APPLICATIONS OF GOODWILLIE CALCULUS

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ABSTRACT. We show that the functor $X \mapsto \overline{A}(\Sigma X)$ splits as a product of its differentials, in the sense of Goodwillie. This splitting is natural in X . Analogous splitting results are shown to hold for the stable homotopy of various functors related to the free loop space of X .

Introduction

This paper represents a collection of results arising from the application of Goodwillie's Calculus of Functors to (relative) Waldhausen K-theory, as well as certain related functors arising in stable homotopy theory.

In the first two chapters, we present a detailed proof of a conjecture due to Goodwillie (stated below), using Goodwillie Calculus and a generalization of the techniques of Waldhausen in his paper [W2]. In this paper, Waldhausen constructed a map

$$W : A(X) \to \Omega^\infty \Sigma^\infty (X_+)$$

(where $A(X)$ denotes the Waldhausen K-theory of the space X) and showed that evaluation on the image of $M : \Omega^\infty \Sigma^\infty (X_+) \to A(X)$ induced by the inclusion of monomial matrices produced a self-map $W \circ M : \Omega^\infty \Sigma^\infty (X_+) \to \Omega^\infty \Sigma^\infty (X_+)$ homotopic to the identity by a homotopy natural in X. This yielded a splitting of $\Omega^\infty \Sigma^\infty (X_+)$ off of $A(X)$ (as well as its stabilization $A^S(X)$), a fact which plays a

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key role in the proof of the fundamental theorem of Waldhausen relating $A(X)$ to pseudo-isotopy theory ([W2], [W4], [W]):

<u>Thm</u>. *[Waldhausen]* $A(X) \simeq \Omega^\infty\Sigma^\infty(X_+) \times Wh^{Diff}(X)$ *where* $\Omega^2 Wh^{Diff}(X) \simeq \wp(X)$ *= the stable pseudo-isotopy space of* X *(as defined by Hatcher-Wagoner-Igusa).*

The construction of W is in stages. It is first shown that fibre $(A(S^n \wedge X_+) \to A(*))$ can be described through a certain range of dimensions (approximately $2n$) in terms of a "cyclic" bar construction. On this cyclic bar construction Waldhausen defines a map to $\Omega^\infty\Sigma^\infty(X_+)$ compatible with stabilization. The result is a map $A^S(X) \to \Omega^\infty\Sigma^\infty(X_+)$ natural in X, and precomposition with the stabilization map $A(X) \to A^S(X)$ yields W. In this paper we construct a generalization of Waldhausen's map W. Specifically let X and Y be pointed simplicial sets, X connected. Then there exists a <u>generalized Waldhausen trace map</u> (2.2.8):

$$\begin{aligned} \overline{Tr}_X(Y) : \varinjlim_n \Omega^n \mathrm{fibre}(\overline{A}(\Sigma(X \vee \Sigma^n Y)) &\to \overline{A}(\Sigma X)) \\ (*) \qquad \to \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} &|X^{[q-1]} \wedge Y|)). \end{aligned}$$

This map is natural in X and Y. The first application of this is to prove a conjecture due to T. Goodwillie:

<u>Theorem A</u>. *For connected* X *there is a weak equivalence*

$$\tilde{\rho} = \prod_{q\geq 1} \tilde{\rho}_q : \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|X|^{[q]})) \xrightarrow{\simeq} \overline{A}(\Sigma X)$$

natural in X.

The action of $\mathbb{Z}/q$ on $|X|^{[q]}$ is given by cyclic permutation, and $\overline{A}(Z)$ denotes fibre $(A(Z) \to A(*))$. Theorem A had been announced previously in [CCGH] as well as by myself in [01]. Unfortunately neither provide complete proofs. The proof of Theorem A we give here follows the line of argument attempted in [CCGH], with technical modification along the lines of [W2]. An outline is as follows: in chapter 1 we recall the necessary results from [W2] and Goodwillie's Calculus of

Functors ([G0] – [G4]), and in this context define the maps $\tilde{\rho}_q$ used in the proof of Theorem A. In chapter 2, we follow the arguments of [W2] in constructing the trace map $\overline{Tr}_X(Y)$ and in section 2.3 we complete the proof of Theorem A by using $\overline{Tr}_X(Y)$ to explicitly compute the 1st differnetial of $\tilde{\rho}_q$ at a connected space X (this 1st differnetial is in the sense of Goodwillie). A crucial ingredient here is the computation, due to Goodwillie, of the derivatives of $A(\Sigma\_)$ on connected spaces. Tom Goodwillie has also been able to prove Theorem A by applying results of G. Carlsson to study the Goodwillie "Taylor series" for the functor $A\Sigma(\_)$.

In chapter 3, we present some results (initially presented in [02]), concerning splittings of homotopy functors. As is often the case, homotopical splittings of infinite loop spaces arise in a natural way from the existence of a suitable weight filtration by homotopy functors. This observation was made by Goodwillie in [G0], where it is shown that the classical splitting of $\Omega^\infty\Sigma^\infty(\Omega\Sigma X)$ due to Snaith can be easily derived by the use of Calculus. By axiomatizing the ingredients necessary to apply his argument, we are able to prove a family of splitting theorems regarding the stable homotopy type of homotopy orbit spaces constructed from the free loop space (c.f. lemma 3.2.2).

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CHAPTER I

§1.1 **Background and Notation for $A(X)$**

We recall the construction of $A(X)$ as given in [W2]. Let X be a pointed, connected simplicial set, GX its Kan loop group. Let $H_k^n(|GX|)$ denote the total singular complex of the topological monoid $Aut_{|GX|}(\overset{k}{\vee} S^n \wedge |GX|_+)$ of $|GX|$-equivariant self-homotopy equivalences of the free basepointed $|GX|$-space $(\overset{k}{\vee} S^n \wedge |GX|_+)$. $H_k^n(|M|)$ is a mapping space (for a simplicial monoid M), for which we will adopt the convention that AB denotes the composition

$$\overset{k}{\vee} S^n \wedge M_+ \xrightarrow{A} \overset{k}{\vee} S^n \wedge M_+ \xrightarrow{B} \overset{k}{\vee} S^n \wedge M_+ \ .$$

$H_k^n(|GX|)$ identifies naturally with a set $\overline{M}_k^n(|GX|_+)$ of path components of

$$M_k^n(|GX|_+) = Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |GX|_+)$$

under the inclusion

$$H_k^n(|GX|) \hookrightarrow Map_{|GX|}(\overset{k}{\vee} S^n \wedge |GX|_+, \overset{k}{\vee} S^n \wedge |GX|_+) \cong Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |GX|_+) \ .$$

One has stabilization maps

$$M_k^n(|GX|_+) \xrightarrow{\imath} M_{k+1}^n(|GX|_+)$$

given by wedge product with the inclusion $S^n \hookrightarrow S^n \wedge |GX|_+$, suspension maps $M_k^n(|GX|_+) \xrightarrow{\Sigma} M_k^{n+1}(|GX|_+)$ given by smash product with the identity map on S^1, as well as pairing maps

$$M_k^n(|GX|_+) \times M_\ell^n(|GX|_+) \to M_{k+\ell}^n(|GX|_+)$$

induced by wedge-sum. Restricted to $\{H_k^n(|GX|)\}_{k\geq 0}$, this pairing gives $\coprod_{k\geq 0} H_k^n(|GX|)$ the structure of a topological permutative category [S] for all $n \geq 0$. These operations – wedge sum, suspension, stabilization – commute up to natural isomorphism. So letting $H_k(|GX|) = \varinjlim_{\Sigma} H_k^n(|GX|)$ and $H(|GX|) = \varinjlim_{k} H_k(|GX|)$ we see that $\coprod_{k\geq 0} H_k(|GX|)$ is also a topological permutative category under wedge-sum. Waldhausen's definition of $A(X)$ is

Definition 1.1.1. $A(X) = \Omega B(\coprod_{k\geq 0} BH_k(|GX|)) \simeq \mathbb{Z} \times BH(|GX|)^+$

where "$\simeq$ "means "weakly equivalent "(i.e., there are maps connecting the two spaces which induce isomorphisms on homotopy groups). If X is a basepointed space, $A(X)$ is defined to be $A(Sing(X))$. Similarly if X is a simplical space, $A(X) \stackrel{\text{def}}{=} A(Sing|X|)$. $A(X)$ is a homotopy functor; this means that if $X \simeq Y$ then $A(X) \simeq A(Y)$.

We will use the notation ΣU to denote the reduced suspension of U. If $|X| \simeq \Sigma|Z|$, where Z is a simplicial space connected in each degree, then GX is weakly equivalent to the simplicial James monoid JZ, which in degree q is the free monoid on the pointed space Z_q. In this case we can use JZ in place of the Kan loop group GX in the above constructions. The result is an equivalence $A(\Sigma Z) \simeq \Omega B(\coprod_{k\geq 0} BH_k(|JZ|))$.

In studying $A(\Sigma Z)$, we will use constructions from §2 of [W2]. The first, due to Segal, generalizes the bar construction which associates to a monoid its nerve. Thus, a <u>partial</u> <u>monoid</u> is a basepointed set M together with a partially defined composition law

$$M \times M \supset M_2 \stackrel{\mu}{\hookrightarrow} M$$

satisfying

(1) $M \vee M \subset M_2$, with $\mu(*, m) = \mu(m, *) = m$ and

(2) $\mu(\mu(m_1, m_2), m_3) = \mu(m_1, \mu(m_2, m_3))$ in the sense that if one is defined then so is the other, and they are equal.

To avoid confusion, we include as part of the data associated to a partial monoid sets

$$\{M_p \subseteq \text{ composable } p\text{-tuples in } M\}_{p\geq 0}$$

where $M_0 = *$, $M_1 = M$, M_2 is as given above, and "composable "means that the iterated product is defined. We require that the face and degeneracy maps, defined in the usual way, induce a simplicial structure on $\{M_p\}_{p\geq 0}$. The <u>nerve</u> of M is then the simplicial set

$$\{[p] \mapsto M_p\}\,.$$

Let M be a monoid, S a set on which M acts on both sides, with $(ms)m' = m(sm')$. Then one can form the cyclic bar construction of M with "coefficients" in S. It is a simplicial set $N^{cy}(M,S)$ which in degree q is $M^q \times S$. The face and degeneracy maps are given by the following formulae (see [W2], §2):

(1.1.1.5)

$$\partial_0(m_1,\ldots,m_q;s) = (m_2,m_3,\ldots,m_q;sm_1)$$

$$\partial_i(m_1,\ldots,m_q;s) = (m_1,\ldots,m_im_{i+1},\ldots,m_q;s), \qquad 1 \le i \le q-1$$

$$\partial_q(m_1,\ldots,m_q;s) = (m_1,\ldots,m_{q-1};m_qs)$$

$$s_i(m_1,\ldots,m_q;s) = (m_1,\ldots,m_i,1,m_{i+1},\ldots;s) \qquad 0 \le i \le q\ .$$

As noted in [W2], the double bar construction is a special case of the cyclic bar construction where S appears as a cartesian product of a left M-set and a right M-set. When M is a grouplike monoid ($\pi_0 M$ is a group) and $S = M$ with induced M action on the left and right, there is a weak equivalence between $N^{cy}(M,M)$ and BM^{S^1} = the free loop-space of BM. The construction of $N^{cy}(M,S)$ extends in the obvious way to a simplicial monoid M acting on a simplicial set S.

It is often the case that S itself is a partial monoid which admits a left and right M-action. In this case one wants to know that the cyclic bar construction $N^{cy}(M,S)$ can be done in such a way as to be compatible with the partial monoid structure on S. So let M be a monoid. By a left M-monoid we will mean a partial monoid E together with a basepointed M-action $M \times E \to E$ compatible with the partial monoid structure on E in the sense that for each $m \in M$ the maps

$$E \to E\ ; \qquad e \mapsto me$$

$$E \to E\ ; \qquad e \mapsto em$$

are homomorphisms of partial monoids. A right M-monoid is similarly defined, and an M-bimonoid is a partial monoid with compatible left and right monoid structures. Given such an M-bimonoid E satisfying an obvious "saturation" condition

([W2], p.367), the semidirect product $M \ltimes E$ is the partial monoid with composition given by $(m,e)(m',e') = (mm', (em')(me'))$ whose nerve is the simplicial set

$$\{[p] \mapsto M^p \times E_p\}_{p \geq 0} \ .$$

When E is commutative, we will write the product in E additively. Clearly this construction can be done degreewise when M and E are simplicial. If the partial monoid structure on E has not been specified, we will assume it is the trivial one. $\{[p] \mapsto \overset{p}{\vee}(E,*)\}$ is a left (resp. right resp. bi-) monoid over M if E is. Iteration of this construction yields an M-monoid structure on a space whose realization is an iterated suspension of $|E|$, and which agrees with that induced by the given action of M on E together with the trivial action on the suspension coordinates. A left (resp. right) <u>M-space</u> is a space equipped with a left (resp. right) action of M, not necessarily basepointed. Obviously, a left (resp. right) M-monoid is a left (resp. right) M-space, not conversely. Similarly, we may define an M-bispace to be one equipped with compatible left and right M actions.

Suppose that A is a monoid, and that we are given an inclusion $A \to M$ of A into an A-bispace which is a morphism of A-bispaces. Identifying A with its image in M , one may define a partial monoid by

$$M_p = \overset{p}{\vee}(M,A) = \bigcup_{j=1}^{p} A^{j-1} \times M \times A^{p-j} \ .$$

The nerve of the resulting partial monoid $\{[p] \mapsto M_p\}$ is called a <u>generalized wedge</u> (this construction is due to Waldhausen). Taking $A = \{pt\}$ yields $\{[p] \mapsto \overset{p}{\vee}(M,*)\}$ whose realization is homeomorphic to $\Sigma|M|$ (as we have seen above). It is often useful to approximate the nerve of a monoid M by generalized wedges. A straightforward argument (Lemma 2.2.1 of [W2]) shows that if $A \to M$ is an $(n-1)$-connected inclusion of monoids, the induced inclusion

$$\{[p] \mapsto \overset{p}{\vee}(M,A)\} \to \{[p] \mapsto \overset{p}{\vee}(M,M)\} = NM$$

is $(2n-1)$-connected. As one can easily see, a fixed monoid may admit many different partial monoid structures. A key result concerning the nerve of a semidirect

product is provided by lemma 2.3.1 of [W2]. It provides a map

$$u : diag(N^{cy}(M, NE)) \to N(M \ltimes E)$$

which is a weak equivalence when $\pi_0(M)$ is a group. Here M is a simplicial monoid, E a simplicial M-bimonoid, and $N^{cy}(M, NE)$ the cyclic bar construction of M acting on the nerve of the partial monoid E. The "diagonal" structure is with respect to the simplicial coordinates coming from $N^{cy}(\_)$ and NE. The saturation condition referred to above, as well as the condition that $\pi_0(M)$ is a group, will always be satisfied in our case. As we will need to know u explicitly later on, we recall that it is given on n-simplices by the formula ([W2], p. 369):

$$\begin{aligned} &u(m_1, \dots, m_n; e_1, \dots, e_n) \\ &= (m_1, (\prod_{i=1}^{n} m_i) e_1 m_1; m_2, (\prod_{i=2}^{n} m_i) e_2 (m_1 m_2); \cdots ; m_n, m_n e_n (\prod_{i=1}^{n} m_i)) \ . \end{aligned} \tag{1.1.2}$$

Let us return to considering JZ and $H_k^n(|JZ|)$ (for connected Z). We will be interested in the case when $Z = X \vee Y$. For a positive integer r, let $F_r(X, Y) \subset J(X \vee Y)$ denote the subset which in each degree consists of elements of word-length at most r in Y. This is clearly a simplicial subset. There is also a natural partial monoid structure on $F_r(X, Y)$, where two elements are composable if their product in $J(X \vee Y)$ lies in $F_r(X, Y)$. We will denote $F_1(X, Y)/F_0(X, Y)$ by $\overline{F}_1(X, Y)$. Note that $\overline{F}_1(X, Y) \cong J(X)_+ \wedge Y \wedge J(X)_+$ for connected Y. Now consider the projection maps

$$\begin{gathered} F_1(X, Y) \twoheadrightarrow F_1(X, *) = JX \\ \text{induced by } \ Y \twoheadrightarrow * \quad , \\ F_1(X, Y) \twoheadrightarrow \overline{F}_1(X, Y) \ . \end{gathered} \tag{1.1.3}$$

Note first that all spaces in sight admit compatible left and right $J(X)$-actions, and these maps commute with the action. Also both projection maps are partial

monoid maps, where we take the trivial monoid structure on $\overline{F}_1(X,Y)$, which with the given left and right actions of $J(X)$ is a $J(X)$- bimonoid. Let

$$\overline{M}^n_k(|F_1(X,Y)|_+) = \overline{M}^n_k(|J(X,Y)|_+) \cap M^n_k(|F_1(X,Y)|_+)\ .$$

<u>Lemma 1.1.4</u>. *The projection maps in (1.1.3) induce maps*

$$\overline{M}^n_k(|F_1(X,Y)|_+) \overset{p_1}{\twoheadrightarrow} \overline{M}^n_k(|J(X)|_+) \cong H^n_k(|J(X)|)\,, \tag{1.1.5}$$

$$\overline{M}^n_k(|F_1(X,Y)|_+) \overset{p_2}{\twoheadrightarrow} M^n_k(|\overline{F}_1(X,Y)|)\ . \tag{1.1.6}$$

All four spaces admit compatible left and right actions of $H^n_k(|J(X)|)$, and these maps commute with the actions. As a result, these projection maps induce a map of generalized wedges

$$\begin{aligned} &\{[p] \mapsto \overset{p}{\vee}\ (\overline{M}^n_k(|F_1(X,Y)|_+), H^n_k(|JX|))\} \\ \to &\{[p] \mapsto \overset{p}{\vee}\ (H^n_k(|JX|) \ltimes M^n_k(|\overline{F}_1(X,Y)|), H^n_k(|JX|))\} \end{aligned} \tag{1.1.7}$$

where the semi-direct product $H^n_k(|J(X)|) \ltimes M^n_k(|\overline{F}_1(X,Y)|)$ is formed using the trivial monoid structure on $\overline{M}^n_k(|\overline{F}_1(X,Y)|)$ together with the given left and right (basepointed) $H^n_k(|J(X)|)$-actions.

<u>*Proof*</u>. The left and right actions of $H^n_k(|J(X)|)$ on $H^n_k(|J(X \vee Y)|)$ induce compatible left and right actions on $\overline{M}^n_k(|F_1(X,Y)|_+)$. These actions are functorial in Y, hence natural with respect to the map $Y \twoheadrightarrow *$ inducing the projection map p_1. It is not hard to see that these actions are also compatible with the collapsing map which induces p_2 . p_1 and p_2 taken together induce a map which on the level of sets is

$$\overline{M}^n_k(|F_1(X,Y)|_+) \overset{p_1 \times p_2}{\longrightarrow} H^n_k(|J(X)|) \times M^n_k(|\overline{F}_1(X,Y)|)\ . \tag{1.1.8}$$

Taking the trivial monoid structure on $M^n_k(|\overline{F}_1(X,Y)|)$, we get an $\overline{H}^n_k(|J(X)|)$ - bimonoid structure on this space, so that the R.H.S. of (1.1.8) is the underlying set of a semi-direct product. The partial composition law in this semi-direct product

amounts to a description of a compatible left and right action of $H_k^n(|J(X)|)$. This action is given by

$$
\begin{aligned}
& x(y,a) = (x,*)(y,a) = (xy, xa) \quad , \\
(1.1.9) \qquad & (y,a)x = (yx, ax) \quad , \\
& \quad x, y \in H_k^n(|J(X)|) \quad , a \in M_k^n(|\overline{F}_1(X,Y)|) \ .
\end{aligned}
$$

The projection maps p_1 and p_2 preserve both the left and right actions of $H_k^n(|J(X)|)$, and therefore so does $p_1 \times p_2$ under the actions described in (1.1.9). This implies that $p_1 \times p_2$ induces a map of generalized wedges, as claimed. $\square$

§1.2 Goodwillie's Calculus

This section recalls results from Goodwillie's Calculus ([G0] – [G3]) that will be used later on. We consider functors $F : \underline{C} \to \underline{D}$, where $\underline{C}$ is either $U, T, U(C)$ or $T(C)$ and $\underline{D}$ is $T, T(C)$ or the category Sp of spectra. Here U is the category of (Hausdorff) topological spaces, T the category of basepointed spaces in U. $U(C)$ denotes the corresponding category of spaces over $C \in obj(U)$, and $T(C)$ the category of basepointed objects in $U(C)$. Note that an object of $T(C)$ is a retractive space Y over C, i.e., $r : Y \to C$ comes equipped with a right inverse i ($r \circ i = id$). Each of these choices of $\underline{C}$ is a closed model category in the sense of Quillen, so one has the usual constructions of homotopy theory. When $\underline{C} = U(C)$ or $T(C)$ we denote by C_n the full subcategory of n-connected objects in $\underline{C}$. We adopt the convention of [G2] that a map of spaces or spectra is called <u>n-connected</u> if each of its homotopy fibres is $(n-1)$-connected. In all of these categories one has a standard notion of weak equivalence, and F is called a <u>homotopy functor</u> if F preserves weak equivalences. We will only be concerned with homotopy functors.

Let S be a finite set, $C(S)$ the category of subsets of S with morphisms corresponding to inclusions. An S-cube in C is a covariant functor $G : C(S) \to \underline{C}$. If $S = \{1, 2 \ldots, n\} = \underline{n}$, G is called an n-cube. Associated to an S-cube is the homotopy-inverse limit $h(G) = holim(G|_{C_0(S)})$ where $C_0(S)$ denotes the full sub-

category of $C(S)$ on all objects except ϕ. The natural coaugmentation map

$$lim(G) \to holim(G)$$

induces a natural transformation

$$a(G) : G(\phi) \to holim(G|_{C_0(S)}) \ .$$

G is <u>h-cartesian</u> if $a(G)$ is a weak equivalence. We say $F : \underline{C} \to \underline{D}$ (as above) is <u>n-excisive</u> if $F \circ G$ is h-cartesian for every strongly homotopy co-cartesian S-cube $G : S \to \underline{C}$ where $|S| = n+1$ ([G2], def.3.1). The condition that F be n-excisive becomes less restrictive as n increases. Thus, if F is n-excisive, it is $(n+1)$-excisive, but not conversely (c.f. prop. 2.3.2 [G2]).

Given a homotopy functor F satisfying certain conditions, there is a natural way of producing a functor P_nF of degree n and a natural transformation $F \to P_nF$. In fact, P_nF can always be constructed. Starting with $X \in$ obj $(\underline{C})$ one can define an $(n+1)$-cube

$$X \underset{C}{*} (\_) : C(\underline{n+1}) \to \underline{C} = U(C) \text{ or } T(C) \ ;$$

this associates to $T \subset \underline{n+1}$ the space $X \underset{C}{*} T$ which is the fibrewise join over C of X with the set T. Now let $(T_nF)(X) = holim(F \circ (X \underset{C}{*} (\_))|_{C_0(\underline{n+1})})$. $a(F \circ (X \underset{C}{*} (\_)))$ defines a transformation $(t_nF)(X) : F(X) \to (T_nF)(X)$. One easily checks that $X \mapsto (T_nF)(X)$ is again a homotopy functor on $\underline{C}$ and that $(t_nF) = (t_nF)(\_)$ defines a natural transformation from F to T_nF. Note that $X \underset{C}{*} (\_) : C(\underline{n+1}) \to \underline{C}$ is a (strongly) homotopy co-cartesian diagram in $\underline{C}$, so that t_nF is an equivalence if F is of degree n. Iteration of this construction yields P_nF which is by definition the homotopy colimit of the directed system $\{T_n^iF, t_nT_n^iF\}$.

The transformations $t_nT_n^iF$ induce a natural transformation $p_nF : F \to P_nF$. Choice of a distinguished element $(m+1) \in \underline{m+1}$ induces an inclusion $\underline{m} \to \underline{m+1}$ and hence a natural transformation $C(\underline{m}) \to C(\underline{m+1})$. This in turn induces a natural transformation of directed systems

$$\{T_n^iF, t_nT_n^iF\} \to \{T_{n-1}^iF, t_{n-1}T_{n-1}^iF\}$$

and hence a natural transformation $P_mF \xrightarrow{q_mF} P_{m-1}F$. Different choices of m yield naturally equivalent choices of q_mF. The <u>Goodwillie Taylor series of F</u> is then by definition the inverse system $\{P_nF, q_nF\}$ which is best viewed as a tower together with the natural transformations p_nF:

(1.2.1)

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_3F"] \\
 & P_2F \arrow[d, "q_2F"] \\
 & P_1F \arrow[d, "q_1F"] \\
F \arrow[uur, "p_2F"] \arrow[ur, "p_1F"] \arrow[r, "p_0F"] & P_0F
\end{tikzcd}$$

The closed diagrams in this tower are homotopy commutative. The nth homogeneous part of F is by definition the homotopy fibre of

$$q_nF : D_nF = hofib(P_nF \xrightarrow{q_nF} P_{n-1}F) \ .$$

We have not yet explained the conditions necessary for P_nF to be n-excisive. This is guaranteed by the following condition on F (compare [G3], Lemma 1.4):

<u>Definition 1.2.2</u>. *(Definition 4.1, [G2]) F is <u>stably n-excisive</u> if the following is true for some numbers c and κ :*

$E_n(c,\kappa)$: If $G : C(\underline{n+1}) \to \underline{C}$ is any strongly co-Cartesian $(n+1)$-cube such that for all $s \in S$ the map $G(\phi) \to G(s)$ is k_s connected and $k_s \geq \kappa$, then the diagram $G(C(\underline{n+1})$ is $(-c + \sum k_s)$-connected.

In this case D_nF is <u>homogeneous</u> of degree n that is, it is stably n-excisive and $P_iD_nF \simeq *$ for $i < n$ (Prop. 1.11 [G3]). We will write $\underline{P_nF}$ for $fibre(F \xrightarrow{p_nF} P_nF)$, and $\underline{P_n^mF}$ for $fibre(P_nF \to P_mF)$ when P_kF has degree k for all k (this will always be the case for the functors in which we are interested). $D_n(F)$ is referred to as

the $\underline{n^{th}\text{ differential}}$ of F (at $*$). One also wants to know not just when P_nF is of degree n, but also when the connectivity of $F \xrightarrow{p_nF} P_nF$ tend to ∞ as n tends to ∞. From Def. 4.2, [G2], F is ρ-analytic if there is some number q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

<u>Theorem 1.1.</u> *(Th. 2.5.21,* [G3]*) The connectivity of p_nF tends to ∞ over the category $\underline{C}_\rho$ where $\rho = \rho(F), F : \underline{C} \to \underline{D}$.*

In analogy with functions, $\underline{C}_\rho$ may be thought of as the <u>disk</u> <u>of</u> <u>convergence</u> of F. In applying this calculus to F, it is natural to restrict one's attention to the subcategory $\underline{C}_{\rho(F)}$ which in general is the largest subcategory of $\underline{C}$ for which the Taylor series of $F|_{\underline{C}_{\rho(F)}}$ converges (in the homotopy-theoretical sense). Within this range it provides a powerful machinery for analyzing F, as well as determining the effect of a natural transformation $\eta : F_1 \to F_2$ on homotopy groups. It is clear from the above theorem that η will induce a weak equivalence when restricted to $\underline{C}_\rho$ ($\rho = max(\rho(F_1), \rho(F_2)), F_i : \underline{C} \to \underline{D}$) if η induces an equivalence on differentials:

$$D_n(\eta) : D_n(F_1) \xrightarrow{\cong} D_n(F_2) \ ,$$

under the condition that $P_0(F_i) \simeq *$. However, there is another way of getting at η. Assume first that $\underline{C} = U(C)$ and that $F_i : \underline{C} \to \underline{D}$ have the same modulus ρ for $i =$ 1,2. Let $(X, p : X \to C)$ be an object in $U(C)$. Then $(X, p : X \to C)$ defines a natural transformation $\nu_{(X,p)} : U(X) \to U(C)$ given on objects by

$$\nu_{(X,p)}(Y, r : Y \to X) = (Y, p \circ r : Y \to C) \ .$$

Analyticity is preserved by the natural transformation $\nu^*_{(X,p)} : F \to F \circ \nu_{(X,p)}$. The next result of Goodwillie's concerns only 1st differentials. The following is contained in theorems 5.3 and 5.7 of [G2].

<u>Theorem 1.2.4.</u> *If $F_1, F_2 : U(C) \to \underline{D}$ are ρ-analytic, and $\eta : F_1 :\to F_2$ is a natural transformation such that the square*

$$\begin{array}{ccc} P_1(\nu^*_{(X,p)}F_1) & \xrightarrow{P_1(\nu^*_{(X,p)}\eta)} & P_1(\nu^*_{(X,p)}F_2) \\ {\scriptstyle q_1(\nu^*_{(X,p)}F_1)}\downarrow & & \downarrow{\scriptstyle q_1(\nu^*_{(X,p)}F_2)} \\ P_0(\nu^*_{(X,p)}F_1) & \xrightarrow{P_0(\nu^*_{(X,p)}\eta)} & P_0(\nu^*_{(X,p)}F_2) \end{array}$$

is homotopy-cartesian for every (X,p) in $U(C)$, then for every $f : Y \to X$ in $U(C)_\rho$ the diagram

$$\begin{array}{ccc} F_1(Y) & \xrightarrow{\eta(Y)} & F_2(Y) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(X) & \xrightarrow{\eta(X)} & F_2(X) \end{array}$$

is homotopy-cartesian.

In the case $C = *$ we will denote $fibre(q_1(\nu^*_{(X,p)}F))$ by $(D_1F)_X$; p in this case is unique. For $F_2 = A(\Sigma\_) = A\Sigma(\_)$, $\rho = 0$ (by theorem 4.6 of [G2]). Theorem 1.2.4 yields

<u>Corollary 1.2.5</u>. *If $\eta : F_1 \to A\Sigma(\_)$ is a natural transformation which induces an equivalence $D_1(\eta)_X : (D_1F_1)_X \xrightarrow{\simeq} (D_1A\Sigma)_X$ for all connected spaces X and F is 0-analytic, then η induces an equivalence*

$$\eta(f) : fibre(F_1(Y) \to F_1(X)) \xrightarrow{\simeq} fibre(A(\Sigma Y) \to A(\Sigma X))$$

for all maps f between connected spaces Y and X.

The result which makes these techniques applicable to the study of $A(X)$ is the computation, due to Waldhausen at $X = pt$, and Goodwillie for general X, of the differentials of $A(X)$: here (Y) denotes the retractive object $(Y \vee X; r : Y \vee X \to X)$ thought of as an object in $T(X)$.

<u>Theorem 1.2.6</u>. *(Waldhausen [W2], [W4]; Goodwillie [G1], Cor. 3.3) For connected X there is an equivalence*

$$(D_1A\Sigma)_X(Y) \simeq \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1}|X^{[q-1]} \wedge Y|))$$

natural in X .

<u>Proof</u>. Goodwillie's computation in [G1] applies to the functor $A(\_)$ rather than $A\Sigma(\_)$. However, there is a natural equivalence between $A\Sigma(\_)$ and the restriction of $A(\_)$ to the subcategory of (basepointed) suspension spaces. By Goodwillie,

the differential $(D_1 A\Sigma)_X(Y)$ may be computed as the differential (at ΣX) of the functor

$$Y \to \Lambda(\Sigma Y \to \Sigma X)$$

which by [G2], §2, together with the Snaith splitting of the functor $\Omega\Sigma(\_)$ yields the result (this computation is also covered in lemma 3.2.2 below).

We have added the realization functor for consistency of notation in the statement of the theorem, as $A(\_)$ was defined on simplical sets in §1.1. Note that as a homotopy functor $A(\_)$ factors by the realization functor and hence can be viewed as a homotopy functor on spaces, which is necessary in order to apply Goodwillie's calculus as it stands.

Remark 1.2.7. *It is an interesting question as to what type of constructions in the calculus of several variables (real or complex) have a suitable analogue in Goodwillie's calculus of functors. For example, it is easy to show that $D_1(F \circ G)_X$ is $(D_1F)_{GX} \wedge (D_1G)_X$ when $G(X) \simeq *$ and F is reduced. This suggests the existence of a chain rule for Goodwillie's Calculus.*

Let Δ_n denote the standard n - simplex. If F is a functor defined on spaces, we will say it is continuous if for each $n \geq 0$ there are natural transformations

$$\Delta_n \times F(\_) \to F(\Delta_n \times \_)$$

compatible with the coface and codegeneracy maps of $\{\Delta_n\}_{n\geq 0}$, such that the transformation corresponding to $n = 0$ is the identity transformation. This collection of transformations induces a natural transformation of realizations

$$\Phi_F : |[k] \mapsto F(\_)| \to F(|[k] \mapsto (\_)|) \ .$$

Here the range of F is either $T, T(C)$ or Sp as before.

Lemma 1.2.8. *If F is a continuous homotopy functor on $U(C)$ then the natural transformation $\Phi_F : |[k] \mapsto F(\_)| \to F(|[k] \mapsto (\_)|)$ induces a weak equivalence over the category of simplicial objects in $U(C)_{\rho(F)}$.*

Proof. The Taylor series of F converges on $U(C)_{\rho(F)}$. Thus the transformation Φ_F induces a map of Taylor series

$$\{|[k] \mapsto P_nF(\_)| \xrightarrow{\Phi_{P_nF}} P_nF(|[k] \mapsto (\_)|)\}_{n\geq 1}$$

and hence for each n a map of homogeneous functors

$$|[k] \mapsto D_nF(\_)| \xrightarrow{\Phi_{D_nF}} D_nF(|[k] \mapsto (\_)|) \ .$$

Goodwillie's classification theorem for homogeneous functors implies that D_nF commutes with realization, that is, Φ_{D_nF} is a weak equivalence for all n. By induction Φ_{P_nF} is an equivalence for all n. As the Taylor series converges this implies Φ_F itself is a weak equivalence. $\square$

There is a slightly more general result one can prove along these lines. Namely, one can consider arbitrary simplicial objects in $U(C)$. Then restricted to such objects there is a weak equivalence

$$|[k] \mapsto \hat{F}(\_)| \xrightarrow{\Phi_{\hat{F}}} \hat{F}(|[k] \mapsto (\_)|) \ ,$$

where $\hat{F} \stackrel{def}{=} \underset{\longleftarrow}{holim}\{P_nF, p_nF\}$ denotes the <u>analytic completion</u> of F (at C). The proof is the same. Now Waldhausen has shown that $A(\_)$ is a continuous homotopy functor ([W]), and by Goodwillie we know that $\overline{A}(\_)$ has modulus 1. Hence we have

<u>Corollary 1.2.9</u>. *If $X = \{X_k\}_{k\geq 0}$ is a simplicial space and X_k is 1-connected for each k, then $\Phi_A : |[k] \mapsto \overline{A}(X_k)| \xrightarrow{\simeq} \overline{A}(|[k] \mapsto X_k|)$.*

<u>§1.3 Elementary Expansions and Representations in $H_q^n(|JX|)$</u>

As in the previous sections X will denote a basepointed connected simplicial set. Our object in the section will be to construct the maps $\tilde{\rho}_q : \tilde{D}_q(X) \to \tilde{A}(\Sigma X)$ of [CCGH] as described in the introduction, to provide some techniques for computing $\tilde{\rho}_q$ on differentials, and to relate certain restrictions of $\tilde{\rho}_q$ to products of elementary expansions. This will be used in section 2.3 where we compute the trace of $\tilde{\rho}_q$. From

the construction of $\tilde{\rho}_q$, it is easy to extend it to a map $\tilde{\rho}_q(JX) : \tilde{D}_q(JX) \to \overline{A}(\Sigma X)$. We do this, and prove analogous results for $\tilde{\rho}_q(JX)$.

Let $\imath : |X| \to |JX|$ denote the standard inclusion. Fixing an indexing of $\overset{q}{\vee} S^n$ and $\overset{q}{\vee} S^n \wedge |JX|_+$ we let $(S^n)_i$ resp. $(S^n \wedge |JX|_+)_i$ denote the ith term in the appropriate wedge for $1 \leq i \leq q$. Given $(x_1 \cdots, x_q) \in |X|^q$ let $\rho_q(x_1, \cdots, x_q)$ be the map which on $(S^n)_i$ is given by the composition

$$\begin{aligned}(S^n)_i = S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{id \vee f_i} S^n \vee (S^n \wedge |JX|) \xrightarrow{inc} \\ (S^n \wedge |JX|_+)_i \vee (S^n \wedge |JX|_+)_{i+1} \ .\end{aligned} \tag{1.3.1}$$

Here subscripts are taken mod q; thus $i + 1 = 1$ if $i = q, i + 1$ otherwise. The basepointed cofibration sequence

$$S^0 \xrightarrow{i} |J(X)|_+ \twoheadrightarrow |J(X)|$$

splits up to homotopy after a single suspension. Fixing $j_1 : \Sigma|JX| \to \Sigma(|JX|_+)$ with $\Sigma p \circ j_1 \simeq id$ and letting $j : \Sigma^n|JX| \to \Sigma^n(|JX|_+)$ be $\Sigma^{n-1}(j_1)$, inc is the map induced by the inclusions

$$\begin{aligned} S^n = S^n \wedge S^0 \hookrightarrow S^n \wedge |JX|_+ \ , \\ S^n \wedge |JX| \overset{j}{\hookrightarrow} S^n \wedge |JX|_+ \ . \end{aligned}$$

$f_i(s) = [s, \imath(x_i)] \in S^n \wedge |JX|$ for $s \in S^n$. "pinch" denotes the pinch map determined by the standard embedding $S^{n-1} \to S^n$, together with a fixed choice of homeomorphism from the cofibre to $S^n \vee S^n$. Clearly ρ_q is continuous and defines a map of spaces

$$\rho_q : |X|^q \to |H_q^n(|JX|)| \cong |\overline{M}_q^n(|JX|_+)| \ .$$

ρ_q is also equivariant with respect to $\mathbb{Z}/q$, where $\mathbb{Z}/q$ acts on $|X|^q$ by cyclically permuting the coordinates and on $H_q^n(|JX|)$ via the standard embedding $\mathbb{Z}/q \to \Sigma_q$ and the usual conjugation action of Σ_q on $H_n^q(|JX|)$.

<u>Proposition 1.3.3</u>. *ρ_q extends to a map $\overline{\rho}_q : E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q \to \Omega\overline{A}(\Sigma X)$, which in turn induces a map $\tilde{\rho}_q : \Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|X|^{[q]})) \to \overline{A}(\Sigma X)$.*

Proof. Taking the direct limit under suspension and stabilization yields a map $|X|^q \to |H(|JX|)|$ which we also denote by ρ_q. This map is still $\mathbb{Z}/q$-equivariant, where $\mathbb{Z}/q$ acts on the second space via the embedding $\mathbb{Z}/q \to \Sigma_q \to \Sigma_\infty$. It suffices to know now that the plus construction $|H(|JX|)| \to \Omega A(\Sigma X)$ can be done so as to be equivariant with respect to the action of Σ_∞ and that the action of Σ_∞ on $\Omega A(\Sigma X)$ is trivial up to homotopy. This follows from [FO]. The result is that $\Omega A(\Sigma X) \xrightarrow{i} E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X)$ admits a left homotopy inverse

$$p : E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X) \to \Omega A(\Sigma X)$$

$(p \circ i \simeq id)$ and we can take $\overline{\rho}_q$ to be the composition

$$E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q \xrightarrow{(1\times\rho_q)} E\Sigma_\infty \times_{\Sigma_\infty} |H(|JX|)| \to E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X) \xrightarrow{p} \Omega A(\Sigma X) \ .$$

Taking the infinite-loop extension of the adjoint of $\overline{\rho}_q$ yields a map

$$\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q)) \to A(\Sigma X) \ .$$

A well-known fact (which we re-prove in section 3.2) is that the projection

$$E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q \to E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|X|^{[q]} = (E\mathbb{Z}/q)_+ \wedge_{\mathbb{Z}/q} |X|^{[q]}$$

admits a stable section s. $\tilde{\rho}_q$ is then the composition

$$\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|X|^{[q]})) \xrightarrow{s} \Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q)) \to A(\Sigma X) \ .$$

Finally we note that all of the constructions are natural in X, and hence factor through $\overline{A}(\Sigma X)$. $\square$

The space $\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|X|^{[q]}))$ will be denoted by $\tilde{D}_q(X)$. $\tilde{D}_q(\_)$ can alternatively be thought of as a functor on connected spaces. The following is more or less contained in ([CCGH],§3). X and Y denote basepointed simplicial sets.

Proposition 1.3.4.

i) $(D_1\tilde{D}_q)_X(Y) \simeq \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|)$.

ii) $(D_1F_q)_X(Y) = \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{i=1}^{q}|X^{[i-1]} \wedge Y \wedge X^{[q-i]}|))$,

where $F_q(Z) = \Omega^\infty\Sigma^\infty(\Sigma|Z^{[q]}|)$. *The natural transformation* $F_q(\_) \to \tilde{D}_q(\_)$ *induces the fold map on 1st derivatives which is the infinite loop extension of the map*

$$\bigvee_{i=0}^{q-1} X^{[q-i-1]} \wedge Y \wedge X^{[i]} \to X^{[q-1]} \wedge Y\,,$$
$$(x_1, \cdots, x_{q-i-1}, y, x'_1, \cdots, x'_i) \mapsto (x'_1, \cdots, x'_i, x_1, \cdots, x_{q-i-1}, y)\,.$$

iii) *The inclusion* $i_q(X,Y) : X^{[q-1]} \wedge Y \to (X \vee Y)^{[q]} \to E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(X \vee Y)^{[q]}$ *induces an equivalence*

$$\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \to \lim_{\overrightarrow{n}} \Omega^m fibre(\tilde{D}_q(X \vee \Sigma^m Y) \to \tilde{D}_q(X)) = (D_1\tilde{D}_q)_X(Y)$$

<u>*Proof*</u>. i) and ii) appear in [CCGH]; the simplest way to see them is to first compute $(D_1F_q)_X(Y)$, which is easy, and then realize that the term $(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(\_))$ simply has the effect of "dividing by q" (in Goodwillie's words) via the fold map. Finally iii) follows from i) and ii) since the inclusion $X^{[q-1]} \wedge Y \to (X \vee Y)^{[q]}$ induces a map $\Omega^\infty\Sigma^\infty(\Sigma(|X^{[q-1]} \wedge Y|)) \to (D_1F_q)_X(Y)$ which agrees up to homotopy with the infinite loop extension of the inclusion of $X^{[q-1]} \wedge Y$ into the last term in the wedge $\bigvee_{i=1}^{q} X^{[i-1]} \wedge Y \wedge X^{[q-i]}$. $\square$

Recall that for a ring R and $r \in R$ the elementary matrix $e_{ij}(r)$ is the matrix $id + \overline{e}_{ij}(r)$ where $(\overline{e}_{ij}(r))_{k\ell} = r$ if $(k,\ell) = (i,j), 0$ otherwise. One should not try to push the analogy between $H_q^n(|JX|)$ and the group $GL_q(\mathbb{Z}[JX])$ too far, especially for finite n. However one can construct elements of $H_q^n(|JX|)$ which behave enough like elementary matrices to be useful. We call these <u>elementary</u> <u>expansions</u> since they correspond to the elementary expansions of classical Whitehead simple homotopy theory.

<u>Definition 1.3.5</u>. *Let X be a connected simplicial set and $\imath : |X| \to |JX|$ the standard inclusion. For $x \in |X|, e_{ij}(\imath(x)) \in |H_q^n(|JX|)|$ is given on $(S^n)_\ell \subset$*

$\bigvee_{k=1}^{q}(S^n)_k$ *by*

$$\underline{\ell \neq i} \qquad (S^n)_\ell \xrightarrow{inc} (S^n \wedge |JX|_+)_\ell$$

$$\underline{\ell = i} \qquad (S^n)_i = S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{id \vee f} S^n \vee (S^n \wedge |JX|) \xrightarrow{inc} (S^n \wedge |JX|_+)_i \vee (S^n \wedge |JX|_+)_j$$

where (as before) we have identified $H_q^n(|JX|)$ with $\overline{M}_q^n(|JX|_+)$. The sequence for $\ell = i$ is exactly as in (1.3.1) with $f(s) = [s, \imath(x)] \in S^n \wedge |JX|$; the only difference is the indexing of the last term. $e_{ij}(-\imath(x))$ is defined the same way, but with $id \vee f$ replaced by $id \vee (-f)$ where $-f$ is f composed with a fixed choice of $S^n \xrightarrow{(-1)} S^n$ representing loop inverse. The <u>reduced</u> <u>elementary</u> <u>expansion</u> $\overline{e}_{ij}(\imath(x))$ is given by

$$\underline{\ell \neq i} \qquad (S^n)_\ell \longrightarrow *$$

$$\underline{\ell = i} \qquad (S^n)_i \xrightarrow{f} (S^n \wedge |JX|) \xrightarrow{inc} (S^n \wedge |JX|_+)_j \ .$$

Similarly one can define $\overline{e}_{ij}(-\imath(x))$.

<u>Remark 1.3.6</u>. *When $i = j$, one could define $e_{ii}(\pm\imath(x))$ (loop inverse). Also, the definition of $e_{ij}(\pm((x))$ depends on a choice of parameters: choice of pinch map, choice of $j : S^n \wedge |JX| \to S^n \wedge |JX|_+$, and choice of $S^n \xrightarrow{(-1)} S^n$ representing -1. These, however, can be fixed so as to be compatible under suspension in the n coordinate and depending in a continuous and natural way on $x \in |X|$ and X. We assume this has been done. All of the manipulations we will do with these elements will be natural in X and $x \in |X|$.*

In a similar vein we will sometimes want to know that two maps depending on $x \in |X|$ (resp. a diagram depending on $X, Y, \dots$) are homotopic by a natural homotopy which depends continuously on $x \in |X|$ (resp. naturally homotopy-commutative by a homotopy which depends continuously on the spaces $X, Y, \dots$). If this can be done, we will say the two maps are <u>canonically</u> homotopic (or that the diagram is <u>canonically</u> h-commutative).

<u>Proposition 1.3.7</u>. *Suppose $f = e_{i_1 j_1}(\imath(x_1)) \cdot \ldots \cdot e_{i_n j_n}(\imath(x_n))$ for $x_i \in |X|$. Then there is a canonical homotopy $f \cdot f^{-1} \simeq *$, where*
$f^{-1} = e_{i_n j_n}(-\imath(x_n)) \cdot \ldots \cdot e_{i_1 j_1}(-\imath(x_1))$.

Proof. There is certainly a homotopy. It can be made canonical by concentrating the homotopy in the spherical coordinates. This involves choosing a homotopy between

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\text{pinch} \vee \text{id}} S^n \vee S^n \vee S^n$$

(1.3.8) and

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\text{id} \vee \text{pinch}} S^n \vee S^n \vee S^n$$

as well as a homotopy between

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\text{id} \vee \text{(-1)}} S^n \vee S^n \xrightarrow{\text{fold}} S^n$$

(1.3.9) and the trivial map

$$S^n \longrightarrow *$$

□

Note that we are not making any claims that such a homotopy is unique, even up to homotopy. We will also need

Proposition 1.3.10. *For $x_1, \ldots, x_{q-1} \in |X|, y \in |Y|$, there is a canonical homotopy between*

$$e_{12}(-\imath(x_1)) \cdot e_{23}(-\imath(x_2)) \cdot \ldots \cdot e_{q-1q}(-\imath(x_{q-1}))\overline{e}_{q1}(\imath(y)) \text{ and}$$
$$\overline{e}_{11}((\prod_{i=1}^{q-1} - \imath(x_i))\imath(y)) + \overline{e}_{21}((\prod_{i=2}^{q-1} - \imath(x_i))\imath(y)) + \cdots + \overline{e}_{q1}(\imath(y))$$

where "+" denotes loop sum.

Proof. On the level of matrices this is clear; the product here is taking place in $|J(X \vee Y)|$. Properly speaking, we should write $\prod_{i=j}^{q-1} - \imath(x_i)$ as $(-1)^{q-1-j}\prod_{i=j}^{q-1} \imath(x_i)$ since $|J(X \vee Y)|$ is a monoid without any strict inverses. To realize that the obvious homotopy is canonical, we note that it involves i) reparamerization to pass between the sequence of pinch maps used to evaluate the compositions and ii) reparametrization to reposition the iterated power of (-1) appearing in the expression $(-1)^{q-1-j}\prod_{i=j}^{q-1} \imath(x_i)$. Both of these can be done in a natural and continuous way with respect to the parameters $x_1, \ldots, x_{q-1}, y$ involved. □

The next result relates the representations ρ_q of (1.3.1) to products of elementary expansions. This is needed for the computation of the trace on $\tilde{\rho}_q$ given in §2.3. We define representations $\overline{\rho}_q^1, \overline{\rho}_q^2$ as follows:

$$\begin{aligned} \overline{\rho}_q^1(x_1, \ldots, x_{q-1}) &= \rho_q(x_1, \ldots, x_{q-1}, *) \quad x_i \in X \\ \overline{\rho}_q^2(y) &= p_2 \rho_q(*, *, \ldots, *, y) \quad y \quad \in Y \end{aligned} \tag{1.3.11}$$

where $p_2 : H_q^n(|J(X \vee Y)|) \to M_q^n(|\overline{F}_1(X, Y)|)$ is as in Lemma 1.1.4, X and Y connected.

<u>Proposition 1.3.12</u>. *As continuous maps $\overline{\rho}_q^1$ and $\overline{\rho}_q^2$ are canonically homotopic to the following products of elementary expansions:*

$$\overline{\rho}_q^1(x_1, \ldots, x_{q-1}) \simeq$$
$$e_{q-1q}(\imath(x_{q-1}))e_{q-2q-1}(\imath(x_{q-2})) \cdot \ldots \cdot e_{12}(\imath(x_1)) \ ;$$
$$\overline{\rho}_q^2(y) \simeq \overline{e}_{q1}(y) \ .$$

<u>Proof</u>. This again only involves a reparametrization in the spherical coordinate independent of X and Y, in the case $\overline{\rho}_q^1$. In the case of $\overline{\rho}_q^2$ we needn't do anything, as the projection map p_2 kills the identity maps along the diagonal and we are left with a single non-zero entry. $\square$

<u>Remark 1.3.13</u>. *The above canonical homotopies arise from Steinberg identities, which hold in $H_k^n(|GX|)$ up to canonical homotopies. Most types of identities among elementary expansions which hold up to homotopy do <u>not</u> hold up to canonical homotopy. For example, it is not true that the entire representation ρ_q is <u>canonically</u> homotopic to a product of elementary expansions. This type of problem arises whenever one tries to analyze such cyclic representations in terms of products of elementary expansions.*

We have stated the above results using elementary expansions with entries in $\imath(|X|) \subset J|X|$, which is all we will need for chapter 2. However all of the above constructions apply to the more general case where one allows arbitrary entries in

$J|X|$ (or even $|GX|$ when $|X|$ is not a suspension). Thus for $y \in J|X| \cong |JX|$, one defines $e_{ij}(y) \in |H_n^q(|JX|)|$ exactly as in definition 1.3.5 where $f : S^n \to S^n \wedge |JX|$ is the map $f(s) = [s, y] \in S^n \wedge |JX|$. Similarly for the reduced elementary expansion $\overline{e}_{ij}(y)$. Remark 1.3.6 and propositions 1.3.7, 1.3.10 and 1.3.12 apply in this more general context.

<u>Proposition 1.3.14</u>. *For $a_1, \ldots, a_{q-1} \in |JX|$, $b \in |\overline{F}_1(X, Y)|$ there is a canonical homotopy between*

$$e_{12}(-a_1) \cdot e_{23}(-a_2) \cdot \ldots \cdot e_{q-1q}(-a_{q-1}) \cdot \overline{e}_{q1}(b) \quad \textit{and}$$
$$\overline{e}_{11}((-1)^{q-1}(\prod_{i=1}^{q-1} a_i)b) + \overline{e}_{21}((-1)^{q-2}(\prod_{i=2}^{q-1} a_i)b) + \cdots + \overline{e}_{q-11}((-1)a_{q-1}b) + \overline{e}_{q1}(b) \ .$$

The representations ρ_q also extend in a natural way to yield a continuous map $\rho_q : |JX|^q \to |H_q^n(|JX|)|$, which on a q-tuple $(a_1, \ldots, a_q) \in |JX|^q$ is given exactly as in (1.3.1) where f_i is now the map $f_i(s) = [s, a_i] \in S^n \wedge |JX|$. Proposition 1.3.3 applies with $|JX|$ in place of $|X|$ for the domain of $\tilde{\rho}_q$; in fact it is easy to see that the map of proposition 1.3.3 factors by this extension.

<u>Proposition 1.3.15</u>. *For $a_1, \ldots, a_{q-1} \in |JX|$, $b \in |\overline{F}_1(X, Y)|$, let*

$$\overline{\rho}_q^1(a_1, \ldots, a_{q-1}) = \rho_q(a_1, a_2, \ldots, a_{q-1}, *)$$
$$\overline{\rho}_q^2(b) = p_2 \rho_q(*, *, \ldots, *, b)$$

as in (1.3.11). Then as continuous maps $\overline{\rho}_q^1$ and $\overline{\rho}_q^2$ are canonically homotopic to the following product of elementary expansions:

$$\overline{\rho}_q^1(a_1, a_2, \ldots, a_{q-1}) \simeq e_{q-1q}(a_{q-1}) e_{q-2q-1}(a_{q-2}) \cdot \ldots \cdot e_{12}(a_1)$$
$$\overline{\rho}_q^2(b) \simeq \overline{e}_{q1}(b) \ .$$

The proofs follow exactly as before.

When $a_1 = a_2 = \cdots = a_{q-1} = *$ the map $\overline{\rho}_q^1(a_1, \ldots, a_{q-1}) = \overline{\rho}_q^1(*, \ldots, *)$ is not the standard inclusion $\overset{q}{\vee}(S^n) \mapsto \overset{q}{\vee}(S^n \wedge |JX|_+)$, but only homotopic to it. This homotopy, which we will need later on (c.f. 2.3.6 below) is a wedge of homotopies

between

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\text{id} \vee *} S^n \vee S^n \xrightarrow{\text{fold}} S^n$$

(1.3.16) and

$$S^n \xrightarrow{\text{id}} S^n$$

This wedge produces a path between the basepoint of $H_q^n(|J(X)|)$ and $\overline{\rho}_q^1(*, \ldots, *)$.

As a final remark, we should note that in the above propositions involving minus signs, we are <u>not</u> requiring any type of coherence conditions to apply for this minus sign with respect to composition product (which in the limiting case $n \to \infty$ will involve the product structure on the generalized ring $\Omega^\infty \Sigma^\infty(|GX|_+)$). We are only stating that certain homotopies can be made canonical. The restriction on the "ring" under consideration that must be made in order for such a coherent (-1) to exist are substantial, as shown by Schwänzl and Vogt in [SV].

CHAPTER II

§2.1 Manipulation in the stable range

We follow closely the argument of Waldhausen ([W2], Th. 3.1) in proving

Theorem 2.1.1. *Let X and Y be pointed simplicial sets, with X connected and Y m-connected. Then the two spaces*

$$NH_k^n(|J(X \vee Y)|) \qquad \textit{and} \qquad N^{cy}(H_k^n(|JX|), M_k^n(\Sigma|\overline{F}_1(X,Y)|))$$

are q-equivalent, where $q = \min(n-2, 2m+1)$ and $n \geq 1$.

Proof. The notation is that of §1.1. Here the monoid structure on $H_k^n(|J(X \vee Y)|)$ and $H_k^n(|JX|)$ is the usual one, while the partial monoid structure on the $H_k^n(|JX|)$-bimonoid $M_k^n(|\overline{F}_1(X,Y)|)$ is trivial. The equivalence follows as in ([W2], Th. 3.1) by the construction of five maps, each of which is suitably connected.

The 1st map $H_k^n(|J(X \vee Y)|)$ admits a partial monoid structure where two elements are composable iff at most one of them lies outside the submonoid $H_k^n(|JX|)$. The nerve of this partial monoid is by definition the generalized wedge

$$\{[p] \mapsto \overset{p}{\vee}(H_k^n(|J(X \vee Y)|), H_k^n(|JX|))\} .$$

As Y is m-connected, the inclusion $H_k^n(|JX|) \to H_k^n(|J(X \vee Y)|)$ is also m-connected. It follows ([W2], Lemma 2.2.1) that the inclusion

$$\{[p] \mapsto \overset{p}{\vee}(H_k^n(|J(X \vee Y)|), H_k^n(|JX|))\} \hookrightarrow NH_k^n(|J(X \vee Y)|)$$

is $(2m+1)$-connected.

The 2nd map The inclusion $F_1(X,Y) \to J(X,Y)$ is $(2m+1)$-connected, hence induces a $(2m+1)$-connected map $\overline{M}_k^n(|F_1(X,Y)|_+) \to \overline{M}_k^n(|J(X \vee Y)|_+)$ of $H_k^n(|JX|)$-bimonoids. This in turn induces an inclusion of generalized wedges

$$\{[p] \mapsto \overset{p}{\vee}(\overline{M}_k^n(|F_1(X,Y)|_+), H_k^n(|JX|))\}$$
$$\hookrightarrow \{[p] \mapsto \overset{p}{\vee}(\overline{M}_k^n(J(X \vee Y)|_+), H_k^n(|JX|))\}$$

which is $(2m+1)$-connected in each degree by the gluing Lemma ([W2], Lemma 2.1.2) and induction on p. It follows that the inclusion of simplicial objects is also $(2m+1)$-connected.

<u>The 3rd map</u> As in ([W2], p.374), we consider the restriction to the path components corresponding to $H_k^n(|J(X \vee Y)|)$ of the inclusion

$$\begin{aligned} M_k^n(|F_1(X,Y)|_+) &= Map(\overset{k}{\vee} S^n,\ \overset{k}{\vee} S^n \wedge |F_1(X,Y)|_+) \\ &\hookrightarrow Map(\overset{k}{\vee} S^n,\ \prod^k S^n \wedge |F_1(X,Y)|_+) \\ &\simeq \prod^k \prod^k \Omega^n\Sigma^n(|F_1(X,Y)|_+)\ . \end{aligned}$$

This is an $(n-1)$-equivalence. Lemma 1, p. 374 of [W2] yields an $(n-2)$-equivalence

$$\Omega^n\Sigma^n(|F_1(X,Y)|_+) \simeq \Omega^n\Sigma^n(|JX_+ \vee \overline{F}_1(X,Y)|) \to \Omega^n\Sigma^n(|JX|_+) \times \Omega^n\Sigma^n(|\overline{F}_1(X,Y)|)\ .$$

The gluing lemma now applies to show that the map on nerves of partial monoids defined in Lemma 1.1.4

$$\begin{aligned} &\{[p] \mapsto \overset{p}{\vee}\ (\overline{M}_k^n(|F_1(X,Y)|_+), H_k^n(|JX|))\} \\ \to &\{[p] \mapsto \overset{p}{\vee}\ (H_k^n(|JX|) \ltimes M_k^n(|\overline{F}_1(X,Y)|), H_k^n(|JX|))\} \end{aligned}$$

is $(n-2)$-connected.

<u>The 4th map</u> Taking the trivial monoid structure on $M_k^n(|\overline{F}_1(X,Y)|)$ and forming its nerve, Lemma 2.3 of [W2] provides an equivalence

$$\begin{aligned} &\text{diag}\ (N^{cy}(H_k^n(|JX|),\ \Sigma.M_k^n(|\overline{F}_1(X,Y)|))) \\ &\xrightarrow[\simeq]{u} N(H_k^n(|JX|) \ltimes M_k^n(|\overline{F}_1(X,Y)|)) \\ &= \{[p] \mapsto \overset{p}{\vee}\ (H_k^n(|JX|) \ltimes M_k^n(|\overline{F}_1(X,Y)|), H_k^n(|JX|))\}\ . \end{aligned}$$

Here $\Sigma.A$ denotes the simplicial space $\{[p] \mapsto \overset{p}{\vee}\ (A,*)\}$ which arises on taking the nerve of a trivial partial monoid.

<u>The 5th map</u> Partial geometric realization produces a map from $\Sigma.M_k^n(|\overline{F}_1(X,Y)|)$ to $S^1 \wedge M_k^n(|\overline{F}_1(X,Y)|)$. The pairing map

$$S^1 \wedge M_k^n(|\overline{F}_1(X,Y)|) \to M_k^n(S^1 \wedge |\overline{F}_1(X,Y)|)$$

together with partial geometric realization produces a map

$$N^{cy}(H_k^n(|JX|), \Sigma.M_k^n(|\overline{F}_1(X,Y)|)) \to N^{cy}(H_k^n(|JX|), M_k^n(S^1 \wedge |\overline{F}_1(X,Y)|)) \ .$$

By the realization lemma, this map is $(2m+1)$-connected.

These 5 maps taken together yield the required sequence connecting $N(H_k^n(|J(X \vee Y)|))$ and $N^{cy}(H_k^n(|JX|), M_k^n(\Sigma|\overline{F}_1(X,Y)|))$. Each of the maps is $\min(n-2, 2m+1)$-connected and the theorem follows. $\square$

The maps constructed in the above theorem are compatible with suspension in the n-coordinate as well as pairing under block sum, by which we will always mean the wedge-sum of section 1.1 for the appropriate monoid in question. Taking the limit as n goes to ∞ yields a sequence of maps connecting

$$\coprod_{k \geq 0} N(H_k(|J(X \vee Y)|)) \qquad \text{and} \qquad \coprod_{k \geq 0} N^{cy}(H_k(|JX|), M_k(\Sigma|\overline{F}_1(X,Y)|)) \ ;$$

each of these maps preserves block-sum and is $(2m-1)$-connected for $(m-1)$-connected Y. We thus get a sequence of maps between their group completions which is also $(2m-1)$-connected.

Denote $\Omega B(\coprod_{k \geq 0} N^{cy}(H_k(|JX|), M_k(\Sigma|\overline{F}_1(X,Y)|))$ by $C(X,Y)$, and $C(X,\_\,)$ by $C_X(\_)$; $C_X(\_)$ is a homotopy functor on the category $T(*)$.

<u>Lemma 2.1.2</u>. *(compare* [W2]*, Lemma 4.2) There is an equivalence of 1st differentials*

$$\begin{aligned} (D_1 A\Sigma)_X(Y) &= \lim_{\substack{\rightarrow \\ n}} \Omega^n(fibre(A(\Sigma(X \vee (S^n \wedge Y))) \to A(\Sigma X))) \\ \simeq (D_1 C_X)_*(Y) &= \lim_{\substack{\rightarrow \\ n}} \Omega^n(fibre(C(X, S^n \wedge Y) \to C(X,*))) \ . \end{aligned}$$

<u>Proof</u>. This is an immediate consequence of the above theorem; for each n, we have an equivalence $A(\Sigma X) \simeq C(X,*)$ as well as a $(2n-1)$-equivalence between $A(\Sigma(X \vee (S^n \wedge Y)))$ and $C(X, S^n \wedge Y)$. This gives a $(2n-1)$-equivalence between $fibre(A(\Sigma(X \vee (S^n \wedge Y))) \to A(\Sigma X))$ and $fibre(C(X, S^n \wedge Y_+) \to C(X,*))$ which in the above limit yields a weak equivalence. $\square$

§2.2 The Generalized Waldhausen Trace Map

In this section we construct a trace map, generalizing the construction of Waldhausen in [W2]. The techniques are essentially those of ([W2], §4).

Let F, F' be basepointed spaces, with F $(m-1)$-connected.

Lemma 2.2.1. *For all integers $k, m, m \geq 0$ the map of spaces*

$$\begin{aligned} &Map(\overset{k}{\vee} S^n, S^{n+m} \wedge F') \wedge Map(S^{n+m}, S^{n+m} \wedge F) \\ &\qquad \xrightarrow{\lambda} Map(\overset{k}{\vee} S^n, S^{n+m} \wedge F' \wedge F) \ , \end{aligned}$$

given by

$$\lambda(f \wedge g) : \overset{k}{\vee} S^n \xrightarrow{f} S^{n+m} \wedge F' \xrightarrow{g \wedge id} S^{n+m} \wedge F \wedge F' \xrightarrow[\cong]{id \wedge \text{switch}} S^{n+m} \wedge F' \wedge F \ ,$$

is $(3m-1)$-connected.

Proof. This is a slight generalization of Lemma 4.3 of [W2], and the proof is the same. Namely, there is a commutative diagram

$$\begin{tikzcd}
Map(S^n, S^{n+m} \wedge F')^k \wedge Map(S^{n+m}, S^{n+m} \wedge F) \arrow[rr] & & Map(S^n, S^{n+m} \wedge F' \wedge F)^k \\
\overset{k}{\vee}(S^0 \wedge S^m \wedge F') \wedge Map(S^{n+m}, S^{n+m} \wedge F) \arrow[u] & & \\
\overset{k}{\vee}(S^0 \wedge S^m \wedge F') \wedge Map(S^m, S^m \wedge F) \arrow[u] & & \overset{k}{\vee} S^0 \wedge (S^m \wedge F' \wedge F) \arrow[uu] \\
\overset{k}{\vee}(S^0 \wedge S^m \wedge F') \wedge F \arrow[u] \arrow[urr, leftrightarrow, "\cong"'] & &
\end{tikzcd}$$

where the top horizontal map corresponds to the map given above, the right vertical map is $(4m-1)$-connected and each of the left vertical maps is $(3m-1)$-connected. $\square$

For connected Y', we have a homeomorphism of $|JX|$-bimonoids

$$|\overline{F}_1(X,Y')| \cong |JX|_+ \wedge |Y'| \wedge |JX|_+ \ .$$

If $Y' = S^m \wedge Y_+$ then the above lemma applies with $F = \overset{k}{\vee} |Y'| \wedge |JX|_+$ and $F' = |JX|_+$. This yields a sequence of maps

$$\begin{array}{c}
H^n_k(|JX|)^p \times Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^{n+m} \wedge |\overline{F}_1(X,Y')|) \\
\cong \updownarrow \varphi^p_{m,n,k} \\
H^n_k(|JX|)^p \times Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^{n+m} \wedge |JX|_+ \wedge |Y'| \wedge |JX|_+) \\
\uparrow f^p_{m,n,k} \\
H^n_k(|JX|)^p \times (Map(\overset{k}{\vee} S^n, S^{n+m} \wedge |JX|_+) \wedge Map(S^{n+m}, \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+)) \\
\downarrow g^p_{m,n,k} \\
Map(S^{n+m}, S^{n+3m} \wedge |JX|_+ \wedge |Y_+|) \ .
\end{array} \tag{2.2.2}$$

$f^p_{m,n,k}$ is induced by the pairing in Lemma 2.2.1, and is $(3m-1)$-connected. $g^p_{m,n,k}$ associates to the $(p+2)$-tuple $(\alpha_1, \ldots, \alpha_p; \beta_1 \wedge \beta_2)$ the composition

$$\begin{array}{l}
S^{n+m} \xrightarrow{\beta_2} \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+ \\
\qquad \xleftrightarrow{\cong} \overset{k}{\vee} (S^n \wedge |JX|_+) \wedge (S^m \wedge |Y'|) \\
\qquad \xrightarrow{((\alpha_1\alpha_2\cdot\ldots\cdot\alpha_p)\wedge id)} \overset{k}{\vee} (S^n \wedge |JX|_+) \wedge (S^m \wedge |Y'|) \\
\qquad \xleftrightarrow{\cong} (\overset{k}{\vee} S^n) \wedge (|JX|_+ \wedge S^m \wedge |Y'|) \\
\qquad \xrightarrow{\beta_1 \wedge id} (S^{n+m} \wedge |JX|_+) \wedge (|JX|_+ \wedge S^m \wedge |Y'|) \\
\qquad \xleftrightarrow{\cong} S^{n+m} \wedge (|JX|_+ \wedge |JX|_+) \wedge (S^m \wedge |Y'|) \\
\qquad \xrightarrow{id \wedge \mu \wedge id} S^{n+m} \wedge |JX|_+ \wedge S^m \wedge |Y'| \\
\qquad \xleftrightarrow{\cong} S^{n+3m} \wedge |JX|_+ \wedge |Y_+| \ .
\end{array} \tag{2.2.3}$$

The equivalence of trivial partial monoids

$$Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^{n+m} \wedge |JX|_+ \wedge |Y'| \wedge |JX|_+) \cong Map(\overset{k}{\vee} S^n, \overset{k}{\vee} S^{n+m} \wedge |\overline{F}_1(X,Y')|)$$

which induces the equivalence $\varphi^p_{m,n,k}$, commutes with the natural left and right actions of $H^n_k(|JX|)$, and therefore is a $H^n_k(|JX|)$-bimonoid equivalence. Now for $f \in H^n_k(|JX|), g \in Map(\overset{k}{\vee} S^n, S^{n+m} \wedge |JX|_+), h \in (S^{n+m}, \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+)$ the pairings

$$(f,g) \mapsto f \cdot g,$$

$$\begin{aligned} f \cdot g : \overset{k}{\vee} S^n &\xleftrightarrow{\cong} \overset{k}{\vee} S^n \wedge S^0 \hookrightarrow \overset{k}{\vee} S^n \wedge |JX|_+ \xrightarrow{f} \overset{k}{\vee} S^n \wedge |JX|_+ \\ &\xleftrightarrow{\cong} |JX|_+ \wedge (\overset{k}{\vee} S^n) \xrightarrow{id \wedge g} |JX|_+ \wedge S^{n+m} \wedge |JX|_+ \xleftrightarrow{\cong} \\ &S^{n+m} \wedge |JX|_+ \wedge |JX|_+ \xrightarrow{id \wedge \mu} S^{n+m} \wedge |JX|_+ \end{aligned} \tag{2.2.4}$$

and

$$(h,f) \mapsto h \cdot f,$$

$$\begin{aligned} h \cdot f : S^{n+m} &\xrightarrow{h} \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+ \\ &\xleftrightarrow{\cong} S^m \wedge |Y'| \wedge (\overset{k}{\vee} S^n \wedge |JX|_+) \\ \xrightarrow{id \wedge f} S^m \wedge |Y'| \wedge (\overset{k}{\vee} S^n \wedge |JX|_+) &\xleftrightarrow{\cong} \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+ \end{aligned}$$

induce a left resp. right $H^n_k(|JX|)$-monoid structure on $Map(\overset{k}{\vee} S^n, S^{n+m} \wedge |JX|_+)$ resp. $Map(S^{n+m}, \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+)$, and hence an $H^n_k(|JX|)$-bimonoid structure on $Map(\overset{k}{\vee} S^n, S^{n+m} \wedge |JX|_+) \wedge Map(S^{n+m}, \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+)$. With this structure, the pairing map of Lemma 2.2.1 (for $F = \overset{k}{\vee} S^m \wedge |JX|_+$ and $F' = |JX|_+ \wedge |Y'|$) becomes an $H^n_k(|JX|)$-bimonoid map (up to natural equivalence involving permutation of spherical coordinates).

All of the above maps are natural with respect to stabilization in the k-coordinate. The equivalence $|\overline{F}_1(X, S^m \wedge Y_+)| \overset{\cong}{=} |JX|_+ \wedge S^m \wedge |Y_+| \wedge |JX|_+$ is compatible with suspension in the m-coordinate and so $\varphi^p_{m,n,k}$ in (2.2.2) is compatible with suspension in both the n and m coordinates. The homeomorphisms appearing in (2.2.3) and (2.2.4) involve a choice of natural equivalences, which can be chosen so as to be compatible with respect to suspension in these coordinates. Also with α_1, β_1 and β_2 as in (2.2.3) it follows directly from (2.2.3) and (2.2.4) that $g^1_{m,n,k}(\alpha_1; \beta_1 \wedge \beta_2) = g^0_{m,n,k}((\alpha_1\beta_1) \wedge \beta_2) = g^0_{m,n,k}(\beta_1 \wedge (\beta_2\alpha_1))$. Moreover by construction $\coprod_{k \geq 0} g^p_{m,n,k}$ maps wedge-sum to loop sum. Putting this all together, we get

Theorem 2.2.5. *For each $n, m, k \geq 1$, $\{\varphi^p_{m,n,k}\}_{p\geq 0}$, $\{f^p_{m,n,k}\}_{p\geq 0}$ and $\{g^p_{m,n,k}\}_{p\geq 0}$ induce well-defined maps of simplicial spaces:*

$$\begin{array}{c} N^{cy}(H^n_k(|JX|), M^n_k(S^m \wedge |\overline{F}_1(X,Y')|)) \\ \cong\updownarrow \varphi^{\cdot}_{m,n,k} \\ N^{cy}(H^n_k(|JX|), M^n_k(S^m \wedge |JX|_+ \wedge |Y'| \wedge |JX|_+)) \\ \uparrow f^{\cdot}_{m,n,k} \\ N^{cy}(H^n_k(|JX|), Map(\overset{k}{\vee} S^n, S^{n+m} \wedge |JX|_+) \wedge Map(S^{n+m}, \overset{k}{\vee} S^{n+m} \wedge |Y'| \wedge |JX|_+)) \\ \downarrow g^{\cdot}_{m,n,k} \\ Map(S^{n+m}, S^{n+3m} \wedge |JX|_+ \wedge |Y_+|) \ , \end{array}$$

where the simplicial structure on the range of $(g^{\cdot}_{m,n,k})$ is trivial, $Y' = S^m \wedge Y_+$, and $f^{\cdot}_{m,n,k}$ is $(3m-1)$-connected. These maps are compatible with suspension in the m and n coordinates, and stabilization in the k-coordinate. They are also natural with respect to X and Y (where X is connected).

Proof. The only point that has not already been covered is the statement concerning the connectivity of $f^{\cdot}_{m,n,k}$. But this follows by the realization lemma, as $f^p_{m,n,k}$ is $(3m-1)$-connected for each p. $\square$

Now $g^{\cdot}$ takes wedge-sum to loop sum, as we have already noted, and thus factors via group completion with respect to wedge-sum. So passing to the limit in m yields a map T:

$$\begin{aligned} &(D_1C_X)_*(Y_+) \\ &= \lim_{\vec{m}} \Omega^m(fibre(C(X, S^m \wedge Y_+) \to C(X,*))) \\ &\quad \xrightarrow{T} \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)) \ . \end{aligned} \tag{2.2.6}$$

Precomposing by the equivalence of Lemma 2.1.2 we get

$$Tr_X(Y) : (D_1A\Sigma)_X(Y_+) \to \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)). \tag{2.2.7}$$

(in the case $X = \{pt\}$ we recover the map constructed in [W2]). This map is natural in both X and Y. Taking the fibre with respect to the map $Y_+ \to \{pt\}$ yields (for

basepointed Y) the (reduced) <u>generalized Waldhausen trace map</u>

$$\overline{Tr}_X(Y) : (D_1 A\Sigma)_X(Y) \to \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y|)) \tag{2.2.8}$$

where on the right we have for $q \geq 1$ composed with the (basepointed) projection $Y_+ \to Y$.

Finally, we can follow by projection to the qth factor $\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|)$; this yields a map $\overline{Tr}_X(Y)_q : (D_1 A\Sigma)_X(Y) \to \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|)$. For connected X , we have $Tr_X(Y) \cong \prod_{q\geq 1} Tr_X(Y)_q$.

§2.3 Computing the trace on $\overline{\rho}_q$

By the results of section 1.3, there is a map

$$\tilde{\rho} = \prod_{q\geq 1} \tilde{\rho}_q : \tilde{D}(X) = \prod_{q\geq 1} \tilde{D}_q(X) \to \overline{A}(\Sigma X)$$

for a connected simplicial set X. This map is natural in X, and is induced (1.3.2) by representations $\rho_q : |X|^q \to |H_k^n(|JX|)|$. The product that appears on the L.H.S. is the weak product; however, we note that as X is connected, the weak product in this case is weakly equivalent to the strong product. Replacing X by $X \vee Y$, we define $\rho_q(X,Y)$ as the restriction of ρ_q to $|X|^{q-1} \times |Y| \subset |X \vee Y|^q$. Note that this inclusion induces the inclusion $i_q(X,Y)$ of proposition 1.3.4 iii) after passing to smash products. Let $\tilde{i}_q(X,Y) = \Omega^\infty\Sigma^\infty(\Sigma i_q(X,Y))$. Then the composition

$$\tilde{\rho}_q(X,Y) : \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \xrightarrow{\tilde{i}_q(X,Y)} \Omega^\infty\Sigma^\infty(\Sigma|X \vee Y|^{[q]}) \to$$
$$\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}|(X \vee Y)^{[q]}|)) = \tilde{D}_q(X \vee Y) \xrightarrow{\tilde{\rho}_q} \overline{A}(\Sigma(X \vee Y))$$

can alternatively be described as the precomposition of $\Omega^\infty\Sigma^\infty(\Sigma\rho_q(X,Y))$ with the stable section $s : \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \to \Omega^\infty\Sigma^\infty(\Sigma|X|^{q-1} \times |Y|)$, followed by the map into $(\overline{A}\Sigma)(X \vee Y)$. Proposition 1.3.4 tells us that the map $\tilde{i}_q(X,Y)$ induces an equivalence $\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \xrightarrow{\cong} (D_1\tilde{D}_q)_X(Y)$, and Goodwillie's results tell us that $\tilde{\rho}$ is an equivalence for connected spaces iff $(D_1\tilde{\rho})_X(Y)$ is an equivalence for all connected X. They also tell us that $(D_1A\Sigma)_X(Y)$ and $(D_1\tilde{D})_X(Y)$ are the same for connected X. The primary task now is to show that $\overline{Tr}_X(Y) \circ (D_1\tilde{\rho})_X(Y)$ is an equivalence for all connected X.

Theorem 2.3.1. *For $p \neq q, \overline{Tr}_X(Y)_p \circ \tilde{\rho}_q(X,Y) \simeq *$. When $p = q$, we have $\overline{T}r_X(Y)_q \circ \tilde{\rho}_q(X,Y) \simeq (-1)^{q-1}$. These homotopies are canonical in X and Y, and hold for all connected X and $q \geq 1$. Thus $\overline{T}r_X(Y) \circ (\prod_{q\geq 1} \tilde{\rho}_q(X,Y))$ is an equivalence for connected X, which implies $\overline{Tr}_X(Y) \circ (D_1\tilde{\rho})_X(Y)$ is an equivalence for connected X.*

Proof. The last implication follows by proposition 1.3.4. Our main objective is the evaluation of the trace map $\overline{Tr}_X(Y)$ on $\tilde{\rho}_q(X,Y)$, which we will do in stages.

First, we determine what happens to the image of the representation $\rho_q(X,Y)$ under the maps constructed in Theorem 2.1.1. This will bring us into the cyclic bar construction. The maps provided by (2.2.2) – (2.2.5) will then determine the composition $\overline{T}r_X(Y)$.

We will assume $Z' = X \vee Y$ where X is connected and Y is m'-connected. ρ_q (resp. its restriction $\rho_q(X,Y)$) is induced by a simplicial representation Z'^q (resp. $X^{q-1} \times Y$) $\to H_q^n(|JZ'|)$ which we will also denote by ρ_q . This map can be represented simplicially by a map of partial monoids:

$$\{[p] \mapsto \overset{p}{\vee}(Z'^q, *)\} \xrightarrow{\{\overset{p}{\vee}\rho_q\}} NH_q^n(|JZ'|) \ .$$

We will construct five diagrams, one for each of the maps in the proof of Theorem 2.1.1.

<u>The 1st diagram</u> The first map in Theorem 2.1.1 was induced by the $(2m'+1)$-connected inclusion of partial monoids:

$$\{[p] \mapsto \overset{p}{\vee}(H_q^n(|JZ'|), H_q^n(|JX|))\} \xrightarrow{\imath_1} NH_q^n(|JZ'|) \ .$$

The generalized wedge on the left contains the image of $\{\overset{p}{\vee}\rho_q\}$ and hence $\{\overset{p}{\vee}\rho_q(X,Y)\}$. Thus $\{\overset{p}{\vee}\rho_q(X,Y)\} = \imath_1 \circ \overline{\rho}_{q,1}$, were $\overline{\rho}_{q,1}$, is a map of generalized wedges, induced in each degree by the representation $\rho_q(X,Y)$, which fits into the commutative diagram:

$$\begin{tikzcd}
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overset{p}{\vee}\rho_q(X,Y)"] \arrow[d, equal] & NH_q^n(|JZ'|) \\
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overline{\rho}_{q,1}"] & \{[p] \mapsto \overset{p}{\vee}(H_q^n(|JZ'|), H_q^n(|JX|))\} \arrow[u, "\imath_1"']
\end{tikzcd} \tag{2.3.2}$$

<u>The 2nd diagram</u> The second map in Theorem 2.1.1 is the $(2m'+1)$-connected map of generalized wedges induced by the $(2m'+1)$-connected inclusion

$$\overline{M}_q^n(|F_1(X,Y)|_+) \to \overline{M}_q^n(|JZ'|_+) \cong H_q^n(|JZ'|) \ .$$

As the image of ρ_q is contained in $\overline{M}_q^n(|F_1(X,Y)|_+)$, we can further factor $\rho_q(X,Y)$ as $\imath_2 \circ \overline{\rho}_{q,2}$. $\overline{\rho}_{q,2}$ is defined exactly as $\overline{\rho}_{q,1}$ – it is the (unique) map of generalized

wedges induced by $\rho_q(X,Y)$ which makes the following diagram commute:

$$
\begin{tikzcd}
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overline{\rho}_{q,1}"] \arrow[d, equal] & \{[p] \mapsto \overset{p}{\vee}(\overline{M}_q^n(|JZ'|_+), H_q^n(|JX|))\} \\
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overline{\rho}_{q,2}"] & \{[p] \mapsto \overset{p}{\vee}(\overline{M}_q^n(|F_1(X,Y)|_+), H_q^n(|JX|))\} \arrow[u, "\imath_2"']
\end{tikzcd}
\tag{2.3.3}
$$

<u>The 3rd diagram</u> The $(n-2)$-connected map

$$
\overline{M}_q^n(|F_1(X,Y)|_+) \xrightarrow{p_1 \times p_2} H_q^n(|JX|)) \times M_q^n(|\overline{F}_1(X,Y)|)
$$

induces the third map in Theorem 2.1.1, where the projections p_1, p_2 are induced by the projections of $F_1(X,Y)$ to JX and $\overline{F}_1(X,Y)$ respectively. Let $\overline{\rho}_q^i = p_i \circ \rho_q(X,Y)$ for $i = 1,2$. Then we have a commuting square

(2.3.4)

$$
\begin{tikzcd}
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overline{\rho}_{q,2}"] \arrow[d, equal] & \{[p] \mapsto \overset{p}{\vee}(\overline{M}_q^n(|F_1(X,Y)|_+), H_q^n(|JX|))\} \arrow[d] \\
\{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} \arrow[r, "\overline{\rho}_{q,3}"] & \{[p] \mapsto \overset{p}{\vee}(\overline{M}_q^n(|JX|_+) \ltimes M_q^n(|\overline{F}_1(X,Y)|), H_q^n(|JX|))\}
\end{tikzcd}
$$

where $\overline{\rho}_{q,3}$ is induced in each degree by the product $\overline{\rho}_q^1 \times \overline{\rho}_q^2$.

<u>The 4th diagram</u> This is the first place where one encounters complications in computing the trace map on arbitrary representations. From equation (1.1.2) we can see the problem – when M is not a group but only grouplike there may be no simple way to choose f^{-1} for $f \in M$, which one needs to do in order to formally invert the equivalence $u : diag(N^{cy}(M, NE)) \xrightarrow{\simeq} N(M \ltimes E)$. In our case by first reducing the representation under consideration to $\rho_q(X,Y)$ we are able to circumvent this difficulty. For by proposition 1.3.12, $\overline{\rho}_q^1$ and $\overline{\rho}_q^2$ are canonically homotopic to a product of elementary expansions:

$$
\begin{aligned}
&\overline{\rho}_q^1(x_1, \ldots, x_{q-1}) \cong e_{q-1q}(\imath(x_{q-1}))e_{q-2q-1}(\imath(x_{q-2})) \cdot \ldots \cdot e_{12}(\imath(x_1)) \\
&\overline{\rho}_q^2(y) \cong \overline{e}_{q1}(\imath(y)) \text{ (the } \underline{\text{reduced}} \text{ expansion with } (q,1) \text{ entry } \imath(y))
\end{aligned}
\tag{2.3.5}
$$

where $\imath(X)$ denotes the image of $x \in |X|$ in $|JX|$ under the natural inclusion $X \to JX$, and similarly for Y (for notational simplicity, we have used $|\overline{\rho}_q^1|$ and $|\overline{\rho}_q^2|$ and $|Y|$

respectively. To recover $\overline{\rho}^1_q$ and $\overline{\rho}^2_q$ as above one applies Sing $(\_)$ and precomposes with the map $A \to Sing(|A|))$. The notation is explained in section 1.3. For such a product of elementary expansions proposition 1.3.7 yields a canonical homotopy between $f^{-1}f, *$ and ff^{-1} where $f^{-1} = e_{12}(-\imath(x_1))e_{23}(-\imath(x_2))\cdot\ldots\cdot e_{q-1q}(-\imath(x_{q-1}))$ for $f = \overline{\rho}^1_q(x_1, \ldots, x_{q-1})$ as above. We can define a map

$$\begin{aligned} |\overline{\rho}^1_{q,4}| : |X|^{q-1} \times |Y| \to & |N^{cy}_1(H^n_q(|JX|), M^n_q(|\overline{F}_1(X,Y)|))| \\ & = H^n_q(|JX|) \times M^n_q(|\overline{F}_1(X,Y)|)| \end{aligned}$$

by $(x_1, \ldots, x_{q-1}, y) \mapsto (f, f^{-1}ef^{-1})$ where $f = \overline{\rho}^1_q(x_1, \ldots, x_{q-1}), e = \overline{e}_{q1}(y)$ as given in (2.3.5). Extending degreewise yields a map $\overline{\rho}_{q,4}$ and a canonically homotopy-commutative diagram

(2.3.6)

$$\begin{array}{ccc} \{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} & \xrightarrow{\overline{\rho}_{q,3}} & \{[p] \mapsto \overset{p}{\vee}(\overline{M}^n_q(|JX|_+) \ltimes M^n_q(|\overline{F}_1(X,Y)|), H^n_q(|JX|))\} \\ \| & & \uparrow \simeq \\ \{[p] \mapsto \overset{p}{\vee}(X^{q-1} \times Y, *)\} & \xrightarrow{\overline{\rho}_{q,4}} & diag(N^{cy}(H^n_q(|JX|), \Sigma.M^n_q(|\overline{F}_1(X,Y)|))) \end{array}$$

where the equivalence on the right hand side is the map u defined in (1.1.2), and the bottom map has been modified by the homotopy of (1.3.16) applied to the first map in (2.3.5) to make it basepoint-preserving. Recall that $\Sigma.A$ is shorthand notation for $\{[p] \mapsto \overset{p}{\vee}(A, *)\}$. The fact that the diagram is canonically homotopy-commutative is important. Note also that $\overline{\rho}_{q,4}$ is given on one-simplices by $\overline{\rho}^1_{q,4}$.

<u>The 5th diagram</u> In Theorem 2.1.1 the fifth map is induced by partial geometric realization

$$r : \Sigma.M^n_q(|\overline{F}_1(X,Y)|) \to S^1 \wedge M^n_q(|\overline{F}_1(X,Y)|)$$

and the pairing

$$p : S^1 \wedge M^n_q(|\overline{F}_1(X,Y)|) \to M^n_q(S^1 \wedge |\overline{F}_1(X,Y)|) \xrightarrow{\cong} M^n_q(|\overline{F}_1(X, S^1 \wedge Y)|) \ .$$

Let $M_q^n(|\overline{F}_1(X,\Sigma.Y)|)$ denote the simplicial object $\{[p] \mapsto M_q^n(|\overline{F}_1(X,\overset{p}{\vee}(Y,*))|)\}$ where the face and degeneracy maps are induced by those of $\Sigma.Y$. There is an obvious map of simplicial objects

$$\Sigma.M_q^n(|\overline{F}_1(X,Y)|) \hookrightarrow M_q^n(|\overline{F}_1(X,\Sigma.Y)|)$$

which in degree p is given by the inclusion

$$\overset{p}{\vee}M_q^n(|\overline{F}_1(X,Y)|) \hookrightarrow M_q^n(|\overline{F}_1(X,\overset{p}{\vee}Y)|)\ .$$

The partial realization map r sends $M_q^n(|\overline{F}_1(X,\Sigma.Y)|)$ to $M_q^n(|\overline{F}(X,S^1\wedge Y)|)$ and the composition

$$\Sigma.M_q^n(|\overline{F}_1(X,Y)|) \to M_q^n(|\overline{F}_1(X,\Sigma.Y)|) \xrightarrow{r} M_q^n(|\overline{F}_1(X,S^1\wedge Y)|)$$

is equivalent to the previous composition of partial realization followed by the pairing p. Note that the partial realization map above is $(n-2)$-connected by the same type of argument used in the construction of the third map in Theorem 2.1.1. Now the map

$$\Sigma.M_q^n(|\overline{F}_1(X,Y)|) \xrightarrow{\alpha} M_q^n(|\overline{F}_1(X,\Sigma.Y)|)$$

is an $H_q^n(|JX|)$-bimonoid map, and so induces a bisimplicial map:

$$N^{cy}(H_q^n(|JX|),\Sigma.M_q^n(|\overline{F}_1(X,Y)|)) \xrightarrow{\beta} N^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|))\ .$$

Let $N_p^{cy}(M,NE)$ denote the simplicial object

$$\{[k] \mapsto N_{p,k}^{cy}(M,NE) = M^p \times (NE)_k\}\ .$$

Then the representation

$$\rho_{q,4}^1 : X^{q-1}\times Y \to H_q^n(|JX|)\times M_q^n(|\overline{F}_1(X,Y)|) = N_{1,1}^{cy}(H_q^n(|JX|),\Sigma.M_q^n(|\overline{F}_1(X,Y)|))$$
$$\xrightarrow{\beta} N_{1,1}^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|))$$

extends uniquely to a map of simplicial objects:

$$\overline{\rho}_{q,5} : X^{q-1} \times \Sigma.Y \to N_1^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, \Sigma.Y)|)) \ .$$

It is not true that there is a map $\Sigma.X^{q-1} \times Y \to X^{q-1} \times \Sigma.Y$ of simplicial objects which makes the appropriate diagram commute (here $X^{q-1} \times \Sigma.Y$ is the simplicial object $\{[p] \mapsto X^{q-1} \times (\overset{p}{\vee}(Y, *))\}$). However there is after passing to smash products. As will be shown in section 3.2, we have stable splittings i_1, i_2 such that $p_1 \circ i_1 \simeq p_2 \circ i_2 \simeq id$ in the square

(2.3.7)

$$\begin{tikzcd}
\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \arrow[r, shift left, "i_1"] \arrow[dd, leftrightarrow, "\cong"'] & \Omega^\infty\Sigma^\infty(\Sigma(|X|^{q-1} \times Y|)) \arrow[l, shift left, "p_1"] \arrow[d, "\tilde{\rho}_{q,4}"] \\
 & \Omega^\infty\Sigma^\infty(|diagN^{cy}(H_q^n(|JX|), \Sigma.M_q^n(|\overline{F}_1(X,Y)|))|) \arrow[dd, bend left, "\tilde{\beta}"] \\
\Omega^\infty\Sigma^\infty(|X|^{[q-1]} \wedge \Sigma|Y|) \arrow[r, shift left, "i_2"] & \Omega^\infty\Sigma^\infty(|X^{q-1} \times \Sigma.Y|) \arrow[l, shift left, "p_2"] \arrow[d, "\tilde{\rho}_{q,5}"] \\
 & \Omega^\infty\Sigma^\infty(|N^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, \Sigma.Y)|))|)
\end{tikzcd}$$

where $\tilde{\rho}_{q,j} = \Omega^\infty\Sigma^\infty|\overline{\rho}_{q,j}|$ for $j = 4, 5$ and $\overline{\beta}$ is induced by (β). By the construction of $\tilde{\rho}_{q,4}$ and $\tilde{\rho}_{q,5}$ it is straightforward to see that the diagram is canonically homotopy-commutative. Note that the space appearing in the lower right-hand corner is $(n-2)$ equivalent to $\Omega^\infty\Sigma^\infty|N^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, S^1 \wedge Y)|)|$. This is our fifth diagram.

Before evaluating the trace we make a useful simplification. In order to be consistent with notation, we will assume $Y = \Sigma^{2m-1} \wedge Z_+$ and use $\Sigma^{2m}F$ to denote $\Sigma|\overline{F}_1(X, Y)|$. There is no loss of generality here, because computation of $\overline{Tr}_X(Y)$ involves passing through a direct limit in which Y becomes more and more highly suspended. Now we know that the partial realization map

$$r : N^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, \Sigma.Y)|)) \to N^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, S^1 \wedge Y)|))$$

commutes with the simplicial structure in the first coordinate (i.e., the face and degeneracy maps of the cyclic bar construction), and that it maps the simplicial

space

$$\{[k] \mapsto N_p^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, \{\Sigma.Y)\}_k|))\}$$

to the space $N_p^{cy}(H_q^n(|JX|), M_q^n(|\overline{F}_1(X, S^1 \wedge Y)|))$. It follows by Theorem 2.2.5 that, upon restricting to the qth component of the trace map $\overline{Tr}_X(Y)$, we have a canonically homotopy commutative diagram

(2.3.8)

$$\begin{tikzcd}
 & & & X^{q-1}\times(\Sigma.Y) \arrow[d, "\rho_{q,5}"] \\
M_q^n(|\overline{F}_1(X,\Sigma.Y)|) \arrow[r, "\cong"] \arrow[d, "r"] & N_0^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|)) \arrow[d, "r"] & & N_1^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|)) \arrow[ll, "\partial_0"'] \arrow[d, "r"] \\
M_q^n(\overline{F}_1(X,S^1\wedge Y)|) \arrow[r, "\cong"] & N_0^{cy}(H_q^n(|JX|),M_q^n(\overline{F}_1(X,S^1\wedge Y)|)) & & N_1^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,S^1\wedge Y)|)) \arrow[ll, "\partial_0"'] \\
N_0^{cy}(H_q^n(|JX|),M_{q,1}^n\wedge M_{q,2}^n) \arrow[u, "\varphi^0_{m,n,q}\circ f^0_{m,n,q}"] \arrow[dr, "\overline{\pi}_p\circ g^0_{m,n,q}"'] & & & N_1^{cy}(H_q^n(|JX|),M_{q,1}^n\wedge M_{q,2}^n) \arrow[ll, "\partial_0"'] \arrow[u, "\varphi^1_{m,n,q}\circ f^1_{m,n,q}"'] \arrow[dl, "\overline{\pi}_p\circ g^1_{m,n,q}"] \\
 & \Omega^{n+m}\Sigma^{n+3m}(|X^{[p-1]}\wedge Z_+|) & &
\end{tikzcd}$$

where

$$M_{q,1}^n = Map(\overset{q}{\vee} S^n, S^{n+m} \wedge |JX|_+ \wedge |Z_+|)$$
$$M_{q,2}^n = Map(S^{n+m}, \overset{q}{\vee} S^{n+2m} \wedge |JX|_+) \ ,$$

and $\overline{\pi}_p$ is the obvious reduced projection onto the pth component

$$\Omega^{n+m}\Sigma^{n+3m}(|X^{[p-1]} \wedge Z_+|) \ .$$

Our object now is to find a map $\rho_{q,6}$ defined on $X^{q-1} \times (\Sigma.Y)$ or its realization, whose range is $N_0^{cy}(H_q^n(|JX|), M_{q,1}^n \wedge M_{q,2}^n)$ such that $\varphi^0_{m,n,q} \circ f^0_{m,n,q} \circ \rho_{q,6}$ is canonically homotopic to $r \circ \partial_0 \circ \overline{\rho}_{q,5}$ (of course, it would suffice to lift $r \circ \overline{\rho}_{q,5}$ directly without using ∂_0, and in fact such a lifting can be written down explicitly. However, it is much simpler to do this <u>after</u> mapping first by ∂_0; this makes the computation

of $\overline{\pi}_p \circ g^0_{m,n,q}$ easier as well). Now $\overline{\rho}_{q,5}$ is the unique extension to $X^{q-1} \times (\Sigma.Y)$ of the representation $\overline{\rho}^1_{q,4}$ on 1-simplices $X^{q-1} \times Y$ given by

$$(x_1, \ldots, x_{q-1}, y) \mapsto (f, f^{-1}ef^{-1}) \; ;$$
$$f = \overline{\rho}^1_q(x_1, \ldots, x_{q-1}) \, , \, e = \overline{\rho}^2_q(y) \; .$$

These are in turn expressed as a product of elementary expansions by (2.3.5). Under ∂_0 this element maps to $(f^{-1}ef^{-1} \cdot f)$ which is canonically homotopic to $(f^{-1}e)$. It follows that we can describe $r \circ \partial_0 \circ \overline{\rho}_{q,5}$ on the realization of $X^{q-1} \times (\Sigma.Y)$ as the map of <u>spaces</u> given by the representation

$$\begin{gathered} \rho^1_{q,6} : |X^{q-1} \times (\Sigma.Y)| \to |M^n_q(|\overline{F}_1(X, \Sigma.Y)|)| \\ (x_1, \ldots, x_{q-1}, \tilde{y}) \mapsto (f^{-1}\tilde{e}); \quad f = \overline{\rho}^1_q(x_1, \ldots, x_{q-1}) \, , \tilde{e} = \overline{e}_{q-1}(\imath(\tilde{y})) \end{gathered} \tag{2.3.9}$$

where f is now viewed as a product of elementary expansions whose range is $|M^n_q(|\overline{F}_1(X, \Sigma.Y)|)|$. Note that $\tilde{y}$ denotes an element of $|\Sigma.Y|$. Writing f^{-1} as $e_{12}(-\imath(x_1))e_{23}(-\imath(x_2)) \cdot \ldots \cdot e_{q-1q}(-\imath(x_{q-1}))$ and applying proposition 1.3.10 yields a canonical homotopy between $\overline{\rho}^1_{q,6}$ and the representation

$$\begin{gathered} \overline{\rho}^2_{q,6} : |X^{q-1} \times (\Sigma.Y)| \to |M^n_q(|\overline{F}_1(X, \Sigma.Y)|)| \\ (x_1, \ldots, x_{q-1}, \tilde{y}) \mapsto \overline{e}_{11}(z_1) + \overline{e}_{21}(z_2) + \cdots + \overline{e}_{q1}(z_q) \end{gathered} \tag{2.3.10}$$

where $z_i = (\prod_{j=i}^{q-1} - \imath(x_i))\imath(\tilde{y}) \in \pm(|\overline{F}_1(X, \Sigma.Y)|) = \pm\Sigma^{2m}F$ and "-" denotes the inverse under loop sum. We can write $\tilde{y} \in \Sigma|Y| \cong S^m \wedge |Z| \wedge S^m$ as $\tilde{y} = (s_1, z, s_2)$. Define $\overline{\rho}^3_{q,6}$ by

$$\begin{gathered} \overline{\rho}^3_{q,6} : |X^{q-1} \times (S^m \wedge |Z|_+ \wedge S^m)| \to |M^n_{q,1} \wedge M^n_{q,2}| \\ (x_1, \ldots, x_{q-1}, s_1, z, s_2) \mapsto (\overline{e}_{11}(z'_1) + \cdots + \overline{e}_{q1}(z'_q)) \wedge \imath^1(s_2) \; . \end{gathered} \tag{2.3.11}$$

Here $M^q_{n,i}$ is as in (2.3.8). $\imath^1(s_2)$ is represented by the composition

$$\begin{gathered} S^{n+m} \to S^{n+m} \wedge S^m \cong S^{n+2m} \wedge S^0 \overset{\text{inc}}{\hookrightarrow} (S^{n+2m} \wedge |JX|_+)_1 \\ \overset{\text{inc}}{\hookrightarrow} \overset{q}{\vee} S^{n+2m} \wedge |JX|_+ \\ s \mapsto (s, s_2) \end{gathered} \tag{2.3.12}$$

$z_i' = (\prod_{j=i}^{q-1} -\imath(x_i))\imath(s_1, z) \in \pm|JX_+| \wedge S^{n+m} \wedge |Z|_+ \cong \pm S^{n+m} \wedge |JX|_+ \wedge |Z_+|$, and the product of reduced elementary expansions in (2.3.11) is viewed as an element of $M^n_{q,1}$. It is straightforward to verify that the diagram

$$\begin{tikzcd} |X^{q-1} \times (S^m \wedge |Z|_+ \wedge S^m)| \arrow[r, "\overline{\rho}^3_{q,6}"] \arrow[d, "\simeq"'] & |M^n_{q,1} \wedge M^n_{q,2}| \arrow[d, "|\varphi^0_{m,n,q} \circ f^0_{m,n,q}|"] \\ |X^{q-1} \times \Sigma.Y| \arrow[r, "\overline{\rho}^2_{q,6}"] & |M^n_q(|\overline{F}_1(X, \Sigma.Y)|)| \end{tikzcd} \tag{2.3.13}$$

is canonically homotopy-commutative. So taking $\overline{\rho}_{q,6}$ to be $\overline{\rho}^3_{q,6}$ provides the necessary lift in order to evaluate $\overline{Tr}_X(Y)$. This evaluation is achieved, according to Theorem 2.2.5, by switching the terms in (2.3.11) and composing. Since $\imath^1$ just involves the standard inclusion to the first factor in the wedge, we get

$$(\overline{e}_{11}(z_1') \cdot \ldots \cdot \overline{e}_{q1}(z_q')) \circ \imath^1(s_2) = \overline{e}_{11}(z_1') \circ \imath^1(s_2)$$

which implies that $\overline{\pi}_p \circ g^0_{m,n,q} \circ \overline{\rho}^3_{q,6}$ is canonically null-homotopic for $p > q$, and that $\overline{\pi}_q \circ g^0_{m,n,q} \circ \overline{\rho}^3_{q,6}$ is the map

$$(|X^{q-1}| \wedge S^m \wedge |Z|_+ \wedge S^m) \to \Omega^{n+m}\Sigma^{n+3m}(|X|^{[q-1]} \wedge |Z|_+)$$

given by

$$(x_1, \ldots, x_{q-1}, s_1, z, s_2) \mapsto (\prod_{j=1}^{q-1} -\imath(x_i), s_1, \imath(z), s_2) \ .$$

Up to reparametrization independent of X and Z this composition is canonically homotopic to $(-1)^{q-1} j_{2m}$, where j_{2m} is the standard inclusion

$$\Sigma^{2m}|X|^{[q-1]} \wedge |Z|_+ \to \Omega^{n+m}\Sigma^{n+3m}(|X|^{[q-1]} \wedge |Z|_+) \ .$$

To complete the proof, note that $\overline{Tr}_X(Y)_p \circ \tilde{\rho}_q(X, Y)$ is induced by a natural transformation of a homogeneous functor of degree q to a homogeneous functor of degree p (evaluated at X), which must be canonically null-homotopic for $q > p$ as it factors through the pth differential of a q-homogeneous functor. $\square$

We now complete the proof of theorem A. By the previous theorem, we may conclude that the map

$$\tilde{\rho} : \tilde{D}(X) \to \overline{A}(\Sigma X)$$

induces a map on first differnetials

$$(D_1\tilde{\rho})_X(Y) : (D_1\tilde{D})_X(Y) \to (D_1\overline{A}\Sigma)_X(Y) \tag{2.3.14}$$

which is split-injective on homotopy groups for all X and Y. In the case that both X and Y are finite complexes, the homotopy groups of both sides of (2.3.14) are finitely generated ; for the right hand side this follows by Goodwillie's theorem (theorem 1.2.6). This implies that $(D_1\tilde{\rho})_X(Y)$ is an equivalence for all finite X and Y. As both functors are also homotopy functors (hence commute up to homotopy with filtered colimits), this implies $(D_1\tilde{\rho})_X(Y)$ is an equivalence for all Y and connected X. Finally, applying Goodwillie's theorem 1.2.4 at $X = *$ implies that $\tilde{\rho}$ is an equivalence.

The equivalence $\tilde{D}(X) \xrightarrow{\rho} \overline{A}(\Sigma X)$ is natural with respect to X, so that if $f : X \to Y$ is a map of connected simplicial sets there is a homotopy-commutative diagram

$$\begin{array}{ccc} \tilde{D}(X) & \xrightarrow{\tilde{\rho}_X} & \overline{A}(\Sigma X) \\ {\scriptstyle \tilde{D}(f)}\downarrow & & \downarrow{\scriptstyle \overline{A}(\Sigma f)} \\ \tilde{D}(Y) & \xrightarrow{\tilde{\rho}_Y} & \overline{A}(\Sigma Y) \end{array} \tag{2.3.15}$$

It also follows that ρ restricts to yield equivalences

$$\prod_{q=m+1}^{n} \tilde{D}_q(X) = p_n^m \tilde{D}(X) \xrightarrow{p_n^m \tilde{\rho}(X)} p_n^m \overline{A}(\Sigma X)$$

natural in X for all $0 \leq m < n \leq \infty$, because $\tilde{\rho}$ is a natural transformation of homotopy functors and hence commutes with Goodwillie Calculus. However, it is <u>not</u> true that $\tilde{\rho}$ or $p_n^m \tilde{\rho}$ are natural with respect to maps $\Sigma X \xrightarrow{g} \Sigma Y$ which do not desuspend up to homotopy.

CHAPTER III

§3.1 Splittings of Homotopy Functors and Weight Filtrations

We will establish a simple criterion for splitting a homotopy functor F (cf. §1.2) as a product of its homogenous components on the subcategory $U(C)_{\rho(F)}$ of $U(C)$, where $\rho(F)$ is the modulus of F. For simplicity we will take $C = \{pt\}$ and assume that F is reduced (i.e. that we have passed to the fibre of $F(X) \to F(*)$ for all $(X \to *)$ in obj$(U(*))$). All of the results however apply with an arbitrary base space C in place of $*$. We leave it to the reader to make the necessary translation. The nth differential of F at $*$ will simply be written as D_nF.

Definition 3.1.1. *A weight filtration of a reduced homotopy functor F is a direct system of reduced homotopy functors $\{\omega_r F\}_{r\geq 0}$ satisfying*

i) *There are compatible natural transformations $\eta_r : \omega_r F \to F$ inducing a weak equivalence of reduced functors*

$$\underset{r}{\overrightarrow{hocolim}}\{\omega_r F\} \xrightarrow{\simeq} F\ .$$

ii) *η_r induces an equivalence of approximations*

$$P_i(\eta_r) : P_i(\omega_r F) \xrightarrow{\simeq} P_i(F) \qquad i \leq r \quad , \text{ for all } r \geq 0.$$

iii) *$\rho(\omega_r F) \geq \rho(F)$ for all $r \geq 0$. $\{\omega_r F\}_{r\geq 0}$ is minimal if*

iv) *$\omega_0 F \simeq *$ and fibre $(\omega_{r-1}F \to \omega_r F)$ is homogeneous of degree r for all $r \geq 1$. We note that inductively this is the same as requiring*

v) *$\omega_r F \to P_r(\omega_r F)$ is an equivalence for all $r \geq 0$ within the disk of convergence of $\omega_r F$.*

The following is implicit in Goodwillie's short proof of the Snaith splitting of $\Omega^\infty\Sigma^\infty(JX)$ (pp. 66–68, [G0]).

Lemma 3.1.2. *If F as above admits a minimal weight filtration $\{\omega_r F\}$, then $F \simeq \prod_{n\geq 1} D_nF$ within the disk of convergence of F.*

Proof. As on p. 68 [G0], consider the diagram

$$\begin{array}{ccccc} \omega_{r-1}F & \xrightarrow{\imath} & \omega_r F & \longrightarrow & \Omega^{-1} fibre(\omega_{r-1}F \to \omega_r F) \\ {\scriptstyle p_{r-1}(\omega_{r-1}F)}\downarrow & & {\scriptstyle p_{r-1}(\omega_r F)}\downarrow & & \downarrow \\ P_{r-1}(\omega_{r-1}F) & \xrightarrow{P_{r-1}(\imath)} & P_{r-1}(\omega_r F) & \longrightarrow & P_{r-1}(\Omega^{-1} fibre(\omega_{r-1}F \to \omega_r F)) \end{array}$$

By ([G0], chap. III), $fibre(\omega_{r-1}F \to \omega_r F)$ is homogeneous and hence canonically deloopable, $p_{r-1}(\omega_{r-1}F)$ is a weak equivalence and $P_{r-1}(\Omega^{-1} fibre(\omega_{r-1}F \to \omega_r F))$ is contractible by iv) or v) and induction on r. Hence $P_{r-1}(\imath)$ is an equivalence. So

$$\omega_r F \simeq \omega_{r-1}F \times D_r(w_r F) \simeq (\prod_{n\geq 1} D_n(\omega_{r-1}F)) \times (D_r \omega_r F) \simeq \prod_{n=1}^{r} D_n(F)$$

within the disk of convergence of F by induction on r, iv'), ii), and iii). $\square$

We will want to apply this lemma to simplicial functors. Recall (e.g., [W3], prop. 6.3) that if $X. \to Y. \to Z.$ is a sequence of simplicial spaces which is a fibration in each degree and Z_r is connected for each r then $|X.| \to |Y.| \to |Z.|$ is a fibration up to homotopy. Although one can do better, this implies (by induction on n) that an n-dimensional cube of simplicial spaces is homotopy cartesian (upon realization) if, in each degree, it is homotopy-cartesian and all of the spaces are $(n-1)$-connected. We can remove the condition on connectivity if we start with a diagram of simplicial spaces which can be sufficiently delooped in a way compatible with the simplicial structure, for then by delooping we can make the diagrams sufficiently connected and proceed as above. Thus we have

Lemma 3.1.3. *Suppose $F. = \{F_r\}_{r\geq 0}$ is a simplicial object in the category of reduced homotopy functors from (spaces) to (∞-loop spaces) = the category of base-pointed infinite loop spaces. Suppose each F_r admits a minimal weight filtration and that the face and degeneracy maps of $F.$ are weight-preserving. Finally assume that $\rho(|F.|) \geq \rho(F_r)$ for each r. Then the Goodwillie Taylor series of $|F.|$ splits as a product of its homogeneous components within the disk of convergence of $|F.|$. Moreover $D_n(|F.|) \simeq |[r] \mapsto D_n(F_r)|$ for each $n \geq 0$.*

Proof. The condition on $\rho(F_r)$ ensures that the Taylor series for F_r converges on the disk of convergence $|F.|$ for each r. Since $F.$ is a simplicial ∞- loop space functor, the delooping argument above shows that $T_n^k(|F.|)$ and $|T_n^k(F.)|$ are weakly equivalent for each r and k, where $T_n^k(F.)$ is the simplicial functor $\{r \mapsto T_n^k(F_r)\}_{r\geq 0}$. Passing to the limit as $k \to \infty$ yields a weak equivalence $P_n(|F.|) \simeq |P_n(F.)|$ for each n. Since the weight filtrations on F_r are compatible with the simplicial structure, Lemma 1.2.8 yields equivalences

$$P_n(|F.|) \simeq |P_n(F.)| \simeq |\prod_{j=1}^{n} D_j(F.)| \simeq \prod_{j=1}^{n} |D_j(F.)| \simeq \prod_{j=1}^{n} D_j(|F.|)$$

for each n. Note that $D_j(|F.|) \simeq |D_j(F.)| = |[r] \mapsto D_j(F_r)|$ by Goodwillie's classification theorem for homogeneous functors. $\square$

§3.2 Some Applications

Let X be a basepointed space, $G \subset \Sigma_n$. We can consider the functor $F_G(X) = \Omega^\infty\Sigma^\infty(EG \times_G X^n)$, which is a homotopy functor on the category of basepointed spaces. G fixes the basepoint $(*, *, *, \ldots, *)$ of X^n, so the fibration $EG \times_G X^n \to BG$ admits a section $s : BG \to EG \times_G X^n$ determined by this basepoint. Let $EG \times_G X^n/BG$ denote the cofiber of s, and let $\overline{F}_G(X) = \Omega^\infty\Sigma^\infty(EG \times_G X^n/BG)$. $\overline{F}_G(X)$ is then a reduced homotopy functor and $F_G(X) \simeq \overline{F}_G(X) \times \Omega^\infty\Sigma^\infty(BG)$.

Proposition 3.2.1. *Over the category of basepointed connected spaces $\overline{F}_G$ has degree n and splits as a product of its differentials. In particular $(D_n\overline{F}_G)(X) = \Omega^\infty\Sigma^\infty(EG\lambda_G X^{[n]})$ naturally splits off of $\overline{F}_G(X)$. This splitting of $\overline{F}_G$ yields a splitting of the nth delooping B^nF_G for all $n \geq 1$.*

Proof. The proof is easy, and typical of the way in which the methods of the previous section apply. Let $\imath_j : X^j \to X^n$ denote the embedding $\imath_j(x_1, \ldots, x_j) = (x_1, \ldots, x_j, *, *, \ldots, *)$. Let $\mathcal{F}_j(X^n)$ denote the union of the orbits containing $\imath_j(X^j)$ in X^n under the usual action of Σ_n which permutes entries. Let $(\omega_j\overline{F}_G)(X) = \Omega^\infty\Sigma^\infty(\overline{\mathcal{F}}_j(X^n)) \subset \overline{F}_G(X)$ where $\overline{\mathcal{F}}_j(X^n) = EG\wedge_G(\mathcal{F}_j(X^n)_+$. Now $\omega_0(F_G)(X) \simeq$

$*$. Assume by induction that $\omega_r(F_G)(X)$ is of degree r and splits as a product of its differentials (at $*$). Then

$$\Omega^{-1} fiber(\omega_r \overline{F}_G(X) \to \omega_{r+1}\overline{F}_G(X)) \simeq \Omega^\infty \Sigma^\infty (EG\lambda_G(\mathcal{F}_{r+1}(X^n)/\mathcal{F}_r(X^n))) \ ,$$

where $\mathcal{F}_{r+1}(X^n)/\mathcal{F}_r(X^n) \simeq \overset{k}{\vee} X^{[r+1]}$, $k = \binom{n}{r+1}$. This is a homogeneous functor of degree $(r+1)$ by ([G0]). By induction the weight filtration is minimal, and after splitting deloopable. The proposition follows. $\square$

Of course, this splitting is known. We have included it as an example, as we have referred to it previous sections.

As another example, let $C_m = \{C_m(n)\}_{n \geq 0}$ denote the little m-cubes operad of Boardman-Vogt. $C_m(n)$ is a topological space via the standard function-space (compact-open) topology. Precomposition with an element of $\sigma \in \Sigma_n$ yields a well-defined action of Σ_n on $C_m(n)$ given by $f \mapsto f \circ \sigma$. Then n ordered inclusions $i^j : \underline{n-1} \to \underline{n}$ = ordered set of n elements induce restriction maps

$$i^j : C_m(n) \to C_m(n-1) \ .$$

The n ordered projection maps

$$h_j : \underline{n}_+ \to \underline{n-1}_+ \qquad (\underline{m}_+ = \underline{m} \coprod \text{ pt }) \text{ given by}$$

$$h_j(k) = \quad i)\, k \text{ if } k < j, \quad ii)\, * \ \text{ if } k = j, \quad iii)\, k-1 \text{ if } k > j$$

yield maps

$$h^j : X^{n-1} = Map_*(\underline{n-1}_+, X) \to X^n = Map_*(\underline{n}_+, X) \qquad \text{given by}$$

$$h^j(g) = g \circ h^j \ .$$

One can then form the configuration space

$$C(\mathbb{R}^m, X) = (\coprod_{n \geq 0} C_m(n) \times X^n / \sim)$$

where $\sim$ is generated by two compatible types of identifications :

$$(f, g \circ \sigma) \sim (f \circ \sigma, g) \, , \qquad \text{and}$$

$$(f, h^j(\overline{g})) \sim (i_j(f), \overline{g})$$

$$\text{for } \sigma \in \Sigma_n \ , f \in C_m(n) \ , g \in X^n \ , \text{ and } \overline{g} \in X^{n-1} \ .$$

The approximation theorem yields a map $C(\mathbb{R}^m, X) \to \Omega^m\Sigma^m(X)$ which is an m-fold loop map and a weak equivalence (as before X is a basepointed connected space). $C(\mathbb{R}^m, X)$ is filtered by $\mathcal{F}^r_m(X) = (\coprod_{n=0}^{r} C_m(n) \times X^n/\sim) \subset C(\mathbb{R}^m, X)$. The inclusion $\mathcal{F}^{r-1}_m(X) \xrightarrow{\imath^r_m} \mathcal{F}^r_m(X)$ is a closed cofibration with cofibre $\imath^r_m \simeq C_m(n)\lambda_{\Sigma_n}X^{[n]}$. Under the associative pairing $C(\mathbb{R}^m, X) \times C(\mathbb{R}^m, X) \xrightarrow{\Phi} C(\mathbb{R}^m, X)$ induced by the action of the operad C_m , $\mathcal{F}^r_m(X) \times \mathcal{F}^s_m(X)$ maps to $\mathcal{F}^{r+s}_m(X)$. Using the monoid $C(\mathbb{R}^m, X)$ is place of $\Omega^m\Sigma^m X$ we have

<u>Lemma 3.2.2</u>. *Let X be a connected basepointed space. The filtration $\{\mathcal{F}^r_m(X)\}$ of $C(\mathbb{R}^m, X)$ induces minimal weight filtrations of the functors $X \mapsto \Omega^\infty\Sigma^\infty(\Omega^m\Sigma^m X)$, $X \mapsto \Omega^\infty\Sigma^\infty((\Omega^{m-1}\Sigma^m X)^{S^1})$ and $X \mapsto \Omega^\infty\Sigma^\infty(ES^1 \underset{S^1}{\times} (\Omega^{m-1}\Sigma^m X)^{S^1}/BS^1)$.*

<u>Proof</u>. : The filtration of $\Omega^\infty\Sigma^\infty(\Omega^m\Sigma^m(X))$ by $\Omega^\infty\Sigma^\infty\mathcal{F}^r_m(X)$ is minimal by the same type of argument as in the previous proposition and yields the Snaith splitting. Recall ([W2], §2) that for a grouplike monoid $M, |N.^{cy}(M)| \simeq (BM)^{S^1}$. Now $C(\mathbb{R}^n, X)$ is grouplike and so $|N^{cy}(C(\mathbb{R}^n, X))| \simeq (\Omega^{m-1}\Sigma^m X)^{S^1}$. $\Omega^\infty\Sigma^\infty(\_)$ commutes with geometric realization ; we need to show that $F.(X) = \{[p] \mapsto \Omega^\infty\Sigma^\infty(N^{cy}_p(C(\mathbb{R}^m, X)))\}$ admits a minimal weight filtration in each degree compatible with the simplicial structures. Let

(3.2.3)
$$\omega t_r(N^{cy}_p(C(\mathbb{R}^m, X))) = \bigcup_{\substack{i_1,\dots,i_{p+1} \\ \sum_{j=1}^{p+1} i_j = r}} \mathcal{F}^{i_1}_m(X) \times \cdots \times \mathcal{F}^{i_{p+1}}_m(X) \subset N^{cy}_p(C(\mathbb{R}^m, X))$$

$\Theta(\mathcal{F}^r_m(X) \times \mathcal{F}^s_m(X)) \subset \mathcal{F}^{r+s}_m$ and so $\{[p] \mapsto \omega t_r N^{cy}_p(C(\mathbb{R}^m, X))\}$ defines a well-defined simplical subset $\omega t_r N^{cy}.(C(\mathbb{R}^m, X))$ of $N^{cy}.(C(\mathbb{R}^m, X))$. The cofibre of

$$(\omega t_{r-1} N^{cy}_p(C(\mathbb{R}^m, X)) \to \omega t_r N^{cy}_p(C(\mathbb{R}^m, X)))$$

is homotopic to $\bigvee_{G \subset \Sigma_r} E\Sigma_r\lambda_G X^{[r]}$ where the wedge is over all G of the form $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_{p+1}} \subset \Sigma_r$, $(\Sigma i_j = r)$, and we have seen that $X \mapsto \Omega^\infty\Sigma^\infty(E\Sigma_r\lambda_G X^{[r]})$ is homogeneous of degree r. Hence the weight filtration is minimal in each degree and Lemma 3.1.3 applies to split $\Omega^\infty\Sigma^\infty((\Omega^{m-1}\Sigma^m X)^{S^1})$ as a product of its homogeneous components. For the last functor, we use the result of Dunn and Fiedorwicz [DF] which

provides a configuration-space model for $(ES^1 \underset{S^1}{\times} X^{S^1})/BS^1$ for connected X. To state their result, let $E\mathbb{Z}/*{+}1$ denote the cyclic space $\{[p] \mapsto E(\mathbb{Z}/p{+}1)\}$ (see [DF], Example 1) with cyclic simplicial structure induced by the cyclic simplicial structure on the crossed simplicial group $\mathbb{Z}/*{+}1 = \{[p] \mapsto \mathbb{Z}/p{+}1)\}$ (in the sense of [FL]) whose standard realization is $\cong S^1$. Any cyclic space can be viewed as a cocyclic space by precomposition with the duality isomorphism $D : \Delta(C_*) \xrightarrow{\simeq} \Delta(C_*)^{op}$ of Connes ([C1], where $\Delta(C_*)$ is denoted Λ). Via this identification, one can form the tensor product (over $\Delta(C_*)$) of $E\mathbb{Z}/*{+}1$ and a cyclic simplicial space S_*, resulting in a space

$$E\mathbb{Z}/*{+}1 \otimes_{\Delta(C_*)} S_* = \coprod_{n\geq 0} E\mathbb{Z}/n+1 \times S_n/\sim\ ,\ (\lambda^*(x), y) \sim (x, (D\lambda)^* y)$$

for $x \in E\mathbb{Z}n+1, y \in S_m$ and $\lambda : [m] \to [n]$ a morphism in $\Delta(C_*)$. Dunn and Fiedorowicz prove

<u>Theorem 3.2.4.</u> *(*[DF]*, p. 8) Let S_* be a cyclic space. Then $E\mathbb{Z}/*{+}1 \otimes_{\Delta(C_*)} S_*$ is equivalent to the pushout of the diagram*

$$\begin{array}{ccc} BS^1 \times Fix(|S_*|) & \longrightarrow & ES^1 \underset{S^1}{\times} |S_*| \\ {\scriptstyle p_2}\big\downarrow & & \\ Fix(|S_*|) & & \end{array}$$

where $Fix(|S_*|)$ is the S^1 fixed-point set of $|S_*|$ (with S^1-action induced by the cyclic structure on S_*, as in [BF]*).*

In particular taking S_* to be $N^{cy}.(C(\mathbb{R}^m, X))$ with the usual cyclic structure we get

$$\begin{aligned} ES^1 \underset{S^1}{\times} (\Omega^{m-1}\Sigma^m X)^{S^1}/BS^1 &\simeq ES^1 \underset{S^1}{\times} |N^{cy}.(C(\mathbb{R}^m, X))|/BS^1 \\ &\simeq E\mathbb{Z}/*{+}1 \otimes_{\Delta(C_*)} N^{cy}_*(C(\mathbb{R}^m, X))\ . \end{aligned}$$

Now $E\mathbb{Z}/*{+}1 \otimes_{\Delta(C_*)} S_*$ is filtered by $\{g^r(S_*) = \coprod_{n=0}^{r} E\mathbb{Z}/n+1 \times S_n/\sim\}$. Denote $N^{cy}.(C(\mathbb{R}^m, X))$ temporarily by S_*. The above filtration (3.2.3) is invariant under the cyclic structure on S_*, and hence induces a cyclic simplicial filtration $\{wt_r S_*\}$

of S_* (functorial in X). In order to show that $\{\Omega^\infty\Sigma^\infty(E\mathbb{Z}/*+1\otimes_{\Delta(C_*)} wt_rS_*)\}$ is a minimal weight filtration we note it suffices to prove the homogeneity of the fibre of

$$\Omega^\infty\Sigma^\infty(g^n(wt_{r-1}S_*)/g^{n-1}(wt_{r-1}S_*)) \to \Omega^\infty\Sigma^\infty(g^n(wt_rS_*)/g^{n-1}(wt_rS_*)).$$

This latter space can be written as

$$\Omega^\infty\Sigma^\infty\left(\bigvee_{\substack{i_1,\ldots,i_{n+1}\\ \Sigma i_j=r, i_j\geq 1}} E\mathbb{Z}/n+1\lambda_{\mathbb{Z}/n+1}\left(C_m(i_1)\lambda_{\Sigma_{i_1}}X^{[i_1]}\right)\wedge\cdots\wedge\left(C_m(i_{n+1})\lambda_{\Sigma_{i_{n+1}}}X^{[i_{n+1}]}\right)\right).$$

Each term in the wedge sum is of the form $E\Sigma_r\lambda_G(A_G\wedge X^{[r]})$ for some $G\subset\Sigma_r$ and A_G a G-space where we take the diagonal G-action on $(A_G\wedge X^{[r]})$. Again, the functors $X\mapsto\Omega^\infty\Sigma^\infty(E\Sigma_r\lambda_G(A_G\wedge X^{[r]}))$ are homogeneous by [G0]. This completes the proof. $\square$

<u>Remarks 3.2.6</u> i) The above theorem applies more generally, by the same arguments, with $\Omega^n\Sigma^{n+1}(X)$ replaced by a functor $F(X)$ satisfying the property that $\Omega F(X)$ admits a filtration $\{\mathcal{F}_r\Omega F(X)\}$ functorial in X and compatible with loop multiplication ($F_r\prod F_s\to F_{r+s}$) such that $\mathcal{F}_rF(X)/\mathcal{F}_{r-1}F(X)\simeq A_r\lambda_{\Sigma_r}X^{[r]}$ for some Σ_r-space A_r.

ii) For $n=1$ the splitting of $\Omega^\infty\Sigma^\infty(ES^1\underset{S^1}{\times}(\Sigma X)^{S^1}/BS^1)$ as a product of derivatives was one the main results of [CC]. Many of these splittings have been also obtained independently by C. F. Bödigheimer.

iii) These techniques can be "equivariantized" to yield equivariant splitting theorems for the functors described in Lemma 3.2. In the simplest case one recovers the equivariant Snaith splitting of $\Omega^v\Sigma^v_G(\Omega^v\Sigma^v(X)$ proved by Lewis, May and Steinberger for a compact Lie group G and basepointed G-space X.

Bibliography

[BCCMGH] M. Bokstedt, G. Carlsson, R. Cohen, I. Madsen, T. Goodwillie, W.-C. Hsiang, *On the algebraic K-theory of simply-connected spaces*, preprint.

[CCGH] G. Carlsson, R. Cohen, T. Goodwillie,W.-C. Hsiang, *The free loop space and the algebraic K-theory of spaces*, K-theory **1** (1987), 53–82.

[CCGH] G. Carlsson, R. Cohen, *The cyclic groups and the free loop space.*, Comm. Mtath. Helv. **62** (1987), 423–449.

[C1] A. Connes, *Cohomologie cyclic et foncteurs* $Ext^{\,n}$, C. R. Acad. Sci. Paris **296** (1983), 953–958.

[DF] G. Dunn, F. Fiedorowicz, *A classifying space construction for cyclic spaces*, preprint (1986), to appear.

[FL] Z. Fiedorowicz and J.-L. Loday, *Crossed simplicial groups and their associated homology*, Trans. of the Amer. Math. Soc. **326** (1991), 57–87.

[FO] Z. Fiedorowicz and C. Ogle, *Algebraic K-theory and configuration spaces*, Topology **29** (1990), 409-418.

[FOV] Z. Fiedorowicz, C. Ogle and R. Vogt, *Volodin K-theory of A^∞ rings*, Topology **32** (1993), 329–352.

[G0] T. Goodwillie, *The calculus of functors*, preprint (1986).

[G1] T. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K-theory **4** (1990), 1–27.

[G2] T. Goodwillie, *Calculus II: Analytic functors* **5** (1992), 295–332.

[G3] T. Goodwillie, *Calculus III*, to appear.

[O1] C. Ogle, *Models for $A(X)$ and stable homotopy theory*, preprint (1985).

[O2] C. Ogle, *Splittings of $A(X)$ and related functors*, preprint (1987).

[S] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293-312.

[W2] F. Waldhausen, *Algebraic K-theory of spaces II*, Lect. Notes in Math. **763** (1979), Springer-Verlag.

[W3] F. Waldhausen, *Algebraic K-theory of generalized freepProducts*, Ann. of Math. **108** (1978), 135–204.

[WM] F. Waldhausen, *Algebraic K-theory of spaces, concordance, and stable homotopy theory*, Ann. of Math. Studies **113** (1987).

[W] F. Waldhausen, *Algebraic K-theory of spaces*, Lect. Notes in Math. **1246** (1985), Springer-Verlag.

[SV] R. Schwänzel, R. Vogt, *E_∞ monoids with coherent homotopy inverses are abelian groups*, Topology **28** (1989), 481-484.