 (in the case that m_{ii} and m_{ij} have the same distribution) where the entries of $\mathbf{f}$ are not necessarily rational.

Notations. Throughout this paper, we regard n as an asymptotic parameter going to infinity (in particular, we will implicitly assume that n is larger than any fixed constant, as our claims are all trivial for fixed n), and allow all mathematical objects in the paper to implicitly depend on n unless they are explicitly declared to be "fixed" or "constant". We write $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote the claim that $|X| \leq CY$ for some fixed C; this fixed quantity C is allowed to depend on other fixed quantities such as K_1, K_2 of ξ unless explicitly declared otherwise. We also use $o(Y)$ to denote any quantity bounded in magnitude by $c(n)Y$ for some $c(n)$ that goes to zero as $n \to \infty$. For a square matrix M_n and a number λ, for short we will write $M_n - \lambda$ instead of $M_n - \lambda I_n$. All the norms in this note, if not specified, will be the usual ℓ_2-norm.

2. Supporting ingredients and existing results

In this section we introduce the necessary tools to prove our main result. First of all, for the rest of the note we will condition on the following event, which is known to hold with probability $1 - \exp(-\Theta(n))$

$$\|M_n\| \leq 10\sqrt{n}. \quad (3)$$

2.1. **Approximate eigenvectors are not asymptotically sparse.** We first need the definition of compressible and incompressible vectors.

Definition 2.1. Let $c_0, c_1 \in (0,1)$ be two numbers (chosen depending on the parameters K_1, K_2, K_1', K_2' of ξ, ζ) A vector $\mathbf{x} \in \mathbb{R}^n$ is called *sparse* if $|\mathbf{supp}(\mathbf{x})| \le c_0 n$. A vector $\mathbf{x} \in S^{n-1}$ is called *compressible* if $\mathbf{x}$ is within Euclidean distance c_1 from the set of all sparse vectors. A vector $\mathbf{x} \in S^{n-1}$ is called *incompressible* if it is not compressible.

The sets of compressible and incompressible vectors in S^{n-1} will be denoted by $\mathbf{Comp}(c_0, c_1)$ and $\mathbf{Incomp}(c_0, c_1)$ respectively.

Regarding the behavior of $M_n\mathbf{x}$ for compressible vectors, the following was proved in [28].

Lemma 2.2. [28, Proposition 4.2] *There exist positive constants c_0, c_1 and α_0 (depending on K_1, K_2 of ξ) such that the following holds for any λ_0 of order $O(\sqrt{n})$. For any fixed $\mathbf{u} \in \mathbb{R}^n$ one has*

$$\mathbf{P}(\inf_{\mathbf{x} \in \mathbf{Comp}(c_0,c_1)} \|(M_n - \lambda_0)\mathbf{x} - \mathbf{u}\| \ll \sqrt{n}) = O(\exp(-\alpha_0 n)).$$

We deduce the following immediate consequence.

Lemma 2.3 (Approximate eigenvectors are not asymptotically sparse)**.** *There exist positive constants c_0, c_1 and α_0 (depending on K_1, K_2 of ξ) such that*

$$\mathbf{P}\left(\exists \text{ a unit vector } \mathbf{v} \in \mathbf{Comp}(c_0, c_1) \text{ and } \lambda = O(\sqrt{n}) \text{ such that } \|(M_n - \lambda)\mathbf{v}\| \ll \sqrt{n}\right) = O(\exp(-\alpha_0 n)).$$

Proof. (of Lemma 2.3) Assuming (3), we can find λ_0 as a multiple of n^{-2} inside $[-10\sqrt{n}, 10\sqrt{n}]$ such that $|\lambda - \lambda_0| \le n^{-2}$. Hence

$$\|(M_n - \lambda_0)\mathbf{v}\| = \|(\lambda - \lambda_0)\mathbf{v}\| \le n^{-2}.$$

On the other hand, for each fixed λ_0, by Lemma 2.2,

$$\mathbf{P}(\exists \mathbf{v} \in \mathbf{Comp}(c_0, c_1) : \|(M_n - \lambda_0)\mathbf{v}\| \le n^{-2}) = O(\exp(-\alpha_0 n)).$$

The claim follows by a union bound with respect to λ_0. □

2.2. **Approximate eigenvectors cannot have structures.** We next introduce a concept developed by Rudelson and Vershynin via the notion of *least common denominator* (see [18]). Fix parameters κ and γ (which may depend on n), where $\gamma \in (0,1)$. For any nonzero vector $\mathbf{x}$ define

$$\mathbf{LCD}_{\kappa,\gamma}(\mathbf{x}) := \inf\left\{\theta > 0 : \operatorname{dist}(\theta\mathbf{x}, \mathbf{Z}^n) < \min(\gamma\|\theta\mathbf{x}\|, \kappa)\right\}.$$

Theorem 2.4 (Small ball probability via LCD)**.** [18] *Let ξ be a sub-gaussian random variable of mean zero and variance one, and let $\xi_1, \dots, \xi_n$ be iid copies of ξ. Consider a vector $\mathbf{x} \in \mathbb{R}^n$. Then, for every $\kappa > 0$ and $\gamma \in (0,1)$, and for*

$$\varepsilon \ge \frac{1}{\mathbf{LCD}_{\kappa,\gamma}(\mathbf{x}/\|\mathbf{x}\|)},$$

we have

$$\rho_\varepsilon(\mathbf{x}) = O\left(\frac{\varepsilon}{\gamma\|\mathbf{x}\|} + e^{-\Theta(\kappa^2)}\right),$$

where the implied constants depend on ξ.

One of the key properties of vectors of small **LCD** is that they accept a fine net of small cardinality (see [19, Lemma 4.7] and also [14, Lemma B6] for the current form).

Lemma 2.5. *Let $D_0 \geq c\sqrt{n}$. Then the set $\{\mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| \leq 1, c\sqrt{m} \leq \mathbf{LCD}_{\kappa,\gamma}(\mathbf{x}/\|\mathbf{x}\|) \leq D_0\}$ has a $(2\kappa/D_0)$-net of cardinality at most $(C_0 D_0/\sqrt{m})^m D_0^2$ for some absolute constant C_0.*

For the rest of our paper $\gamma = 1/2$ and $\kappa = n^{2c}$ for some constant c chosen sufficiently small (compared to all other parameters).

To deal with symmetric or Hermitian Wigner matrices, it is more convenient to work with the so-called *regularized least common denominator*. Let $\mathbf{x} = (x_1, \dots, x_n) \in S^{n-1}$. Let $c_0, c_1 \in (0,1)$ be given constants, and assume $\mathbf{x} \in \mathbf{Incomp}(c_0, c_1)$. It is not hard to see that (see for instance [18, Section 3]) there are at least $c_0 c_1^2 n/2$ coordinates x_k of $\mathbf{x}$ which satisfy

$$\frac{c_1}{\sqrt{2n}} \leq |x_k| \leq \frac{1}{\sqrt{c_0 n}}. \tag{4}$$

Thus for every $x \in \mathbf{Incomp}(c_0, c_1)$ we can assign a subset $\mathbf{spread}(\mathbf{x}) \subset [n]$ such that (4) holds for all $k \in \mathbf{spread}(\mathbf{x})$ and

$$|\mathbf{spread}(\mathbf{x})| = \lceil c' n \rceil,$$

where we set

$$c' := c_0 c_1^2/4. \tag{5}$$

Definition 2.6 (Regularized LCD, see also [28])**.** Let $\alpha \in (0, c'/4)$. We define the *regularized LCD* of a vector $\mathbf{x} \in \mathbf{Incomp}(c_0, c_1)$ as

$$\widehat{\mathbf{LCD}}_{\kappa,\gamma}(\mathbf{x}, \alpha) = \max\Big\{\mathbf{LCD}_{\kappa,\gamma}\big(\mathbf{x}_I/\|\mathbf{x}_I\|\big) : I \subseteq \mathbf{spread}(\mathbf{x}), |I| = \lceil \alpha n \rceil\Big\}.$$

Roughly speaking, the reason we choose to work with $\widehat{\mathbf{LCD}}$ is that we want to detect structure of $\mathbf{x}$ in sufficiently small segments. From the definition, it is clear that if $\mathbf{LCD}(x)$ is small (i.e. when x has strong structure), then so is $\widehat{\mathbf{LCD}}(x, \alpha)$.

For given D, κ, γ and α, we denote the set of vectors of norm $1 + o(1)$ with bounded regularized LCD by

$$T_{D,\kappa,\gamma,\alpha} := \{\mathbf{x} \in \mathbf{Incomp}(c_0, c_1) : \widehat{\mathbf{LCD}}_{\kappa,\gamma}(\mathbf{x}, \alpha) \leq D\}.$$

The following is [14, Lemma 5.9].

Lemma 2.7. *Assume that M_n is a random Wigner matrix with subgaussian entries. Then there exist $c > 0, \alpha_0 > 0$ depending on c_0, c_1 from Lemma 2.3 such that the following holds with $\kappa = n^{2c}$ and $\gamma = 1/2$. Let α, D be such that*

$$n^{-c} \leq \alpha \leq c'/4, \text{ and } 1 \leq D \leq n^{c/\alpha}.$$

Then for any fixed $\mathbf{u} \in \mathbb{R}^n$ and any real number λ_0 of order $O(\sqrt{n})$, with $\beta = \frac{\kappa}{\sqrt{\alpha}D}$ we have

$$\mathbf{P}\left(\exists \mathbf{x} \in T_{D,\kappa,\gamma,\alpha} : \|(M_n - \lambda_0)\mathbf{x} - \mathbf{u}\| = o(\beta\sqrt{n})\right) = O(\exp(-\alpha_0 n)),$$

We remark that, while Lemma 2.3 and Lemma 2.7 were proved for unit vectors $\mathbf{x}$, the proofs automatically extend to vectors of norm $1 \pm n^{-2c}$. For instance Lemma 2.7 can be extended to

$$\mathbf{P}\left(\exists \mathbf{x} : 1 - n^{-2c} \le \|\mathbf{x}\| \le 1 + n^{-2c} \wedge \mathbf{x}/\|\mathbf{x}\| \in T_{D,\kappa,\gamma,\alpha} : \|(M_n - \lambda_0)\mathbf{x} - \mathbf{u}\| = o(\beta\sqrt{n})\right) = O(\exp(-\alpha_0 n)).$$

Indeed, the event $\|(M_n - \lambda_0)\mathbf{x} - \mathbf{u}\| = o(\beta\sqrt{n})$ implies $\|(M_n - \lambda_0)\mathbf{x}/\|\mathbf{x}\| - \mathbf{u}/\|\mathbf{x}\|\| = o(\beta\sqrt{n})$, and the later implies that $\|(M_n - \lambda_0)\mathbf{x}/\|\mathbf{x}\| - \mathbf{u}_i\| = o(\beta\sqrt{n})$ for some deterministic $\mathbf{u}_i$ appropriately chosen to approximate $\mathbf{u}/\|\mathbf{x}\|$ with an error, say, at most β. As one can easily construct a set of size $n^{O(1)}/\beta$ for the $\mathbf{u}_i$'s, taking union bound over these approximating points will not dramatically change the exponential bound $O(\exp(-\alpha_0 n))$ of the right hand side of Lemma 2.7 as $\beta \ge \exp(-n^c)$.

We deduce the following crucial consequence from Lemma 2.3 and Lemma 2.7.

Corollary 2.8. *Let $\mathbf{u} \in \mathbb{R}^n, \lambda_0$ be fixed, and D, β be as above. Let $\mathcal{E}_{\mathbf{u},\lambda_0}$ be the event that for any $\mathbf{x}$ with $1 - n^{-2c} \le \|\mathbf{x}\| \le 1 + n^{-2c}$, if $\|(M_n - \lambda_0)\mathbf{x} - \mathbf{u}\| = o(\beta\sqrt{n})$ then $\mathbf{x}/\|\mathbf{x}\| \notin T_{D,\kappa,\gamma,\alpha}$ and $\mathbf{x}/\|\mathbf{x}\| \in$* $\mathbf{Incomp}(c_0, c_1)$. *We then have the bound*

$$\mathbf{P}(\mathcal{E}_{\mathbf{u},\lambda_0}) \ge 1 - O(\exp(-\alpha_0 n)).$$

Finally, together with the structural results above, we will also need the following result (see [18, Lemma 2.2]) to pass from small ball bounds to a total bound.

Theorem 2.9. *Let $\zeta_1, \cdots, \zeta_n$ be independent nonnegative random variables, and let $K, t_0 > 0$. If one has*

$$P(\zeta_k < t) \le Kt$$

for all $k = 1, \cdots, n$ and all $t \ge t_0$, then for all $t \ge t_0$

$$P(\sum_{k=1}^n \zeta_k^2 < t^2 n) \le O((Kt)^n).$$

We remark that all of the results in this section including Lemma 2.2, Lemma 2.3, Lemma 2.7 and Theorem 1.4 hold for matrices where the entry distributions are not necessarily symmetric.

3. Proof of Theorem 1.5

3.1. **Extra randomness.** A key observation, by using the fact that ξ is symmetric, is that if $\varepsilon_1, \dots, \varepsilon_n$ are iid Bernoulli random variables independent of M_n, then M_n and $M_n' = (\varepsilon_i \varepsilon_j m_{ij})$ have the same matrix distribution. Furthermore, a quick calculation shows that $M_n \mathbf{u} = \lambda \mathbf{u}$ if and only if $M_n' \mathbf{u}' = \lambda \mathbf{u}'$, where $\mathbf{u}' = (\varepsilon_1 u_1, \dots, \varepsilon_n u_n)$. So the eigenvalues of M_n and M_n' are identical, and the spectrum of M_n is simple if and only if the spectrum of M_n' is simple.

Lemma 3.1. [16, Lemma 10.2] *Conditioning on the event $\mathcal{E}$ that the spectrum of M_n is simple. For any $\delta > 0$ and any deterministic vector $\mathbf{f}$ we have*

$$\mathbf{P}\big(|\langle \mathbf{u}, \mathbf{f} \rangle| \le \delta | \mathcal{E}) = \mathbf{P}(|\langle \mathbf{u}', \mathbf{f} \rangle| \le \delta | \mathcal{E}\big).$$

Consequently, by Theorem 1.4,

$$\mathbf{P}\big(\sup_i |\langle \mathbf{u}_i, \mathbf{f} \rangle| \le \delta\big) \le \mathbf{P}\big(\sup_i |\langle \mathbf{u}_i', \mathbf{f} \rangle| \le \delta\big) + \exp(-n^c).$$

As the proof of this lemma is short but crucial, we insert it here for the reader's convenience.

Proof. (of Lemma 3.1) Let λ be the eigenvector associated to both $\mathbf{u}$ and $\mathbf{u}'$. Let P_λ denote the orthogonal projection of M_n onto the eigenspace associated with λ, and let P_λ' denote the orthogonal projection of M_n' onto the eigenspace associated with λ. From the fact that M_n and M_n' have the same distribution, P_λ and P_λ' also have the same distribution. Also, when our spectrum is simple, we have that $P_\lambda(\cdot) = \langle \mathbf{u}, \cdot\rangle \mathbf{u}$ and $P_\lambda'(\cdot) = \langle \mathbf{u}', \cdot\rangle \mathbf{u}'$. It thus follows that

$$\begin{aligned}\mathbf{P}(|\langle \mathbf{u}, \mathbf{f}\rangle| \le \delta \,|\, \mathcal{E}) &= \mathbf{P}(|\langle \mathbf{u}, \mathbf{f}\rangle||\mathbf{u}| \le \delta \,|\, \mathcal{E})\\ &= \mathbf{P}(|P_\lambda(\mathbf{f})| \le \delta \,|\, \mathcal{E})\\ &= \mathbf{P}(|P_\lambda'(\mathbf{f})| \le \delta \,|\, \mathcal{E})\\ &= \mathbf{P}(|\langle \mathbf{u}', \mathbf{f}\rangle| \le \delta \,|\, \mathcal{E}),\end{aligned}$$

i.e. $\mathbf{P}(|\langle \mathbf{u}, \mathbf{f}\rangle| \le \delta \cap \mathcal{E}) = \mathbf{P}(|\langle \mathbf{u}', \mathbf{f}\rangle| \le \delta \cap \mathcal{E})$. Hence

$$\mathbf{P}(|\langle \mathbf{u}, \mathbf{f}\rangle| \le \delta) \le \mathbf{P}(|\langle \mathbf{u}', \mathbf{f}\rangle| \le \delta \cap \mathcal{E}) + \exp(-n^c) \le \mathbf{P}(|\langle \mathbf{u}', \mathbf{f}\rangle| \le \delta) + \exp(-n^c),$$

as desired. □

It is remarked that one can deduce from here an almost optimal analog of (1) of Theorem 1.2, say, for the sequence $\mathbf{f} = (1, \dots, 1)$. Indeed, by Lemma 3.1 it suffices to show the comparison for $\mathbf{u}' = (\varepsilon_1 u_1, \dots, \varepsilon_n u_n)$. To this end, by the classical Berry-Esseen bound, as $\sum_i (f_i u_i)^2 = \sum_i u_i^2 = 1$ and $\max_i |u_i| \le n^{-1/2+o(1)}$ (see for instance [9, 10, 29])

$$\mathbf{P}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_i \varepsilon_i u_i f_i \le x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt + \sup_i |f_i u_i| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt + O(n^{-1/2+o(1)}).$$

3.2. **Starting from controlled sets.** Now suppose $|\langle \mathbf{u}, \mathbf{f}\rangle| = |u_1 f_1 + \dots + u_n f_n| \le \delta$ for some unit eigenvector $\mathbf{u}$ of M_n. By Lemma 3.1, the probability of this event is bounded above by the probability of the event $|\varepsilon_1 u_1 f_1 + \dots + \varepsilon_n u_n f_n| \le \delta$ for some unit eigenvector $\mathbf{u}$ of M_n and for some Bernoulli vector $(\varepsilon_1, \dots, \varepsilon_n)$. This extra randomness allows us to study our main problem as follows.

(1) (Randomness on M_n) show that with respect to M_n, the eigenvectors $\mathbf{u} = (u_1, \dots, u_n)$ of M_n does not have structure.

(2) (Randomness on $\varepsilon_1, \dots, \varepsilon_n$) conditioned on the event above, the proof is concluded by applying Theorem 2.4.

Now we look at the first step more closely. Without loss of generality we assume that $n^{-c} \le |f_1|, \dots, |f_{n_0}| \le n^c$ for $n_0 = (1-c)n$. For now we fix a parameter i and let $\mathbf{u}$ be the i-th eigenvector. Assume otherwise that

$$\mathbf{P}_{\varepsilon_1,\dots,\varepsilon_n}(|\sum_i \varepsilon_i f_i u_i| \le \delta) \ge n^{2c}\delta.$$

We are not ready to apply Theorem 2.4 yet as $\sum_i (u_i f_i)^2$ is not necessarily 1. However, by Condition 1.1 and by Lemma 2.2, provided that c is sufficiently small, it suffices to consider the case

$$n^{-c} \ll \sqrt{\sum_{i=1}^{n_0} (u_i f_i)^2} \ll n^c.$$

Approximate $\sqrt{\sum_{i=1}^{n_0} (u_i f_i)^2}$ by $\sqrt{p_j}$ where $p_j \in [-n^c, n^c]$ is an integral multiple of n^{-5c},

$$1-n^{-4c} \le \sum_i (\frac{1}{\sqrt{p_j}} f_i u_i)^2 \le 1+n^{-4c}. \quad (6)$$

Thus the event $|\sum_i \varepsilon_i f_i u_i - u| \le \delta$ implies that

$$|\sum_{i=1}^{n_0} \varepsilon_i \frac{1}{\sqrt{p_j}} f_i u_i - \frac{u}{\sqrt{p_j}}| \le n^c \delta.$$

In other words, there exists some p_j such that, with $\delta' = n^c\delta$

$$\sup_u \mathbf{P}(|\sum_i \varepsilon_i (\frac{f_i u_i}{\sqrt{p_j}} - u)| \le \delta') \ge n^c \delta'.$$

Let $\mathbf{x} = (\frac{f_1 u_1}{\sqrt{p_j}}, \dots, \frac{f_{n_0} u_{n_0}}{\sqrt{p_j}})$. By theorem 2.4, the above implies that

$$D = \mathbf{LCD}_{\gamma,\kappa}(\frac{\mathbf{x}}{\|\mathbf{x}\|}) \le \delta'^{-1}.$$

Notice that as there are many non-zero $u_i, 1 \le i \le n_0$ by Lemma 2.2 and by the assumption $\delta \ge \exp(-n^c)$,

$$\sqrt{n} \ll D \ll \exp(n^c) := D_0.$$

By Lemma 2.5, there is a set $\mathcal{S}_{j,D_0}$ (corresponding to p_j) of cardinality at most $(CD_0/\sqrt{n})^{n_0}$ which is a $(2\kappa/D_0)$-net for the set of $\mathbf{x}$ above.

For each $\mathcal{S}_{j,D_0}$, we consider the scaling map from $\mathbf{x} = (x_1, \dots, x_{n_0})$ to $\mathbf{v}' = (v_1, \dots, v_{n_0})$:

$$\mathbf{v}' := \frac{(\sqrt{p_j} x_1/f_1, \dots, \sqrt{p_j} x_{n_0}/f_{n_0})}{\|\mathbf{x}\|}.$$

This creates a new set $\mathcal{V}_{j,D_0}$ of vectors $\mathbf{v}'$ which well approximates the truncated vectors $\mathbf{u}' = (u_1, \dots, u_{n_0})$ of our eigenvector $\mathbf{u}$

$$\|\mathbf{u}'-\mathbf{v}'\| = \sqrt{\sum_{i=1}^{n_0} (u_i - \frac{\sqrt{p_j} x_i}{\|\mathbf{x}\| f_i})^2} \le \sqrt{n^{2c} \|\mathbf{x}\|^2 \sum_{i=1}^{n_0} (\frac{\|\mathbf{x}\| f_i u_i}{\sqrt{p_j}} - x_i)^2} \le n^c (1+n^{-4c}) \frac{2\kappa}{D_0} \le n^c \frac{4\kappa}{D_0}$$

We can also κ/D_0-approximate the remaining $n-n_0$ coordinates trivially by a set of size $(D_0/\kappa)^{n-n_0} = (D_0/\kappa)^{cn}$. Append this to $\mathcal{V}_{j,D_0}$ above, and take union over p_j, we obtain the following.

Theorem 3.2. *There exists a deterministic set $\mathcal{V}$ of size $n^{O(1)}(CD_0/\sqrt{n})^n (\sqrt{n}/\kappa)^{cn}$ such that for any unit vector $\mathbf{u} \in S^{n-1}$ with $\sup_u \mathbf{P}(|\sum_i \varepsilon_i f_i u_i - u| \le \delta') \ge n^c \delta'$, there exists $\mathbf{v} \in \mathcal{V}$ such that*

$$\|\mathbf{u}-\mathbf{v}\| \ll n^c \kappa / D_0.$$

Notice that by the approximation, for any $\mathbf{v} \in \mathcal{V}$

$$1 - O(n^c \kappa/D_0) \le \|\mathbf{v}\| \le 1 + O(n^c\kappa/D_0).$$

Using this approximation, if $(M_n - \lambda)\mathbf{u} = 0$ then by (3), with $\beta_0 = \kappa n^c / D_0$,

$$\|(M_n-\lambda)\mathbf{v}\| \le \sqrt{n} \beta_0.$$

From now on, let $t_i := i/D_0$. We say that $\mathbf{v}$ is an *approximate vector* of M_n if there exists i such that

$$\|(M_n - t_i)\mathbf{v}\| = O(\sqrt{n}\beta_0).$$

3.3. **Concluding the proof of Theorem 1.5.** In what follows we will choose $\alpha = n^{-6c}$, for a constant c to be chosen sufficiently small. Our main goal is to show the following.

Theorem 3.3. *With $\mathcal{V}$ from Theorem 3.2,*

$$\mathbf{P}\Big(\exists i, \exists \mathbf{v} \in \mathcal{V}, \|(M_n - t_i)\mathbf{v}\| \le \beta_0 n^{1/2}\Big) \le \exp(-\alpha_0 n).$$

It is clear that Theorem 1.5 follows from Theorem 3.3. It remains to prove Theorem 3.3 for a fixed t_i, and then take union bound over t_i (the factor of D_0 will be absorbed by $\exp(-c_0 n)$). Recall that $\beta_0 = \kappa n^c / D_0$ and $\alpha = n^{-6c}$. We now condition on the event $\mathcal{E}_{\mathbf{0},t_i}$ of Corollary 2.8 with $D = D_0$ and $\beta_1 = \kappa/\sqrt{\alpha} D_0$. On this event, if $\|(M_n - t_i)\mathbf{v}\| \le \beta_0 n^{1/2} = o(\beta_1 n^{1/2}), \mathbf{v} \in S^{n-1}$, then

$$\mathbf{v}/\|\mathbf{v}\| \in \mathbf{Incomp}(c_0, c_1) \text{ and } \widehat{\mathbf{LCD}}_{\kappa,\gamma}(\mathbf{v}/\|\mathbf{v}\|, \alpha) \ge D_0. \tag{7}$$

Consequently, on $\mathcal{E}_{\mathbf{0},t_i}$, for any $\mathbf{v} \in \mathcal{V}$ we either have $\|(M_n - t_i)\mathbf{v}\| > \beta_0 n^{1/2}$ or (7) holds for $\mathbf{v}$. So to prove Theorem 3.3 for t_i one just need to focus on these vectors $\mathbf{v}$.

Set $n' = \alpha n$. For $\mathbf{v} = (v_1, \dots, v_n)$, let $p_{\alpha,\beta}(\mathbf{v})$ be as below

$$p_{\alpha,\beta}(\mathbf{v}) = \inf_{i_1,\dots,i_{n'}} \sup_x \mathbf{P}(|\xi_{i_1} v_{i_1} + \dots + \xi_{i_{n'}} v_{i_{n'}} - x| \le \beta).$$

By splitting M_n accordingly,

$$M_n = \begin{pmatrix} M_{n-n'} & B \\ B^* & M_{n'} \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} \mathbf{v}' \\ \mathbf{v}'' \end{pmatrix},$$

where $M_{n'}$ is the $n' \times n'$ principle minor of M_n with indices $i_1, \dots, i_{n'}$ and $M_{n-n'}$ is the remaining principle minor. Here $\mathbf{v}' \in \mathbb{R}^{n-n'}$ and $\mathbf{v}'' \in \mathbb{R}^{n'}$.

So $\|(M_n - t_i)\mathbf{v}\| \le \beta_0 \sqrt{n}$ implies that

$$\|B\mathbf{v}'' - (M_{n-n'} - t_i)\mathbf{v}'\| \le \beta_0 \sqrt{n}.$$

We will condition on the matrix $M_{n-n'}$. Using Theorem 2.9, we thus have

$$\mathbf{P}(\|(M_n - t_i)\mathbf{v}\| \le n^{1/2}\beta_0) \le (2\rho_{\alpha,\beta_0}(\mathbf{v}))^{n-n'}.$$

Indeed, we will consider $\mathbf{P}(\sum r_i^2 \le \beta_0^2 n)$, where

$$r_i = b_{i,1} v_{n-n'+1} + \dots + b_{i,n'} v_n - (m_{i,1} v_1 + \dots + (m_{i,i} - t_i) v_i + \dots + m_{i,n-n'} v_{n-n'})$$

denotes the i^{th} row of $B\mathbf{v}'' - (M_{n-n'} - t_i)\mathbf{v}'$. Conditioning on B, we have that $\mathbf{P}(|r_i| \le \beta_0) \le \rho_{\alpha,\beta_0}$ by the definition of $\rho_{\alpha,\beta}$. We claim that $\mathbf{P}(|r_i| \le t)$ is true for every $t \ge t_0$ with $t_0 = \beta_0$ and $K = \rho_{\alpha,\beta_0}/\beta_0$. Indeed, breaking the interval $[0, t)$ into $\lceil t/\beta_0 \rceil$ intervals each of length at most β_0, we have that

$$\mathbf{P}(|r_i| \le t) \le (t/\beta_0 + 1)\rho_{\alpha,\beta_0} \le 2Kt$$

and we are done via Theorem 2.9.

Now we estimate the event considered in Theorem 3.3 for a fixed t_i conditioning on $\mathcal{E}_{\mathbf{0},t_i}$

$$\mathbf{P}\Big(\exists \mathbf{v} \in \mathcal{V}, \mathbf{v} \text{ satisfies (7)}, \|(M_n - t_0)\mathbf{v}\| \le \beta_0 n^{1/2}\Big) \le \sum_{\mathbf{v}\in\mathcal{V}, \mathbf{v}\in(7)} (2\rho_{\alpha,\beta_0}(\mathbf{v}))^{n-n'}.$$

To this end, as $\mathbf{v}$ satisfies (7)

$$\widehat{\mathbf{LCD}}_{\kappa,\gamma}(\mathbf{v}/\|\mathbf{v}\|, \alpha) \ge D_0.$$

By definition, there exists $I \subseteq \mathbf{spread}(\mathbf{v})$, $|I| = \lceil \alpha n \rceil$ such that

$$\mathbf{LCD}_{\kappa,\gamma}\big(\mathbf{v}_I/\|\mathbf{v}_I\|\big) \ge D_0 = n^c \beta_0^{-1}.$$

Thus

$$\rho_{\alpha,\beta_0}(\mathbf{v}) \le \rho_{\beta_0/\sqrt{\alpha}}(\mathbf{v}_I/\|\mathbf{v}_I\|) = O(\beta_0 n^{4c}),$$

where in the last estimate we apply Theorem 2.4 as $\beta_0 n^{4c} > 1/D_0$. So

$$\begin{aligned}
\sum_{\mathbf{v}\in\mathcal{V}} (2\rho_{\alpha,\beta_0}(\mathbf{v}))^{n-n'} &\le (\beta_0 n^{4c})^{(1-\alpha)n} |\mathcal{V}| \\
&\le (C'\beta_0 n^{4c})^{(1-\alpha)n} n^{O(1)} (CD_0/\sqrt{n})^n (\sqrt{n}/\kappa)^{cn} \\
&\le (C'\beta_0 n^{4c})^{(1-\alpha)n} n^{O(1)} (Cn^c\beta_0^{-1}/\sqrt{n})^n (\sqrt{n}/n^{2c})^{cn} \\
&\le \beta_0^{-\alpha n} n^{-(1/2-6c)n} \\
&\le e^{n^c n^{-6c} n} n^{-(1/2-6c)n} \\
&\le n^{-(1/2-6c)n}
\end{aligned}$$

provided that n is sufficiently large, where we noted that $\beta_0 > 1/D_0 = \exp(-n^c)$ and c is sufficiently small.

The proof of Theorem 3.3 is then complete where the bound $\exp(-\alpha_0 n)$ comes from the complement of the event of Corollary 2.8 we conditioned on.

References

[1] P. Bourgade and H.-T. Yau, The eigenvector moment flow and local quantum unique ergodicity, *Communications in Mathematical Physics*, **350** (2017), no. 1, 231-278.

[2] R. Band, I. Oren, and U. Smilansky, Nodal domains on graphs: how to count them and why?, *Analysis on graphs and its applications*, volume 77 of Proc. Sympos. Pure Math., 5-27, Amer. Math. Soc., Providence, RI, 2008.

[3] R. Coifman, S. Lafon, A. B. Lee, M. Maggioni, B. Nadler, F. Warner, and S. Zucker. Geometric diffusion as a tool for harmonic analysis and structure definition of data, part 1: Diffusion maps, *Proc. Natl. Acad. Sci.*, **102** (21):7426-7431, 2005.

[4] R. R. Coifman, S. Lafon, A. B. Lee, M. Maggioni, B. Nadler, F. Warner, and S. Zucker. Geometric diffusion as a tool for harmonic analysis and structure definition of data, part 2: Multiscale methods, *Proc. Natl. Acad. Sci.,* **102** (21):7432-7437, 2005.

[5] Y. Dekel, J. Lee, and N. Linial, Eigenvectors of random graphs: Nodal domains, *Random Structures & Algorithms*, **39** (2011), no. 1, 39-58.

[6] P. Erdős, On a lemma of Littlewood and Offord, *Bulletin of the American Mathematical Society*, **51** (1945), 898–902.

[7] L. Erdős, Universality of Wigner random matrices: a survey of recent results, `arxiv.org/abs/1004.0861`.

[8] L. Erdős and H.-T. Yau, Universality of local spectral statistics of random matrices, *Bulletin of the American Mathematical Society*, **49** (2012), 377-414.

[9] L. Erdős, B. Schlein, and H. T. Yau, Local semicircle law and complete delocalization for Wigner random matrices, *Communications in Mathematical Physics*, **287**(2):641-655, 2009.

[10] L. Erdős, B. Schlein, and H. T. Yau, Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices, *Ann. Probab.*, **37**(3):815-852, 2009.

[11] G. Halász, Estimates for the concentration function of combinatorial number theory and probability, *Periodica Mathematica Hugarica*, **8** (1977), 197-211.

[12] H. Nguyen and V. Vu, Small ball probability, inverse theorems, and applications, *Erdős Centennial Proceeding*, Eds. L. Lovász et. al., Springer 2013.

[13] H. Nguyen and V. Vu, Optimal Littlewood-Offord theorems, *Advances in Mathematics,* Vol. 226, **6** (2011), 5298-5319.

[14] H. Nguyen, T. Tao, and V. Vu, Random Matrices: Tail Bounds for Gaps Between Eigenvalues, *Probability Theory and Related Fields*, (2017), Vol. 167, **3**, 777-816.

[15] S. O'Rourke and B. Touri, On a conjecture of Godsil concerning controllable random graphs, *SIAM Journal on Control and Optimization,* **54** (2016), no. 6, 3347-3378.

[16] S. O'Rourke and B. Touri, Controllability of random systems: universality and minimal controllability, `arxiv.org/abs/1506.03125`.

[17] S. O'Rourke, V. Vu and K. Wang, Eigenvectors of random matrices: a survey, *Journal of Combinatorial Theory Series A,* **144** (2016), issue C, 361-442.

[18] M. Rudelson and R. Vershynin, The Littlewood-Offord problem and invertibility of random matrices, *Advances in Mathematics*, **218** (2008), no. 2, 600-633.

[19] M. Rudelson and R. Vershynin, Smallest singular value of a random rectangular matrix, *Communications on Pure and Applied Mathematics*, **62** (2009), 1707-1739.

[20] M. Rudelson and R. Vershynin, No-gaps delocalization for general random matrices, *Geometric and Functional Analysis*, **26** (2016), no. 6, 1716-1776.

[21] D. A. Spielman and S. H. Teng, Smoothed analysis of algorithms, *Proceedings of the International Congress of Mathematicians*, Vol. I, 597-606, Higher Ed. Press, Beijing, 2002.

[22] T. Tao and V. Vu, From the Littlewood-Offord problem to the circular law: universality of the spectral distribution of random matrices, *Bulletin of the American Mathematical Society*, **46** (2009), 377-396.

[23] T. Tao and V. Vu, Inverse Littlewood-Offord theorems and the condition number of random matrices, *Annals of Mathematics* (2), **169** (2009), no 2, 595-632.

[24] T. Tao and V. Vu, A sharp inverse Littlewood-Offord theorem, *Random Structures & Algorithms* , **37** (2010), no. 4, 525-539.

[25] T. Tao and V. Vu, Smooth analysis of the condition number and the least singular value, *Mathematics of Computation*, **79** (2010), 2333-2352.

[26] T. Tao and V. Vu, Random matrices: universal properties of eigenvectors, *Random Matrices Theory Appl.* **1** (2012), no. 1.

[27] T. Tao and V. Vu, Random matrices have simple spectrum, *Combinatorica*, to appear, `arxiv.org/abs/1412.1438`.

[28] R. Vershynin, Invertibility of symmetric random matrices, *Random Structures & Algorithms*, **44** (2014), no. 2, 135-182.

[29] V. Vu and K. Wang, Random weighted projections, random quadratic forms and random eigenvectors, *Random Structures and Algorithms*, to appear.

Department of Mathematics, The Ohio State University, Columbus, OH

Department of Mathematics, The Ohio State University, Columbus, OH

E-mail address: `nguyen.1261@math.osu.edu`