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ALGEBRAIC K-THEORY AND CONFIGURATION SPACES

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INTRODUCTION

The approximation theorem in stable homotopy states that for a connected space X, there is an equivalence

$$\Omega^\infty S^\infty X \simeq \coprod_{n\geq 0} C(n,\mathbb{R}^\infty)\times \Sigma_n X^n/\approx$$

where $C(n,\mathbb{R}^\infty)$ denotes the configuration space of n-tuples of distinct points in $\mathbb{R}^\infty$ and the equivalence relation identifies

$$((u_1,\ldots,u_n)(x_1,\ldots,\underset{\substack{\uparrow\\ i}}{*},\ldots,x_n))\approx((u_1,\ldots,\hat{u}_i,\ldots,u_n),(x_1,\ldots,*,\ldots,x_n))$$

Its importance lies in the fact that the right hand side of the above equivalence (which we will briefly refer to as a configuration space model) is a much smaller space than the function space on the left, and thus is much easier to deal with.

The second author noted that in the configuration space model, one can replace the X^n by spaces like $BGl_n(R)$. For there is an obvious Σ_n action on $BGl_n(R)$ and there are obvious inclusions i_n^j: $BGl_{n-1}(R)\to BGl_n(R)$, $1\leq j\leq n$ which give sense to the equivalence relation defining the configuration space model. A natural question thus poses itself: what is the homotopy type of the resulting configuration space model

$$\coprod_{n\geq 0} C(n,\mathbb{R}^\infty)\times_{\Sigma_n} BGl_n(R)/\approx ?$$

An easy argument in invariant theory over $\mathbb{Q}$ shows that the rational homotopy type is the same as that of Quillen's plus construction $BGl(R)^+$. The second author conjectured that the global homotopy types were also the same.

In this paper we verify a generalized form of this conjecture. We first consider a general framework where the configuration space model makes sense and has a natural E_∞ structure (that is an H-space structure which is coherently associative and commutative, giving an infinite loop space structure upon group completion). This is the notion of a permutative monoid: a graded space $M_* = \coprod_{n\geq 0} M_n$, together with symmetric group actions $\Sigma_n\times M_n\to M_n$, injections i_n^j: $M_{n-1}\to M_n$, $1\leq j\leq n$, and direct sum operations $\oplus$: $M_m\times M_n\to M_{m+n}$, all suitably compatible. (The example to keep in mind is, of course, $M_n = BGl_n(R)$ or more generally $M_n = BEnd(A^n)$ where A is an object in a permutative category.)

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For such a permutative monoid M_*, the operation of direct sum gives M_* an E_∞ structure (but certainly not an infinite loop space structure since $\pi_0(M_*)$ is obviously not a group, as it surjects onto the additive monoid of natural numbers). In the configuration space model, if one replaces the configuration spaces $C(n,\mathbb{R}^\infty)$ by $C(n)$, the nth space of an E_∞ operad C one obtains an E_∞ structure on a space equivalent to the configuration space model. In most relevant examples, this structure is group complete and thus gives an infinite loop structure on the configuration space model.

Our main theorem below relates the infinite loop space obtained by group completing M_* with that associated to the configuration space model of M_*. We use the notation Γ to denote any infinite loop space machine which converts E_∞ spaces into infinite loop spaces by group completing them. By a theorem of May and Thomason [4], all such machines are essentially equivalent.

Theorem A. *Assume that Σ_∞ acts trivially on the homology of*

$$M_\infty = \varinjlim_n M_n.$$

(We take the direct limit of the M_n's with respect to the inclusions i_n^n.) Then there is a fibration sequence of infinite loop spaces

$$\Gamma\left(\coprod_{n\geq 0} C(n)\times_{\Sigma_n} M_n/\approx\right)\to\Gamma(M_*)\to Z.$$

Thus the group completion of the configuration space model of M_* is a collection of path components of the group completion of M_*. As we mentioned in the previous paragraph, the configuration space model is frequently group complete, in which case the fibration sequence simplifies to

$$\coprod_{n\geq 0} C(n)\times_{\Sigma_n} M_n/\approx\to\Gamma(M_*)\to Z.$$

In particular, if $M_n = BEnd(A^n)$, A an object in a permutative category, we obtain that the configuration space model is equivalent to the basepoint component of $\Gamma(M_*)$, i.e. Quillen's plus construction, thus verifying the conjecture posed by the second author.

The outline of the paper is as follows. In Section 1 we introduce the notion of a permutative monoid. By using a convenient choice of operad with $C(n)=E\Sigma_n$ and the homotopy theory of permutative categories we show that there is a fibration sequence of infinite loop spaces

$$Q(S^0)\to\Gamma\left(\coprod_{n\geq 0} E\Sigma_n\times_{\Sigma_n} M_n\right)\to\Gamma\left(\coprod_{n\geq 0} E\Sigma_n\times_{\Sigma_n} M_n/\approx\right) \tag{0.1}$$

On the other hand, by applying the results of McDuff and Segal [2] we obtain, under the hypothesis of Theorem A, a homotopy cartesian diagram

$$\begin{array}{ccc} \Gamma\left(\coprod_{n\geq 0} E\Sigma_n\times_{\Sigma_n} M_n\right) & \to & Q(S^0) \\ \downarrow & & \downarrow \\ \Gamma(M_*) & \longrightarrow & Z \end{array} \tag{0.2}$$

with splitting map

$$Q(S^0)\to\Gamma\left(\coprod_{n\geq 0} E\Sigma_n\times_{\Sigma_n} M_n\right).$$

Using (0.1) and (0.2) to analyze the infinite loop space cofiber of this map we obtain the fibration sequence of Theorem A.

In Section 2 we use the full strength of Theorem A to construct Volodin type configuration space models for $\Omega BGl(R)^+$ and $\Omega A(X)$, where $A(X)$ denotes Waldhausen's algebraic K-theory of the space X. This model is used in [5] to prove a splitting theorem (Theorem 3.6 of [5]) for $\Omega A(\Sigma X)$ when X is connected. In another paper we will show how these models for $\Omega A(\Sigma X)$ can be used to verify Waldhausen's result that the "mystery" homology theory vanishes: i.e., $\mu(X) \simeq *$ for all X, where

$$A^s(X) \simeq \mu(X) \times Q(X)$$

is Waldhausen's stabilization of $A(X)$.

§1

We first establish some conventions concerning the symmetric groups. As usual, $\Sigma_n = \mathrm{Aut}(\mathbf{n})$, $\mathbf{n} = \{1, 2, \ldots, n\}$. We choose Σ_n to act on the right of $\mathbf{n}$, so that for $\alpha, \beta \in \Sigma_n$, $\alpha\beta$: $\mathbf{n} \to \mathbf{n}$ is given by $\mathbf{n} \xrightarrow{\alpha} \mathbf{n} \xrightarrow{\beta} \mathbf{n}$. Let i_j: $\mathbf{n} \to \mathbf{n+1}$, $1 \leqslant j \leqslant n+1$ be the ordered inclusion which misses $j \in \mathbf{n+1}$; then i_j determines maps i_j^Σ: $\Sigma_n \to \Sigma_{n+1}$ and π_j^Σ: $\Sigma_{n+1} \to \Sigma_n$ which are uniquely defined by requiring that the diagrams

$$\begin{tikzcd} \mathbf{n} \arrow[r, "i_j"] \arrow[d, "\sigma"'] & \mathbf{n+1} \arrow[d, "i_j^\Sigma(\sigma)"] \\ \mathbf{n} \arrow[r, "i_j"'] & \mathbf{n+1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{n} \arrow[r, "i_j"] \arrow[d, "\pi_j^\Sigma(\tau)"'] & \mathbf{n+1} \arrow[d, "\tau"] \\ \mathbf{n} \arrow[r, "i_{j\tau}"'] & \mathbf{n+1} \end{tikzcd}$$

commute for all $\sigma \in \Sigma_n$, $\tau \in \Sigma_{n+1}$ (here $j\tau$ denotes the image of $j \in \mathbf{n+1}$ under the action of $\tau \in \Sigma_{n+1}$). i_j^Σ is a homomorphism, while π_j^Σ satisfies the identity $\pi_j^\Sigma(\alpha\beta) = \pi_j^\Sigma(\alpha)\pi_{j\alpha}^\Sigma(\beta)$. Moreover, $\pi_j^\Sigma i_j^\Sigma = id$ for each j.

Definition 1.1. A permutative monoid $S_* = \coprod_{n \geq 0} S_n$ is a graded set together with an associative pairing $\oplus = \{\oplus_{mn}: S_m \times S_n \to S_{m+n}\}$, symmetric group actions $\Sigma_n \times S_n \to S_n$ written as $(\sigma, s) \to {}^\sigma s$, and distinguished elements $e_i \in S_i$, $i = 0, 1$, satisfying the following properties:

(P1) e_0 is the identity element with respect to $\oplus$.

(P2) If $\sigma(m, n) \in \Sigma_{m+n}$ is the permutation which switches the blocks $\{1, 2, \ldots, m\}$, $\{m+1, \ldots, m+n\} \subset \mathbf{m+n}$, then ${}^{\sigma(m,n)}(x \oplus y) = (y \oplus x)$ for all $x \in S_m$, $y \in S_n$.

(P3) If $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$ and $\sigma \oplus \tau \in \Sigma_{m+n}$ is their block sum, then ${}^{(\sigma \oplus \tau)}(x \oplus y) = {}^\sigma x \oplus {}^\tau y$ for all $x \in S_m$, $y \in S_n$.

(P4) $S_0 = \{e_0\}$

(P5) i_1^s: $S_{n-1} \to S_n$, given by $i_1^s(x) = e_1 \oplus x$, is injective for all n.

(P6) If $x \in S_m$, $y \in S_n$ and $x \oplus y = i_1^s(z)$ for some $z \in S_{m+n-1}$, then $x = i_1^s(w)$ for some $w \in S_{m-1}$.

A map of permutative monoids f: $\coprod S_n \to \coprod S'_n$ is a collection of maps $\{f_n: S_n \to S'_n\}_{n \geqslant 0}$ such that $f(e_i) = e'_i$ for $i = 0, 1$, f_n is Σ_n-equivariant for each n, and such that f preserves the sum operation.

Let $\sigma_j \in \Sigma_n$ denote the permutation $(1, j)$ which sends $(\mathbf{n}-\{1\}) \subset \mathbf{n}$ to $(\mathbf{n}-\{j\}) \subset \mathbf{n}$, and define $i_j^s\colon S_{n-1} \to S_n$ by $i_j^s(x) = {}^{\sigma_j}(i_1^s(x))$ for $1 \leqslant j \leqslant n$. By definition and (P3) one has

1.2. $i_j^s({}^{\sigma}x) = i_j^{\Sigma}(\sigma)\,(i_j^s(x))$

and

$${}^{\tau}(i_j^s(x)) = i_{j\tau}^s(\pi_j^{\Sigma}(\tau)(x)) \quad \text{for} \quad x \in S_{n-1},\ \sigma \in \Sigma_{n-1} \text{ and } \tau \in \Sigma_n.$$

The maps $i_j^s\colon S_{n-1} \to S_n$ are analogous to stabilization maps. We will call an element $y \in S_n$ *reduced* if $y \notin \bigcup_{j=1}^{n} i_j^s(S_{n-1}) \subset S_n$. Properties (P5) and (P6) imply that any non-reduced element $x \in S_n$ satisfies the equation $x = i_j^s(y)$ for some unique reduced $y \in S_m$ and iterated stabilization map

$$i_J^s\colon S_m \xrightarrow{i_{j_1}^s} S_{m+1} \xrightarrow{i_{j_2}^s} S_{m+2} \to \dots \xrightarrow{i_{j_{n-m}}^s} S_n.$$

Also, (P6) implies that $x \oplus y$ is reduced if both x and y are reduced.

Definition 1.3. Let $S_* = \coprod S_n$ be a permutative monoid. The *configuration space* of S_*, $\bar{C}(S_*)$ is the simplicial set

$$\bar{C}(S_*) = \{[k] \to \coprod (E\Sigma_n)_k \times_{\Sigma_n} S_n / \sim\},$$

where

$$((\sigma_1, \sigma_2, \dots, \sigma_{k+1}),\ i_j^s(x)) \sim ((\pi_{j_1}^{\Sigma}(\sigma_1),\ \pi_{j_2}^{\Sigma}(\sigma_2), \dots, \pi_{j_{k+1}}^{\Sigma}(\sigma_{k+1})),\ x)$$

and

$$((\sigma \cdot \sigma_1, \sigma \cdot \sigma_2, \dots, \sigma \cdot \sigma_{k+1}), y) = ((\sigma_1, \dots, \sigma_{k+1}), {}^{\sigma}y) \text{ in } (E\Sigma_n)_k \times_{\Sigma_n} S_n.$$

Here $(E\Sigma_n)$, denotes the homogeneous bar resolution of Σ_n, and j_m is defined recursively by

$$j_1 = j,\ j_m = j_{m-1}\sigma_{m-1} \quad \text{for} \quad 1 < m \leqslant k+1.$$

(If $S_*^{(m)}$ is an m-simplicial permutative monoid, $\bar{C}(S_*^{(m)})$ is the $(m+1)$-simplicial set formed by choosing the simplicial direction of $(E\Sigma_n)$, to be independent of the m-simplicial directions of $S_n^{(m)}$ for each n, and making the above identifications degree-wise.)

Definition 1.4. The (unreduced) configuration category $\mathscr{C}(S_*)$ of a permutative monoid S_* has objects $= S_* = \coprod S_n$. If $x \in S_m$, $y \in S_n$ then there are no morphisms from x to y unless $m = n$, in which case the morphisms are $\sigma \in \Sigma_n$ such that $y = {}^{\sigma}x$. Composition is defined by composing permutations. *The reduced configuration category* $\bar{\mathscr{C}}(S_*)$ is then the full subcategory of $\mathscr{C}(S_*)$ with obj $(\bar{\mathscr{C}}(S_*)) = \{x \in S_* \mid x \text{ is reduced}\}$.

LEMMA 1.5. (i) $B\mathscr{C}(S_*) \cong \coprod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} S_n$

(ii) $B\bar{\mathscr{C}}(S_*) \cong \bar{C}(S_*)$.

Proof. As S_* is a set, both $\mathscr{C}(S_*)$ and $\bar{\mathscr{C}}(S_*)$ are small categories. The isomorphism in (i) is given on k-simplicies by

$$(x \xrightarrow{\sigma_1} x_1 \xrightarrow{\sigma_2} x_2 \to \dots \xrightarrow{\sigma_k} x_k) \to ((1, \sigma_1, \dots, \sigma_k), x)$$

for $x \in S_n$. Restricting to x such that x is reduced produces the isomorphism on k-simplicies $B\bar{\mathscr{C}}(S_*)_k \xrightarrow{\cong} \bar{C}(S_*)_k$. One easily verifies that these maps preserve simplicial structure. □

Remark 1.6. Since all morphisms in $\mathscr{C}(S_*)$ and $\bar{\mathscr{C}}(S_*)$ are isomorphisms, choosing an object in each isomorphism class yields equivalences of categories

$$\mathscr{C}(S_*) \cong \coprod_{x \in S_*} G_x,\ \bar{\mathscr{C}}(S_*) \cong \coprod_{y \in S_*} G_y,\ y \text{ reduced}$$

where $G_w \subset \Sigma_n$ denotes the 1-object category whose morphisms are the isotropy subgroup of w (see, for example, [6], §1).

LEMMA 1.7. (i) *$\mathscr{C}(S_*)$ and $\bar{\mathscr{C}}(S_*)$ are permutative categories.*

(ii) *There is an isomorphism of permutative categories*

$$\mathscr{C}(S_*) \cong \left(\coprod_{n \geq 0} \Sigma_n\right) \times \bar{\mathscr{C}}(S_*).$$

(iii) *If Γ: (permutative categories)$\to$(infinite loop spaces) is any infinite loop space machine satisfying the May–Thomason axioms (which are unique by [4]) then $\Gamma(\mathscr{C}(S_*)) \cong Q(S^0) \times \Gamma(\bar{\mathscr{C}}(S_*))$.*

Proof. By properties (*P*1)–(*P*3), the operation $\oplus = \{\oplus_{mn}: S_m \times S_n \to S_{m+n}\}$ and the block sum operation $\oplus^\Sigma = \{\oplus^\Sigma_{mn}: \Sigma_m \times \Sigma_n \to \Sigma_{m+n}\}$ define permutative structures on $\mathscr{C}(S_*)$ and $\bar{\mathscr{C}}(S_*)$. By remark 1.6, it suffices to show how to uniquely decompose automorphisms in $\mathscr{C}(S_*)$ into pairs of automorphisms in $(\coprod \Sigma_n) \times \bar{\mathscr{C}}(S_*)$. For $x \in S_n$, let $y_x \in S_m$ denote the unique reduced element which stabilizes to x (y_x exists and is unique by (*P*5)). For $\sigma \in G_x \subset \Sigma_n$, the identity $x = {}^\sigma x$ implies (by (1.2) above) that

$$x = i^s_J(y_x) = {}^\sigma(i^s_J(y_x)) = i^s_{J\sigma}(\pi^\Sigma_J(\sigma)y_x);$$

thus $y_x = \pi^\Sigma_J(\sigma)y_x$ (since the choice of y_x is unique) and $i^s_J = i^s_{J\sigma}$. Without loss of generality, we can assume that $i^s_J = (i^s_1)^{n-m}$; then the equation $i^s_j = i^s_{j\sigma}$ implies that σ preserves the image of $\mathbf{n}-\mathbf{m}$ in $\mathbf{n}$ under the standard inclusion. So σ decomposes uniquely as a block sum $\sigma = \sigma' \oplus (\pi^\Sigma_1)^{n-m}(\sigma)$, and $(\sigma', (\pi^\Sigma_1)^{n-m})\ (\sigma)) \in (\Sigma_{n-m}) \times (G_{y_x})$ is an automorphism in $(\coprod \Sigma_n) \times (\bar{\mathscr{C}}(S_*))$, proving (ii). Finally, Γ preserves products (up to homotopy) and $\Gamma\left(\coprod_{n \geq 0} \Sigma_n\right) \simeq Q(S^0)$ which implies (iii). □

The map $\mathscr{C}(S_*) \xrightarrow{p} \bar{\mathscr{C}}(S_*)$ given by $p(x) = y_x$ and $p(\sigma) = \pi^\Sigma_j(\sigma)$ defines a functor. Given a map of permutative monoids f: $S_* \to S'_*$ the induced functor $\mathscr{C}(S_*) \xrightarrow{\mathscr{C}(f)} \mathscr{C}(S'_*)$ does *not* in general restrict to a functor $\bar{\mathscr{C}}(S_*) \to \bar{\mathscr{C}}(S'_*)$, since f may send reduced objects to unreduced objects. However, by projecting to reduced objects, we can construct a functor $\bar{\mathscr{C}}(f)$ so that

$$\begin{tikzcd} \mathscr{C}(S_*) \arrow[r, "p"] \arrow[d, "\mathscr{C}(f)"] & \bar{\mathscr{C}}(S_*) \arrow[d, "\bar{\mathscr{C}}(f)"] \\ \mathscr{C}(S'_*) \arrow[r, "p"] & \bar{\mathscr{C}}(S_*) \end{tikzcd}$$

commutes. One can check that this defines a functor from (permutative monoids) to (permutative categories).

LEMMA 1.8. *There is a fibration sequence of infinite loop spaces:*

$$Q(S^0) \to \Gamma(\mathscr{C}(S_*)) \to \Gamma(\bar{\mathscr{C}}(S_*))$$

which is natural with respect to maps of permutative monoids, and unnaturally split.

Proof. This is a restatement of lemma 1.7 (iii) above, since the functor

$$P\colon \mathscr{C}(S_*) \to \bar{\mathscr{C}}(S_*)$$

induces, after passing to classifying spaces, the quotient map

$$|\mathscr{C}(S_*)| = |\coprod E\Sigma_n \times_{\Sigma_n} S_n| \to |\bar{\mathscr{C}}(S_*)| = |\coprod E\Sigma_n \times_{\Sigma_n} S_n/\sim|$$

and this map is natural with respect to maps of permutative monoids. □

We now extend the above to m-simplicial permutative monoids $S_*^{(m)} = \coprod S_n^{(m)}$. Lemma 1.5 extends degree-wise in the obvious way. The following lemma extends the fibration sequence of lemma 1.8 to simplicial permutative monoids; the analogous statement for m-simplicial permutative monoids is exactly the same (as is the proof, which we omit).

LEMMA 1.9. *Let $S_*^{(1)} = \{[k] \to S_{*k}\}$ be a simplicial permutative monoid. Then there is a natural fibration sequence of infinite loop spaces:*

$$Q(S^0) \to |[k] \to \Gamma(\mathscr{C}(S_{*k}))| \to |[k] \to \Gamma(\bar{\mathscr{C}}(S_{*k}))|.$$

Proof. The connective delooping of the fibration in lemma 1.8 produces for each $k \geqslant 0$ a fibration $BQ(S^0) \to B\Gamma(\mathscr{C}(S_{*k})) \to B\Gamma(\bar{\mathscr{C}}(S_{*k}))$. This yields a simplicial sequence which at each level is a fibration with connected base. By ([9], Prop. 6.3) one gets a fibration

$$BQ(S^0) \to |[k] \to B\Gamma(\mathscr{C}(S_{*k}))| \to |[k] \to B\Gamma(\bar{\mathscr{C}}(S_{*k}))|$$

after passing to geometric realization. The loop space functor commutes with geometric realization for simplicial fibrations which are level-wise connected, so looping this fibration yields the desired result. □

We now can state and prove Theorem A in the context of m-simplicial permutative monoids:

THEOREM A′. *Let $S_*^{(m)}$ be an m-simplicial permutative monoid. If $|\mathscr{C}(S_*^{(m)})|$ is group-complete, then $\Gamma(|\bar{\mathscr{C}}(S_*^{(m)})|) \simeq |\bar{\mathscr{C}}(S_*^{(m)})|$. Furthermore, if Σ_∞ acts trivially on the homology of $|S_\infty^{(m)}| = \varinjlim_n |S_n^{(m)}|$, then $\Gamma(S_*^{(m)}) \simeq \mathbb{Z} \times |C(S_*^{(m)})|$ where $\Gamma(S_*^{(m)})$ is the Γ-space whose underlying space is $\Gamma(1) = \coprod_{n \geq 0} |S_n^{(m)}|$.*

Proof. By definition, $\Gamma(\bar{\mathscr{C}}(S_*^{(m)}))$ is the group-completion of $B\bar{\mathscr{C}}(S_*^{(m)})$ up to homotopy, and so the first statement follows by Lemma 1.5 (ii) for m-simplicial $S_*^{(m)}$ after passing to realization. Also, $\Gamma(|\mathscr{C}(S_*^{(m)})|)$ is weakly equivalent to the group completion of $\coprod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} |S_n^{(m)}|$, where the monoid structure is induced by block sum. Now suppose that Σ_∞ acts trivially on $H_*(|S_\infty^{(m)}|)$. Then a homotopy co-limit argument shows that the inclusion $|S_\infty^{(m)}| \to |\bar{C}(S_*^{(m)})|$ induces an isomorphism on H_0 and hence π_0. Since $|\bar{C}(S_\infty^{(m)})|$ is group-complete, $\pi_0(|S_\infty^{(m)}|)$ is a group, with the group structure induced by the block-sum operation on $S_*^{(m)}$. Since the action of Σ_∞ is trivial on $H_0(|S_\infty^{(m)}|)$, it is trivial on $\pi_0(|S_\infty^{(m)}|)$; by property $(P2)$ this implies $\pi_0(|S_\infty^{(m)}|)$ is an abelian group.

Let $e_* \in H_o(S_*^{(m)})$ be the element in homology corresponding to $[e_1^{(m)}] \in \pi_o(S_*^{(m)}, e_o^{(m)})$ where $e_i^{(m)} \in S_i^{(m)}$ is given as in definition 1.1 for $i=0,1$. Note that since an m-simplicial monoid preserves all the structure, the elements $e_i^{(m)}$ are m-simplicial 1-element sets with trivial simplicial structure, and so correspond naturally to points in $|S_*^{(m)}|$. Now since Σ_∞ acts trivially on $H_*(|S_\infty^{(m)}|)$, it follows that for any iterated Pontrjagin product $x_1 * \ldots * x_n \in H_*(|S_*^{(m)}|)$, there exists an integer $\phi(n)$ such that

$$x_1 * \ldots * x_n * (e_*)^{\phi(n)} = x_{\sigma(1)} * \ldots * x_{\sigma(n)} * (e_*)^{\phi(n)}$$

for any $\sigma \in \Sigma_n$. This in turn implies that for any multiplicative subset $M \subset \pi_0(|S_*^{(m)}|)$, the localized Pontrjagin ring $H_*(|S_\infty^{(m)}|)[M^{-1}]$ can be constructed by right fractions ([2], p. 279). The result of proposition 1 of [2] now applies to show that there is an isomorphism of Pontrjagin rings

$$H_*(|S_*^{(m)}|)\,[\pi_0(S_*^{(m)})^{-1}] \cong H_*(\Omega B(|S_*^{(m)}|)).$$

Moreover, the argument in proposition 2 of [2] yields an equivalence $H_*(|S_\infty^{(m)}|)[e_*^{-1}] \cong H_*(\mathbb{Z} \times |S_\infty^{(m)}|)$ of Pontrjagin rings. But $\pi_0(\mathbb{Z} \times |S_*^{(m)}|) = \mathbb{Z} \times \pi_0(|S_\infty^{(m)}|)$ is a group. Since the second isomorphism is induced on the level of spaces by the maps $|S_n^{(m)}| \to |S_\infty^{(m)}|$, $n \geqslant 0$, it follows that there are isomorphisms

$$H_*(|S_\infty^{(m)}|)\,[\pi_0(S_*^{(m)})^{-1}] \cong H_*(\mathbb{Z} \times |S^{(m)}|)\,[\pi_0(S^{(m)})^{-1}] \cong H_*(\mathbb{Z} \times |S_\infty^{(m)}|)$$

and so the resulting map $\mathbb{Z} \times |S_\infty^{(m)}| \to \Omega B(S_*^{(m)})$ induces an isomorphism of Pontrjagin rings $H_*(\mathbb{Z} \times |S_*^{(m)}|) \cong H_*(\Omega B(S_*^{(m)}))$. This implies that there is a homotopy-equivalence $\mathbb{Z} \times |S_*^{(m)}|^+ \cong \Omega B(S_*^{(m)})$, where the "plus" construction is done on each path component of $|S_\infty^{(m)}|$. Again, the triviality of the action of Σ_∞ on $\pi_0(|S_\infty^{(m)}|)$ implies that there is an isomorphism $\pi_0(|S_\infty^{(m)}|) \cong \pi_0(E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)$; exactly the same arguments as above now apply to show that there is an isomorphism of Pontrjagin rings

$$H_*(\mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)) \cong H_*(\textstyle\coprod E\Sigma_n \times_{\Sigma_n} |S_n^{(m)}|)[e_*^{-1}] \cong$$

$$H_*(\textstyle\coprod E\Sigma_n \times_{\Sigma_n} |S_n^{(m)}|)\,[\pi_0^{-1}] \cong H_*(\Omega B(\coprod E\Sigma_n \times_{\Sigma_n} |S_n^{(m)}|))$$

and hence a weak equivalence

$$\mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)^+ \cong \Omega B(\textstyle\coprod E\Sigma_n \times_{\Sigma_n} |S_n^{(m)}|).$$

There is a map of fibration sequences

$$\begin{tikzcd}
|S_\infty^{(m)}| \arrow[r] \arrow[d, "\iota"'] & (E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|) \arrow[r] \arrow[d] & (B\Sigma_\infty) \arrow[d] \\
F \arrow[r, hook, "s_2"] & (E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)^+ \arrow[r, "s_1"] & (B\Sigma_\infty)^+
\end{tikzcd}$$

where F is the fibre after the plus construction. The triviality of the action of Σ_∞ on $H_*(|S_\infty^{(m)}|)$ implies by a standard spectral sequence comparison argument that the map ι induces an isomorphism in homology. By property $(P2)$ and $(P5)$, the action of Σ_∞ fixes the image of $e_1^{(m)}$ in $S_\infty^{(m)}$, so the top fibration and hence the bottom admit sections. It follows that

$$(E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)^+ \simeq (B\Sigma_\infty)^+ \times F \simeq (B\Sigma_\infty)^+ \times |S_\infty^{(m)}|^+,$$

since the section $(B\Sigma_\infty)^+ \xrightarrow{s_1} (E\Sigma_\infty \times_{\Sigma_\infty} |S_\infty^{(m)}|)^+$ is a section of H-spaces. Now the m-simplicial analogue of lemma 1.8 (see lemma 1.9) yields a fibration sequence $Q(S^0) \to \Gamma(\mathscr{C}(S_*^{(m)})) \to \Gamma(\overline{\mathscr{C}}(S_*^{(m)}))$ which is natural with respect to infinite loop maps out of $Q(S^0)$. Such maps identify with maps out of S^0, and this identifies the fibration sequence of lemma 1.8 with the fibration sequence

$$\mathbb{Z}\times(B\Sigma_\infty)^+ \xrightarrow{1\times s_1} \mathbb{Z}\times(E\Sigma_\infty\times_{\Sigma_\infty}|S_\infty^{(m)}|)^+ \xrightarrow{s_2} |S_\infty^{(m)}|)^+.$$

Thus

$$|S_\infty^{(m)}|^+\simeq|\bar{C}(S_\infty^{(m)})| \text{ and so } \Gamma(S_*^{(m)})\simeq\mathbb{Z}\times|S_\infty^{(m)}|^+\simeq\mathbb{Z}\times|\bar{C}(S_\infty^{(m)})|^+,$$

proving the Theorem. □

To illustrate Theorem A, we note that if $\coprod_{n\geq 0} P_n^{(m)}$ is an m-simplicial permutative category, then $\coprod_{n\geq 0} BP_n^{(m)}$ is an $(m+1)$-simplicial permutative monoid and $BP_\infty^{(m)}=\varinjlim_n BP_n^{(m)}$ is connected. It follows that

Corollary B. $\Gamma_0(\coprod P_n^{(m)})\simeq|\bar{C}(\coprod BP_n^{(m)})|$, *where* $\Gamma=$*Segal's machine on m-simplicial permutative categories, and* Γ_0 *denotes the path component of the identity.* □

It is worth remarking that a more general statement than Theorem A applies where one does not assume that $|\bar{C}(S^{(m)})|$ is group-complete. However, this more general case does not as yet have any applications not covered by Theorem A.

§2

We apply Theorem A to construct configuration-spaces for $\Omega K(R)$ and $\Omega A(X)$ where $A(X)$ denotes Waldhausen's K-Theory of X.

Thus if α is a partial ordering of $\mathbf{n}=\{1,2,\ldots,n\}$, a subgroup $T_n(R)\subset GL_n(R)$ is α-triangular if $g_{ij}=0$ for $i \overset{\alpha}{\not<} j$ and $g_{ii}=1$, for all $g\in T_n(R)$. If $T_n(R)$ is α-triangular, it will be written as $T_n^\alpha(R)$. One can form the union of classifying spaces $\bigcup_\alpha BT_n^\alpha(R)\subset BGL_n(R)$, where the (non-empty) union is taken over all partial orderings α of $\mathbf{n}$. These are well-defined simplicial subsets of $BGL_n(R)$ for each n.

Definition 2.1. $V(GL_n, R)$ is the pullback of the diagram

$$\begin{array}{ccc} & & EGL_n(R) \\ & & \downarrow \\ \bigcup_\alpha BT_n^\alpha(R) & \rightarrow & BGL_n(R) \end{array}$$

$V(GL_n, R)=\left(\bigcup_{g\in GL_n(R)} \bigcup_\alpha ET_n^\alpha(R)\cdot g\right)$ is a union of contactible right coset spaces $ET_n^\alpha(R)$ which are simplicial subsets of the bar resolution of $GL_n(R)$. Thus $\coprod V(GL_n, R)\subset\coprod EGL_n(R)$ is a well-defined simplicial subset of the simplicial permutative monoid $\coprod EGL_n(R)$.

Proposition 2.2. *The permutative monoid structure on* $\coprod EGL_n(R)$ *restricts to one on* $\coprod VGL_n(R)$.

Proof. The simplicial permutative monoid structure on $\coprod EGL_n(R)$ is induced by the degree-wise extension of the permutative monoid structure on $\coprod GL_n(R)$. This structure is induced by the block sum operation $\oplus$, $\oplus_{mn}$: $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$, the conjugation action of Σ_n on $GL_n(R)$ and $e_1 = id \in R^* = GL_l(R)$. As the subset $\coprod VGL_n(R)$ is closed under the degree-wise extension of this structure, it is a simplicial permutative submonoid of $\coprod EGL_n(R)$. □

LEMMA 2.3. *There are weak equivalences* $\Omega K(R) \simeq |V(GL, R)| \simeq |\bar{C}(\coprod V(GL_n, R))|$, *where* $\bar{C}(\coprod V(GL_n, R))$ *is the (bi-simplicial) configuration space of* $\coprod V(GL_n, R)$.

Proof. The equivalence $\Omega K(R) \simeq |V(GL, R)|$ is proved in [8], where $V(GL, R) = \varinjlim_n V(GL_n, R)$. The action of Σ_∞ on $|V(GL, R)|$ is trivial up to homotopy, as it corresponds to the restriction of the action of $GL(R)$ on $\Omega K(R)$ via conjugation, which is trivial up to homotopy. In particular, Σ_∞ acts trivially on $H_*(|V(GL, R)|)$. For each n, there is a fibration sequence with section $V(GL_n, R) \xrightarrow{i} E\Sigma_n \times_{\Sigma_n} V(GL_n, R) \to B\Sigma_n$, and so i_n induces an isomorphism on π_0 and H_0 for all n. Moreover, all of the distinct stabilization maps $V(GL_{n-1}, R) \xrightarrow{i_j^s} V(GL_n, R)$ $1 \leq j \leq n$ are conjugate to the standard one i_n^s by conjugation by an element of Σ_n. It follows that the natural inclusion of coends $|V(GL, R)| = (\coprod V(GL_n, R)/\sim) \to \coprod E\Sigma_n \times_{\Sigma_n} V(GL_n, R)/\sim) = \bar{C}(\coprod V(GL_n, R))$ induces an isomorphism on H_0 and hence π_0. As $\pi_0(|V(GL, R)|) = K_1(R)$ is a group (with group structure induced by simplicial block sum in $\coprod V(GL_n, R)$) it follows that $|\bar{C}(\coprod V(GL_n, R))|$ is group-complete. Theorem A now applies, yielding a weak equivalence

$$\Omega K(R) \simeq |V(GL, R)| \simeq |V(GL, R)|^+ \simeq |\bar{C}(\coprod V(GL_n, R))|. \qquad \square$$

In [10], Waldhausen defines the algebraic K-theory of a space X (for connected X) as follows: for each $n, k \geq 0$, $H_n^k(GX) = Aut_{GX}(V^n S^k \wedge GX_+)$ where GX is the Kan loop group of X (by space we mean simplicial set). Suspension defines a map $H_n^k(GX) \xrightarrow{\Sigma} H_n^{k+1}(GX)$ and $H_n(GX) = \varinjlim_{\Sigma} H_n^k(GX)$; $H_n(GX)$ is group-like and Waldhausen defines $A(X) = \Gamma\left(\coprod_{n \geq 0} BH_n(GX)\right)$ = the Γ-space with underlying space $\coprod_{n \geq 0} |BH_n(GX)|$.

In [5], a Volodin-type model for $\Omega A(X)$ is constructed. Given a partial ordering α, one has a suitable notion of an α-triangular monoid $M_n^\alpha(GX) \subset H_n(GX)$, corresponding to α-triangular "matrices" in $H_n(GX)$.

As for discrete rings, one can form $\bigcup_\alpha BM_n^\alpha(GX) \subset BH_n(GX)$ and

$$V(H_n, GX) = \left(\bigcup_{\substack{\alpha \\ g \in H_n(GX)}} EM_n^\alpha(GX) \cdot g\right) \subset EH_n(GX);$$

$V(H_n, GX)$ is the pull-back of

$$\begin{array}{c} EH_n(GX) \\ \downarrow \\ \bigcup_\alpha BM_n^\alpha(GX) \to BH_n(GX). \end{array}$$

The following Theorem is proved in section 1 of [5].

THEOREM 2.4. *There is a weak equivalence* $|V(H, GX)| \xrightarrow{\cong} \Omega A(X)$ *for connected* X, *natural in* X, *where* $V(H, GX) = \varinjlim_n V(H_n, GX)$. *For* $X = \Sigma Y$, *the simplicial monoid equivalence* $J(Y) \xrightarrow{\cong} G\Sigma Y$ *induces an equivalence* $|V(H, J(Y))| \xrightarrow{\cong} \Omega A(\Sigma Y)$ *natural in* Y *for connected* Y, *where* $J(Y)$ *is the simplicial James monoid on* Y. □

As a corollary, Theorem A applies to yield

COROLLARY 2.5. *There are equivalences of infinite loop spaces*

$$|\bar{C}(\coprod V(H_n, G(X)))| \xrightarrow{\cong} \Omega A(X)$$

$$|\bar{C}(\coprod V(H_n, J(X)))| \xrightarrow{\cong} \Omega A(\Sigma X)$$

natural in X *for connected* X.

Proof. $\coprod V(H_n, GX)$ is a bi-simplicial permutative monoid via the simplicial permutative monoid structure on $\coprod H_n(GX)$, induced by the wedge-sum maps $H_n(GX) \times H_m(GX) \to H_{n+m}(GX)$ and the natural action of Σ_n on $H_n(GX)$ for each n. The rest of the proof follows exactly as in the proof of lemma 2.3 by the above Theorem, with an extra simplicial dimension occuring throughout. □

Remark 2.6. There is a close relationship between the Volodin model for $\Omega A(X)$ and Waldhausen's expansion space $E(X)$ ([10], §3); this is not surprising as $E(X) \simeq \Omega Wh^{comb}(X)$ and there is a natural map $\Omega A(X) \to Wh^{comb}(X)$.

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