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VOLODIN K-THEORY OF A_∞-RING SPACES

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0. INTRODUCTION

IN [15], Suslin showed that a certain simplicial set $V(G, R)$, constucted by Volodin, was weakly equivalent to $\Omega(BG(R)^+)$, where $G(R) = GL(R)$, $E(R)$ or $St(R)$, and R is a discrete ring with unit. This model for the loop space of K-theory has had a number of important applications, ranging from the stability problem in algebraic K-theory [16] to the construction of invariants for stable pseudio-isotopy theory [20]. More recently, it has been used in determining the homotopy-type of n-relative K-theory ([6], [9]), and in showing that the Waldhausen K-theory of various diagrams of simplicial rings can be computed degreewise [6]. The existence of a Volodin model for A_∞ rings is presupposed in a crucial reduction step required to prove the equivalence of Waldhausen's stable K-theory and Bökstedt's topological Hochschild homology [10].

In this paper, we construct a Volodin model for the loop space of $K(R)$, where R is an A_∞ ring. It is worth noting that even in the case of simplicial rings, the proof that the Volodin model has the right homotopy type does not follow directly by the arguments of [15], and as far as we are aware of there is no rigorous proof of this fact in the existing literature. A key point in Suslin's argument for discrete rings is that the fundamental group of the Volodin space acts trivially on the higher homotopy groups. This is not true for simplicial rings. For consider the natural map $R \to \pi_0 R$ for a simplicial ring. The induced map on Volodin spaces $\mathscr{V}\mathscr{S}(R) \to \mathscr{V}\mathscr{S}(\pi_0 R)$ ($\mathscr{V}\mathscr{S}(\,)$ is denoted $X(\,)$ in [15]) is an isomorphism in homology (both spaces are acyclic) and on fundamental groups ($St(\pi_0 R)$ in both cases). $St(\pi_0 R)$ acts trivially on the higher homotopy groups of $\mathscr{V}\mathscr{S}(\pi_0 R)$. If it acts trivially on the higher homotopy groups of $\mathscr{V}\mathscr{S}(R)$ too, then $\mathscr{V}\mathscr{S}(R) \to \mathscr{V}\mathscr{S}(\pi_0 R)$ would be a nilpotent homology equivalence and hence a homotopy equivalence, which is not true in general.

In the ratioal case as well as for simplicial rings with torsion-free π_0, Song [14] proved that the Volodin model has the right homotopy type using methods of homological algebra to reduce the proof to Suslin's arguments. For general A_∞ rings, the proof is much harder. For this reason, we have devoted this paper to proving the existence and the relevant properties of our Volodin model, deferring to other papers the applications which provided the initial motivation for the construction of such a model for $\Omega K(R)$.

After the introduction of A_∞ rings with involution in Section 1 (which we will need for the proof of our main result) we introduce the Volodin construction in Section 2 and prove our main theorem based on three lemmas. In Section 3 we give alternative models for this construction which we will need for the proofs of the three lemmas in Sections 4, 5, and 6. Finaly, §7 is devoted to generalizing the group completion theorem of [4] and casting it in homotopy invariant form. In the process we also simplify a number of arguments in that

paper. While the main application of the theorem is the construction of an infinite loop model of Volodin K-theory for A_∞ rings, the result is of independent interest.

1. A_∞ RINGS WITH INVOLUTION

Throughout this paper we use the definitions and notation of [11]. Besides the theories Θ_m of monoids and Θ_r of semirings, we will use the theories $\Theta_{r,i}$ of semirings with involution and $\Theta_{m,i}$ of monoids with involution. The morphisms $n \to p$ in $\Theta_{r,i}$ are p-tuples of polynomials in $2n$ non-commuting variables $x_1, \ldots, x_n$ and their involutes $x_1^*, \ldots, x_n^*$ with nonnegative integer coefficients. Composition is defined using substitution (as in Θ_r), the usual algebraic identities in a non-commutative ring and the additional relations

$$1^* = 1, \quad (x_{i_1}^{(l_1)} \cdot \ldots \cdot x_{i_q}^{(l_q)})^* = x_{i_q}^{(-l_q)} \cdot \ldots \cdot x_{i_1}^{(-l_1)}, \quad (a+b)^{(-1)} = a^{(-1)} + b^{(-1)},$$

where for notational convenience we have denoted $x_i^{(1)} = x_i$ and $x_i^{(-1)} = x_i^*$ and a and b stand for monomials. We take $\Theta_{m,i}$ to be the subcategory of morphisms whose components are simple monomials. A morphism in $\Theta_{r,i}$ will be called *simple* iff all its polynomial components have coefficients which are 0 or 1. A_∞ monoids and rings with involution are then defined as in [11, §1]. An extensive discussion of A_∞ monoids and rings with involution may be found in [5].

1.1 *Convention.* Throughout this paper we only consider ringlike A_∞ rings X, i.e. A_∞ rings for which $\pi_0 X$ is a genuine ring rather than a semiring.

1.2 *Remark.* In [12] A_∞ rings were defined to be continuous functors $X: \Theta \to Top$, Θ an A_∞ ring theory, such that $X(0)$ is contractible and

$$(X(\pi_1^*), \ldots, X(\pi_n^*)): X(n) \to (X(1))^n$$

is an equivalence for all $n \geq 1$, where $\pi_i^*: n \to 1$ is the i-th projection set operation. Our results extend to this more general notion as follows: The A_∞ analogue of May's rectification process [8] associates with each ringlike A_∞ ring X of the more general kind an equivalent A_∞ ring $\hat{X}$ which is product preserving. Since X and $\hat{X}$ have equivalent K-theories we can take the Volodin construction of $\hat{X}$ as Volodin construction of X. The corresponding rectification exists also for A_∞ rings with involution [5].

1.3 *Remark.* The standard CW-approximation functor preserves products and equivalences. Hence we may assume that our A_∞ monoids and rings (with involution) are structured by CW-theories. Moreover, up to weak equivalence, we can substitute our A_∞ monoids and rings (with involution) by their CW-approximations.

2. THE VOLODIN CONSTRUCTION

Let R be a genuine ring, $n \in \mathbb{N}$ and α a partial ordering of $\underline{n} = \{1, 2, \ldots, n\}$. Then by $T_n^\alpha R$ we will denote the subgroup of $Gl_n R$ consisting of matrices $A = (a_{ij})$ such that $a_{ii} = 1$ and $a_{ij} = 0$ unless $i \leq_\alpha j$. Such a matrix will be called *α-triangular*. If $n \leq m$ and $\Delta(\underline{m})$ denotes the diagonal in $\underline{m} \times \underline{m}$, then $\alpha \cup \Delta(\underline{m})$ is a partial ordering of $\underline{m}$ via the inclusion $\alpha \cup \Delta(\underline{m}) \subset \underline{n} \times \underline{n} \cup \Delta(\underline{m}) \subset \underline{m} \times \underline{m}$. Hence there is a stabilization map

$$T_n^\alpha R \to T_m^{\alpha \cup \Delta(\underline{m})} R, \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & E_{m-n} \end{pmatrix}$$

compatible with the natural embedding $Gl_nR \subset Gl_mR$. (Here E_k is the $k \times k$ unit matrix). The inclusion $T_n^\alpha R \subset Gl_nR$ induces a map of two-sided bar constructions

$$B(Gl_nR, T_n^\alpha R, *) \subset B(Gl_nR, Gl_nR, *)$$

and we define $\mathscr{V}\mathscr{M}_n(R)$ to be the union over all partial orderings α of the images. Stabilization defines a sequence

$$\mathscr{V}\mathscr{M}_1(R) \to \mathscr{V}\mathscr{M}_2(R) \to \cdots$$

whose colimit is the Volodin model $\mathscr{V}\mathscr{M}(R)$.

We extend this construction to A_∞ rings. For this purpose we describe the diagram of all groups $T_n^\alpha R$ and Gl_nR and their natural embeddings in the language of universal algebra.

Let $\mathscr{P}_n$ denote the poset of partial orderings of $\underline{n}$ considered as category and $\mathscr{P}_{n*}$ the category $\mathscr{P}_n$ with an additional terminal object ω_n. The inclusion $\underline{n} \subset \underline{n+1}$ induces a functor $\mathscr{P}_{n*} \to \mathscr{P}_{n+1*}$ mapping ω_n to ω_{n+1}, so that we arrive at a functor $\mathbb{N} \to \mathscr{C}at$, $n \mapsto \mathscr{P}_{n*}$, into the category of small categories ($\mathscr{P}_{0*}$ is the trivial category with one object). A standard categorical construction (e.g. [17, (1.1)]) combines all the $\mathscr{P}_{n*}$ to a single category $\mathscr{P}_*$. Its objects are pairs $(\underline{n}, \alpha)$, $n \in \mathbb{N}$, $\alpha \in ob\mathscr{P}_{n*}$. The set of morphisms from $(\underline{n}, \alpha)$ to $(\underline{m}, \beta)$ is empty if $n > m$ and consists of the morphisms $\alpha \cup \Delta(\underline{m}) \to \beta$ in $\mathscr{P}_{m*}$ otherwise.

If Θ is a theory and $\mathscr{A}$ a small category let $\Theta \diamond \mathscr{A}$ denote the quotient of $\Theta \times \mathscr{A}$ obtained by identifying all objects $(\underline{0}, A)$ to a single terminal object 0. We construct a functor

$$\Phi\colon \Theta_m \diamond \mathscr{P}_* \to \Theta_r.$$

For an object α in $\mathscr{P}_{n*}$, $\alpha \neq \omega_n$, let $\langle\alpha\rangle = \operatorname{card}\alpha - n$, where $\operatorname{card}\alpha$ is the cardinality of $\alpha \subset \underline{n} \times \underline{n}$. The functor Φ sends $(p; \underline{n}, \omega_n)$ to $p \cdot n^2$ and $(p; \underline{n}, \alpha)$ to $p \cdot \langle\alpha\rangle$. Think of $p \cdot n^2$ as the entries a_{rij}, $1 \le r \le p$, $1 \le i, j \le n$ of a p-tuple of $n \times n$-matrices $(A_1, \ldots, A_p)$ and of $p \cdot \langle\alpha\rangle$ as the non-specified entries b_{rij}, $1 \le r \le p$, $(i,j) \in \alpha - \Delta(\underline{n})$ of a p-tuple of α-triangular matrices $(B_1, \ldots, B_p)$. These entries are arranged according to lexicographical ordering of the indexes.

If $\mu_p \in \Theta_m(p, 1)$ denotes the monomial $x_1 \cdot \ldots \cdot x_p$, then every morphism of Θ_m can be uniquely decomposed into the form

$$(\mu_{p_1} \times \ldots \times \mu_{p_r}) \circ \sigma^*$$

with σ^* a set operation. Hence it suffices to specify Φ on morphisms (μ_p, id), (σ^*, id), (id, f) of $\Theta_m \diamond \mathscr{P}_*$. If $\sigma\colon \underline{q} \to \underline{p}$ is a map of sets, then $\Phi(\sigma^*, id)$ is the tuple of monomials specified by the entries of the target matrices

$$(A_1, \ldots, A_p) \to (A_{\sigma_1}, \ldots, A_{\sigma_q})$$

in matrix notation (the left side is the collection of variables arranged as matrices). $\Phi(\mu_p, id)$ is p-fold matrix multiplication

$$(A_1, \ldots, A_p) \to A_1 \cdot A_2 \cdot \ldots \cdot A_p.$$

This is the usual fomula for $\Phi(\mu_p, id_{(\underline{n}, \omega_n)})$, but for $\Phi(\mu_p, id_{(\underline{n}, \alpha)})$ we have to make changes since only entries $(i,j) \in \alpha - \Delta(\underline{n})$ are undetermined. E.g. if $p = 2$, the (i,k)-th entry of the product

$$(B, C) \to B \cdot C$$

is the polynomial $b_{ik} + \Sigma b_{ij} \cdot c_{jk} + c_{ik}$, summed over all j with $(i,j), (j,k) \in \alpha - \Delta(\underline{n})$, in the $2 \cdot \langle\alpha\rangle$ variables $b_{ij} \in B$ and $c_{ij} \in C$, $(i,j) \in \alpha - \Delta(\underline{n})$. Iterated multiplication of α-triangular

matrices is defined correspondingly. Finally, let $f:\alpha \cup \Delta(\underline{m}) \to \beta$ in $\mathscr{P}_{m*}$ represent a morphism $(\underline{n}, \alpha) \to (\underline{m}, \beta)$ in $\mathscr{P}_*$. If $\beta \neq \omega_m$ then f is an inclusion and

$$\Phi(id_1, f): \langle a \rangle \to \langle \beta \rangle$$

maps the (i,j)-th entry, $(i,j) \in \alpha$, identically to the (i,j)-th entry, while the remaining $\langle \beta \rangle - \langle \alpha \rangle$ entries consist of the constant monomial 0, i.e. we fill up an α-triangular $(n \times n)$-matrix with zeros in the additional undetermined entries to obtain a β-triangular $(m \times m)$-matrix. If $\beta = \omega_m$ we must also allow ω_n as source of f. We make the analogous construction by filling up the diagonal with 1's and the other undetemined entries with 0's to end up with an $(m \times m)$-matrix. By componentwise extension we obtain $\Phi(id_p, f)$.

Note that the image of Φ consists of simple morphisms only and that $\Phi(-, id_{(\underline{n},\alpha)})$ is product preserving for all objects α in $\mathscr{P}_{n*}$.

Now given an A_∞ ring $X: \Theta \to \mathscr{T}op$ with A_∞ ring theory $F: \Theta \to \Theta_r$ we consider the pullback

$$\begin{array}{ccccc} \Psi & \xrightarrow{H\Phi} & \Theta & \xrightarrow{X} & \mathscr{T}op \\ \downarrow{\scriptstyle \nu} & \text{pullback} & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \pi_0} \\ \Theta_m \circ \mathscr{P}_* & \xrightarrow{\Phi} & \Theta_r & \xrightarrow{\pi_0 X} & \mathscr{S}ets \end{array}$$

Since ν is an equivalence, Ψ is a $\mathscr{P}_*$-indexed A_∞ monoid theory in the sense of [11], and $X \circ H\Phi$ is a $\mathscr{P}_*$-indexed A_∞ monoid.

Since $\pi_0 X$ is a genuine ring, $\pi_0 X \circ \Phi(-, (\underline{n}, \alpha)): \Theta_m \to \mathscr{S}ets$ is the monoid of α-triangular matrices with entries in $\pi_0 X$ and $\pi_0 X \circ \Phi(-, (\underline{n}, \omega_n))$ is the multiplicative monoid of $n \times n$-matrices with entries in $\pi_0 X$. Hence it makes sense to talk of the subfunctor $G(\pi_0 X \circ \Phi)$ taking only invertible matrices as values. We restrict $X \circ H\Phi$ to these components to obtain a subfunctor $G(X \circ H\Phi)$. We call this process "restriction to invertible components". Note that $G(X \circ H\Phi(-, (\underline{n}, \omega_n))): \Psi \to \mathscr{T}op$ is Waldhausen's space $\widehat{GL}_n(X)$ of homotopy invertible matrices [19], while $G(X \circ H\Phi(-, (\underline{n}, \alpha)): \Psi \to \mathscr{T}op$ structures the α-triangular matrices $T_n^\alpha(X)$ over X as an A_∞ monoid.

(2.1) We now apply the rectification process of [2, (4.49)]. This procedure can be viewed as a diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{Y} & \mathscr{T}op \\ \downarrow{\scriptstyle \varepsilon} & & \\ \Theta_m \circ (\mathscr{P}_* \times \mathscr{L}_1) & & \end{array} \qquad (0)$$

where $\mathscr{L}_1$ is the category $0 \xrightarrow{\iota} 1$, Λ a $(\mathscr{P}_* \times \mathscr{L}_1)$-indexed A_∞ monoid theory and Y a Λ-space such that

(1) $\varepsilon^{-1}(\Theta_m \circ (\mathscr{P}_* \times 1)) \cong \Theta_m \circ \mathscr{P}_*$

(2) there is a map of theories

$$\varepsilon^{-1}(\Theta_m \circ (\mathscr{P}_* \times 0)) \to \Psi$$

which upon composition with $G(X \circ H\Phi)$ yields the restriction of diagram (2.1.0) to $\varepsilon^{-1}(\Theta_m \circ (\mathscr{P}_* \times 0))$.

(3) Y applied to any morphism in Λ lying over a morphism of the form (id, id, ι) is an equivalence.

We thus obtain a functor

$$Y_1 = Y|_{\Theta_m \circ (\mathscr{P}_* \times 1)} : \mathscr{P}_* \to \mathscr{M}onoids$$

where $\mathscr{M}onoids$ is the category of topological monoids.

The monoids $Y_1(\underline{n}, \alpha)$ and $Y_1(\underline{n}, \omega_n)$ will be denoted by $\tilde{T}_n^\alpha(X)$ and $\widetilde{Gl}_n(X)$ respectively (note that $\pi_0 \tilde{T}_n^\alpha(X) = T_n^\alpha(\pi_0 X)$ and $\pi_0 \widetilde{Gl}_n(X) = Gl_n(\pi_0 X)$). By (2.1.3) there are A_∞ monoid equivalences (by homotopy homomorphisms) $T_n^\alpha(X) \to \tilde{T}_n^\alpha(X)$ and $\widehat{Gl}_n(X) \to \widetilde{Gl}_n(X)$ compatible with the various stabilization maps. By restriction to the path components belonging to the elementary matrices $E_n(\pi_0 X)$ the homotopy homomorphism $\widehat{Gl}_n(X) \to \widetilde{Gl}_n(X)$ restricts to a homotopy homomorphism $\hat{E}_n(X) \to \tilde{E}_n(X)$, where $\hat{E}_n(X)$ is the A_∞ monoid of homotopy elementary matrices in Waldhausen's sense.

Remark on notation: Plain symbols Gl_n, E_n, and T_n^α denote the usual linear group, group of elementary matrices, and group resp. A_∞ monoid of upper triangular matrices. Following Waldhausen, the A_∞ monoids $\widehat{Gl}_n(X)$ and $\hat{E}_n(X)$ are the pullbacks

$$\begin{array}{ccc} \widehat{Gl}_n(X) & \longrightarrow & M_n(X) \\ \downarrow & & \downarrow \\ Gl(\pi_0 X) & \longrightarrow & M_n(\pi_0 X) \end{array} \qquad \begin{array}{ccc} \hat{E}_n(X) & \longrightarrow & \widehat{Gl}_n(X) \\ \downarrow & & \downarrow \\ E_n(\pi_0 X) & \longrightarrow & Gl_n(\pi_0 X) \end{array}$$

where $M_n(X) = X^{n^2}$ is the space of all matrices over X. The symbols $\widetilde{Gl}_n(X)$, $\tilde{E}_n(X)$, and $\tilde{T}_n^\alpha(X)$ are reserved for the rectified versions (cf. 2.1). Note that the elements of $\widetilde{Gl}_n(X)$, $\tilde{E}_n(X)$, and $\tilde{T}_n^\alpha(X)$ have lost any resemblance to matrices, e.g. one cannot speak of matrix entries any more. $\widetilde{Gl}_n(X)$, $\tilde{E}_n(X)$, and $\tilde{T}_n^\alpha(X)$ are just grouplike topological monoids.

Since all monoids under consideration are grouplike, we have fibration sequences

$$\begin{aligned} B(\tilde{Gl}_n(X), \tilde{T}_n^\alpha(X), *) &\to B\tilde{T}_n^\alpha(X) \to B\tilde{Gl}_n(X) \\ B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *) &\to B\tilde{T}_n^\alpha(X) \to B\tilde{E}_n(X) \end{aligned} \tag{2.2}$$

where B is the classifying space functor, respectively the two-sided bar construction. These fibration sequences glue together to form fibration sequences

$$\begin{aligned} \operatorname{hocolim} B(\tilde{Gl}_n(X), \tilde{T}_n^\alpha(X), *) &\to \operatorname{hocolim} B\tilde{T}_n^\alpha(X) \to B\tilde{Gl}(X), \\ \operatorname{hocolim} B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *) &\to \operatorname{hocolim} B\tilde{T}_n^\alpha(X) \to B\tilde{E}(X), \end{aligned} \tag{2.3}$$

the homotopy colimit taken over the subcategory $\mathscr{P} \subset \mathscr{P}_*$ obtained by deleting the objects $(\underline{n}, \omega_n)$.

2.4. *Definition.* We define $\mathscr{V}\mathscr{M}^{Gl}(X) = \operatorname{hocolim} B(\tilde{Gl}_n(X), \tilde{T}_n^\alpha(X), *)$, $\mathscr{V}\mathscr{M}^{E}(X) = \operatorname{hocolim} B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *)$, and call both the *Volodin models*. We define

$$\mathscr{V}\mathscr{S}(X) = \operatorname{hocolim} B\tilde{T}_n^\alpha(X)$$

and call it the *Volodin space* of X. For technical reasons (see 2.5 below) it is easier to work with $\mathscr{V}\mathscr{M}^{E}(X)$ rather than $\mathscr{V}\mathscr{M}^{Gl}(X)$, and we will denote $\mathscr{V}\mathscr{M}^{E}(X)$ simply as $\mathscr{V}\mathscr{M}(X)$. Since the rectification process (2.1) produces "cofibrant" diagrams $\operatorname{hocolim} B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *)$ is equivalent to $\operatorname{colim} B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *)$. So we can work with hocolim or with colim whichever is more suitable.

2.5 LEMMA: *$\mathscr{V}\mathscr{M}(X)$ is connected and there is an equivalence*

$$\mathscr{V}\mathscr{M}^{Gl}(X) \simeq K_1(\pi_0 X) \times \mathscr{V}\mathscr{M}(X)$$

Proof. It is readily seen that the natural maps $\mathscr{V}\mathscr{M}^{Gl}(X) \to \mathscr{V}\mathscr{M}^{Gl}(\pi_0 X)$ and $\mathscr{V}\mathscr{M}(X) \to \mathscr{V}\mathscr{M}(\pi_0 X)$ induce isomorphisms on π_0, so the result follows from [15, p. 1561].

2.6 PROPOSITION. *$\mathscr{V}\mathscr{S}(X)$ is the fiber of the plus construction $B\tilde{E}(X) \to (B\tilde{E}(X))^+$.*

2.7 THEOREM. *Let X be a ringlike A_∞ ring and $K(X)$ its K-theory. Then*

$$\Omega K(X) \simeq K_1(\pi_0 X) \times \mathscr{V}\mathscr{M}(X) \simeq \mathscr{V}\mathscr{M}^{Gl}(X).$$

The proof of the theorem is based on three lemmas the proofs of which are postponed to the following sections.

2.8 LEMMA A. *$\pi_1(B\tilde{E}(X)) \cong E(\pi_0 X)$ operates trivially on $H_*(\mathscr{V}\mathscr{M}(X))$.*

2.9 LEMMA B. *$\mathscr{V}\mathscr{M}(X)$ is nilpotent.*

2.10 LEMMA C. *$\mathscr{V}\mathscr{S}(X)$ is acyclic and $\pi_1(\mathscr{V}\mathscr{S}(X))$ is the Steinberg group $St(\pi_0 X)$.*

Proof of Theorem 2.7. Consider the fibration sequence (2.3)

$$\mathscr{V}\mathscr{M}(X) \to \mathscr{V}\mathscr{S}(X) \to B\tilde{E}(X).$$

Since $\mathscr{V}\mathscr{M}(X)$ is nilpotent, $\mathscr{V}\mathscr{M}(X)^+ \simeq \mathscr{V}\mathscr{M}(X)$ [1, (4.11)]. Hence, by Lemma A and [1, (6.4)]

$$\mathscr{V}\mathscr{M}(X)^+ \to \mathscr{V}\mathscr{S}(X)^+ \to (B\tilde{E}(X))^+$$

is a fibration sequence. Lemma C implies that $\mathscr{V}\mathscr{S}(X)^+$ is contractible. Hence

$$\mathscr{V}\mathscr{M}(X) \simeq \mathscr{V}\mathscr{M}(X)^+ \simeq \Omega(B\tilde{E}(X)^+).$$

Next apply the plus construction to the fibration sequence

$$B\tilde{E}(X) \to B\widetilde{Gl}(X) \to K(K_1(\pi_0 X), 1).$$

Since the Eilenberg–MacLane space $K(K_1(\pi_0 X), 1)$ has an abelian fundamental group, we obtain a fibration sequence

$$B\tilde{E}(X)^+ \to B\widetilde{Gl}(X)^+ \to K(K_1(\pi_0 X), 1)$$

[1, (6.4)], and upon looping, a fibration sequence

$$\Omega(B\tilde{E}(X)^+) \to \Omega(B\widetilde{Gl}(X)^+) \to K_1(\pi_0 X).$$

Since $\Omega(B\tilde{E}(X)^+)$ is connected, $K_1(\pi_0 X)$ discrete, $B\widetilde{Gl}(X) \simeq B\widehat{Gl}(X)$, and by definition $KX = K_0(\pi_0 X) \times B\widehat{Gl}(X)^+$, the result follows. □

Proof of (2.6). From the proof of the theorem we obtain the following diagram of fibration sequences

$$\begin{array}{ccccc} * & \to & \mathscr{V}\mathscr{S}(X) & \to & Fiber \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{V}\mathscr{M}(X) & \to & \mathscr{V}\mathscr{S}(X) & \to & B\tilde{E}(X) \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow \\ \mathscr{V}\mathscr{M}(X) & \to & * & \to & (B\tilde{E}(X))^+ \end{array}$$

which implies the proposition. □

We close this section with a remark on products. Let $X, Y: \Theta \to \mathscr{T}op$ be A_∞ rings. Their product

$$X \times Y: \Theta \xrightarrow{(X,Y)} \mathscr{T}op \times \mathscr{T}op \xrightarrow{\times} \mathscr{T}op$$

is again an A_∞ ring. The rectification (2.1) preserves products up to homotopy. Restriction to elementary components is product preserving and so is the two-sided bar construction. Hence, the projections induce a homotopy equivalence

$$q: B(\tilde{E}_n(X \times Y), \tilde{T}_n^\alpha(X \times Y), *) \to B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *) \times B(\tilde{E}_n(Y), \tilde{T}_n^\alpha(Y), *).$$

Unfortunately, $\operatorname{hocolim}_{\mathscr{P}}$ is not product preserving, so that we cannot conclude that $\mathscr{V}\mathscr{M}(X \times Y) \simeq \mathscr{V}\mathscr{M}(X) \times \mathscr{V}\mathscr{M}(Y)$ (although this is true by Theorem 2.7), but we note for later use

2.11 LEMMA. *The projection $\mathscr{V}\mathscr{M}(X \times Y) \to \mathscr{V}\mathscr{M}(X)$ is a homotopy retraction.*

Proof. Let $e \in B(\tilde{E}_n(Y), \tilde{T}_n^\alpha(Y), *)$ be the unit of the monoid $\tilde{E}_n(Y)$ in the 0-skeleton. Then $x \mapsto (x, e)$ defines a section of the projection

$$B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *) \times B(\tilde{E}_n(Y), \tilde{T}_n^\alpha(Y), *) \to B(\tilde{E}_n(X), \tilde{T}_n^\alpha(X), *)$$

compatible with the diagram structure. □

3. ALTERNATIVE DESCRIPTIONS OF THE VOLODIN MODEL

We describe a different model of $\mathscr{V}\mathscr{M}(X)$ which puts us into the context of the machinery of [12]. We use to the terminology of that paper. In particular $\mathscr{C}at(\Sigma \int \Theta_r)$ denotes the category of category objects in $\Sigma \int \Theta_r$, where $\Sigma \int \Theta_r$ is obtained from Θ_r by adjoining arbitrary sums. We construct a functor

$$\lambda: \mathscr{P} \to \mathscr{C}at(\Sigma \int \Theta_r)$$

as follows. $(\underline{n}, \alpha)$ is mapped to the category object $\mathscr{C}_n^\alpha$ with $ob(\mathscr{C}_n^\alpha) = n^2 \in \Theta_r \subset \Sigma \int \Theta_r$, to be considered as the set of $(n \times n)$-matrices and $mor(\mathscr{C}_n^\alpha) = n^2 + \langle \alpha \rangle$, considered as the set of all pairs (A, B) with A an $(n \times n)$-matrix and B an α-triangular matrix. The source map is $(A, B) \mapsto A$, the target map $(A, B) \mapsto A \cdot B$ with the necessary adaptations of the matrix multiplication described in §2. The inclusion of the units is the obvious one. A morphism $f: \alpha \cup \Delta(\underline{m}) \subset \beta$ representing a morphism $(\underline{n}, \alpha) \to (\underline{m}, \beta)$ in $\mathscr{P}$ induces an obvious functor $\mathscr{C}_n^\alpha \to \mathscr{C}_m^\beta$. Applying the nerve functor and taking pullbacks we obtain a diagram

$$\begin{array}{ccccc} \mathscr{Q} & \xrightarrow{H\lambda} & \Sigma \int \Theta & \xrightarrow{\amalg X} & \mathscr{T}op \\ \downarrow \nu & \text{pullback} & \downarrow \Sigma \int F & & \\ \mathscr{P} \times \Delta^{op} & \xrightarrow{\Delta\lambda} & \Sigma \int \Theta_r & & \end{array}$$

with v an equivalence (compare [12, §5]). Segal's homotopy pushdown construction [13, App. B] produces a $\mathscr{P}$-diagram of simplicial spaces

$$S(\lambda) = S(\lambda, X): \mathscr{P} \to \mathscr{T}op^{\Delta^{op}}$$

whose value at $(\underline{n}, \alpha)$ is the simplicial space whose k-simplices have the homotopy type of $M_n(X) \times T_n^{\alpha}(X)^k$. By restricting to elementary components we obtain a subfunctor

$$E(\lambda): \mathscr{P} \to \mathscr{T}op^{\Delta^{op}}$$

whose value at $(\underline{n}, \alpha)$ is the simplicial space whose k-simplices have the homotopy type of $\hat{E}_n(X) \times T_n^{\alpha}(X)^k$. Let $\mathscr{E}(\lambda): \mathscr{P} \to \mathscr{T}op$ denotes its realization. We define $\Gamma(\lambda) = \text{hocolim } \mathscr{E}(\lambda)$.

3.1 Lemma. $\mathscr{V}\mathscr{M}(X) \simeq \Gamma(\lambda)$

The proof is a consequence of the naturality of Segal's pushdown and the fact that it is equivalent to the rectification process (2.1). (For additional information see [12, §8]).

We shall need variations of this construction: Let $\mathscr{P}_n \subset \mathscr{P}$ be the full subcategory of all objects $(\underline{n}, \sigma)$ and let

$$u: \mathscr{P}_n \to \mathscr{P}_{n+1}$$

be a functor of the form $(\underline{n}, \alpha) \mapsto (\underline{n+1}, \alpha')$ where $\alpha \cup \Delta(\underline{n+1}) \subset \alpha'$. The inclusions $\alpha \cup \Delta(\underline{n+1}) \subset \alpha'$ induce an obvious natural transformation of functors

$$\lambda | \mathscr{P}_n \to (\lambda | \mathscr{P}_{n+1}) \circ u: \mathscr{P}_n \to \mathscr{C}at(\Sigma \int \Theta_r)$$

or equivalently a functor

$$\rho = \rho(u): \mathscr{P}_n \times \mathscr{L}_1 \to \mathscr{C}at(\Sigma \int \Theta_r).$$

Let $j_0, j_1: \mathscr{P}_n \to \mathscr{P}_n \times \mathscr{L}_1$ be the inclusions to 0 and 1. By naturality of Segal's pushdown we obtain a diagram

$$S(\lambda | \mathscr{P}_n) = S(\rho \circ j_0) \xrightarrow{\alpha} S(\rho) \circ j_0 \longrightarrow S(\rho) \circ j_1 \xleftarrow{\beta} S(\rho \circ j_1) \xrightarrow{\gamma} S(\lambda | \mathscr{P}_{n+1} \circ u)$$

where α, β, γ are weak equivalences, i.e. homomorphisms of simplicial $\mathscr{P}_n$ diagrams which are at each object of $\mathscr{P}_n$ degreewise homotopy equivalences. Define

$$\mathscr{V}\mathscr{M}_n(X) = \Gamma(\lambda | \mathscr{P}_n) = \text{hocolim } \mathscr{E}(\lambda | \mathscr{P}_n).$$

By choosing homotopy inverses we obtain a map

$$\tilde{u}: \mathscr{V}\mathscr{M}_n(X) \to \mathscr{V}\mathscr{M}_{n+1}(X)$$

uniquely defined up to homotopy. If $v: \mathscr{P}_{n+1} \to \mathscr{P}_{n+2}$ is another such functor, then for any choice of $\tilde{v}$ and $\tilde{u}$

$$\tilde{v} \circ \tilde{u} \simeq \widetilde{(v \circ u)}$$

by an argument similar to [12, (5.5)]. Now let $i_n: \mathscr{P}_n \to \mathscr{P}_{n+1}$ be the inclusion of §2 and $\rho = \rho(i_n)$. Then ρ factors through λ_n and Segal's pushdown yields a commutative diagram

$$\begin{array}{ccc} S(\lambda) | \mathscr{P}_n & \to & S(\lambda) | \mathscr{P}_{n+1} \\ \uparrow & & \uparrow \\ S(\rho) \circ j_0 & \to & S(\rho) \circ j_1 \end{array} \tag{3.2}$$

By [17, Thm 1.2] $\mathscr{V}\mathscr{M}(X) = \text{hocolim}_{\mathbb{N}}\ \text{hocolim}(\mathscr{E}(\lambda)|\mathscr{P}_n)$. Hence (3.2) implies

$$\mathscr{V}\mathscr{M}(X) \simeq \text{hocolim}(\cdots \to \mathscr{V}\mathscr{M}_n(X) \xrightarrow{\tilde{in}} \mathscr{V}\mathscr{M}_{n+1}(X) \to \cdots) \tag{3.3}$$

4. PROOF OF LEMMA *A*

For $u \in \pi_0 X$ let $e_{n,i}(u)$ denote the elementary $(n \times n)$-matrix which differs from the identity matrix by having u in the (n, i)-entry. If R is a genuine ring, the action of $e_{n,i}(r)$, $r \in R$, on $H_*(\mathscr{V}\mathscr{M}(R))$ is induced by the action of $(R, +)$ on $\mathscr{V}\mathscr{M}(R)$ given by left multiplication with $e_{n,i}(r)$ on the left slot of $B(E_n(R), T_n^\alpha(R), *)$. Using the machinery of §3 we describe the corresponding action in the A_∞-case.

Let $\mathscr{R}$ be the category object in $\Sigma \int \Theta_r$ with $ob(\mathscr{R}) = \text{mor}(\mathscr{R}) = 1$ and structure maps the identities. Define a functor

$$\rho_n : \mathscr{P}_n \times \mathscr{L}_1 \to \mathscr{C}at(\Sigma \int \Theta_r)$$

by $((\underline{n}, \alpha), 0) \mapsto \mathscr{R} \times \mathscr{C}_n^\alpha$, $\rho_n|\mathscr{P}_n \times 1 = \lambda|\mathscr{P}_n$, and $\rho_n(id_{(\underline{n},\alpha)}, \iota)$ is the functor $\mathscr{R} \times \mathscr{C}_n^\alpha \to \mathscr{C}_n^\alpha$, $(r, A) \mapsto e_{n,i}(r)\cdot A$ in matrix notation. As in §3 we obtain an induced map

$$\tilde{\rho}_n : \Gamma(\rho_n \circ j_0) \to \Gamma(\rho_0 \circ j_1) = \mathscr{V}\mathscr{M}_n(X).$$

The projections $\mathscr{R} \leftarrow \mathscr{R} \times \mathscr{C}_n^\alpha \to \mathscr{C}_n^\alpha$ induce an equivalence

$$\Gamma(\rho_n \circ j_0) \to X \times \mathscr{V}\mathscr{M}_n(X)$$

[12, (5.7)] so that we have a map

$$\tilde{e}_{n,i} : X \times \mathscr{V}\mathscr{M}_n(X) \to \mathscr{V}\mathscr{M}_n(X)$$

uniquely defined up to homotopy. This map induces the action. The proof of Lemma A is based on the matrix identity over rings

$$e_{n+1,i}(r)\cdot\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ rA_i & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} E_n & 0 \\ rA_i & 1 \end{pmatrix}$$

where A is an $(n \times n)$-matrix and A_i its i-th row, $i \le n$. Hence the correspondence

$$\tau_n^\alpha : ob(\mathscr{R} \times \mathscr{C}_n^\alpha) \to mor(\mathscr{C}_{n+1}^{\alpha'}), \quad (r, A) \mapsto \left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} E_n & 0 \\ rA_i & 1 \end{pmatrix}\right)$$

defines a natural transformation in $\mathscr{C}at(\Sigma \int \Theta_r)$ from

$$F_n^\alpha : \mathscr{R} \times \mathscr{C}_n^\alpha \to \mathscr{C}_{n+1}^{\alpha'}, \qquad (r, A) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

to

$$G_n^\alpha : \mathscr{R} \times \mathscr{C}_n^\alpha \to \mathscr{C}_{n+1}^{\alpha'}, \qquad (r, A) \mapsto e_{n+1,i}(r)\cdot\begin{pmatrix} A & 0 \\ 1 & 1 \end{pmatrix}$$

where α' is the partial ordering of $\underline{n+1}$ defined by α and the condition that $n+1$ is minimal. The functors F_n^α and G_n^α combine to give diagrams

$$F, G : \mathscr{P}_n \times \mathscr{L}_1 \to \mathscr{C}at(\Sigma \int \Theta_r).$$

By §3, F induces the map

$$X \times \mathscr{V}\mathscr{M}_n(X) \xrightarrow{\text{proj}} \mathscr{V}\mathscr{M}_n(X) \xrightarrow{\tilde{u}_n} \mathscr{V}\mathscr{M}_{n+1}(X)$$

where $u_n: \mathscr{P}_n \to \mathscr{P}_{n+1}$ sends $(\underline{n}, \alpha)$ to $(\underline{n}, \alpha')$, and G induces the map

$$X \times \mathscr{V}\mathscr{M}_n(X) \xrightarrow{id \times \tilde{u}_n} X \times \mathscr{V}\mathscr{M}_{n+1}(X) \xrightarrow{\tilde{e}_{n+1,i}} \mathscr{V}\mathscr{M}_{n+1}(X).$$

By (4.2) below we therefore have

$$\begin{array}{ccc} \mathscr{V}\mathscr{M}_n(X) & \xrightarrow{\tilde{u}_n} & \mathscr{V}\mathscr{M}_{n+1}(X) \\ & \searrow^{\tilde{u}_n} & \downarrow{\scriptstyle \tilde{e}_{n+1,i}(u)} \\ & & \mathscr{V}\mathscr{M}_{n+1}(X) \end{array} \tag{4.1}$$

commutes up to homotopy for $i \leq n$.

4.2 LEMMA. *The maps induced by F and G are homotopic.*

Proof. The natural transformation τ_n^α induces a functor

$$\tau_\alpha^n: \mathscr{R} \times \mathscr{C}_n^\alpha \times \mathscr{L}_1 \to \mathscr{C}_{n+1}^{\alpha'}$$

extending the functors F_α^n, G_α^n. The τ_n^α combine to a diagram

$$\mathscr{P}_n \times \mathscr{L}_1 \to \mathscr{C}at(\Sigma \int \Theta_r)$$

and the argument of [12, (5.8)] applies. □

4.3 LEMMA. $\tilde{i}_n \simeq u_n$. *Hence $\tilde{e}_{n+1,i}(u)$ acts trivially on the image of $\mathscr{V}\mathscr{M}_n(X)$.*

Proof. The inclusions $(\underline{n+1}, \alpha) \to (\underline{n+1}, \alpha')$ define a natural transformation of functors $i_n \to u_n: \mathscr{P}_n \to \mathscr{P}_{n+1}$. By general properties of homotopy colimits (e.g. see [7, (3.17)] the result holds. □

We now show that $E(\pi_0 X)$ acts trivially on $H_*\mathscr{V}\mathscr{M}(X)$ under the additional hypothesis that X is an A_∞ ring with involution. Our starting point is the following.

4.4 LEMMA. *Let X be an A_∞ ring with involution. Then there is a $\mathbb{Z}/2$ action on the fibration sequence*

$$\mathscr{V}\mathscr{M}(X) \to \mathscr{V}\mathscr{S}(X) \to B\tilde{E}(X)$$

which induces the involution $A \mapsto (A^*)^{-1}$ on $\pi_1 B\tilde{E}(X) = \tilde{E}(\pi_0 X)$ and which restricts to a $\mathbb{Z}/2$ action on the fibration sequence

$$\mathscr{V}\mathscr{M}_n(X) \to \mathscr{V}\mathscr{S}_n(X) \to B\tilde{E}_n(X)$$

for each n.

Proof Sketch. We first enlarge the category $\mathscr{P}_*$ to a category $\mathscr{Q}_*$ having the same objects but with a new class of morphisms

$$\begin{aligned} &(\underline{n}, \alpha) \to (\underline{m}, \beta) && \text{if } n \leq m \text{ and } \alpha^* \cup \Delta(\underline{m}) \subseteq \beta \\ &(\underline{n}, \alpha) \to (\underline{m}, \omega_m) && \text{if } n \leq m \\ &(\underline{n}, \omega_n) \to (\underline{m}, \omega_m) && \text{if } n \leq m \end{aligned}$$

which we will refer to as *antimorphisms* where α^* denotes the order opposite to α. Composition is specified by the rule that the composite of a morphism in $\mathscr{P}_*$ and an antimorphism is again an antimorphism, and that of two antimorphisms is a morphism in $\mathscr{P}_*$.

We denote by $\Theta_m \bar{\times} \mathscr{Q}_*$ the twisted product of Θ_m and $\mathscr{Q}_*$: the objects and morphisms are ordered pairs, but the usual composition rule is modified to

$$(\kappa_1, \rho_1) \circ (\kappa_2, \rho_2) = (\kappa_1 \kappa_2', \rho_1 \rho_2)$$

if ρ_1 is an antimorphism in $\mathscr{Q}_*$. Here κ_2' denotes the morphism in Θ_m obtained by reversing the order of multiplication in all component monomials κ_2. Again we denote by $\Theta_m \tilde{\diamond} \mathscr{Q}_*$ the quotient category obtained by identifying all objects $(\underline{0}, Q)$ to a single terminal object 0. The functor $\Phi\colon \Theta_m \diamond \mathscr{P}_* \to \Theta_r$ can be enlarged to a functor

$$\hat{\Phi}\colon \Theta_m \tilde{\diamond} \mathscr{Q}_* \to \Theta_{r,i}$$

with the antimorphisms in $\mathscr{Q}_*$ specifying formally the conjungate transpose operation $A \mapsto A^*$ on a matrix A, possibly followed by inclusion into a larger matrix group. We note that the image of $\hat{\Phi}$ consists of simple morphisms. For an A_∞ ring with involution $X\colon \Theta \to \mathscr{T}op$, we form the pullback diagram

$$\begin{array}{ccccc} \Psi & \xrightarrow{H\hat{\Phi}} & \Theta & \xrightarrow{X} & \mathscr{T}op \\ \downarrow v & \text{pullback} & \downarrow F & & \\ \Theta_m \tilde{\diamond} \mathscr{Q}_* & \xrightarrow{\hat{\Phi}} & \Theta_{r,i} & & \end{array}$$

with v an equivalence. Construction 2.1 can be promoted to replace the appropriate subfunctor of $X \circ H\hat{\Phi}\colon \Psi \to \mathscr{T}op$ by an equivalent functor

$$\mathscr{Q}_* \to \mathscr{A}ntimonoids$$

where $\mathscr{A}ntimonoids$ denotes the category of topological monoids with both homomorphisms and antihomomorphisms allowed as morphisms. We then apply a functor which converts monoids to groups and which is an equivalence on group-like monoids (e.g. [3, (1.8)]) to obtain a functor

$$Z\colon \mathscr{Q}_* \to \mathscr{A}ntigroups$$

Thus

$$Z(\underline{n}, \alpha) \simeq T_n^a(X),\ Z(\underline{n}, \omega_n) \simeq \tilde{E}_n(X)$$

and we will denote the antihomomorphisms $Z(\underline{n}, \alpha) \to Z(\underline{n}, \alpha^*)$, $Z(\underline{n}, \omega_n) \to Z(\underline{n}, \omega_n)$ coming from the antimorphisms in $\mathscr{Q}_*$ by $g \mapsto g^*$. Then it is easily seen that we have an involution of diagrams

$$B(Z(\underline{n}, \omega_n), Z(\underline{n}, \omega_n), *) \to B(Z(\underline{n}, \omega_n), Z(\underline{n}, \alpha^*), *)$$

$$BZ(\underline{n}, \alpha) \to BZ(\underline{n}, \alpha^*)$$

$$BZ(\underline{m}, \omega_n) \to BZ(\underline{n}, \omega_n)$$

specified by the homorphisms $g \mapsto (g^*)^{-1}$. This induces the desired $\mathbb{Z}/2$ action on the fibration sequence. □

Denote the involution of Lemma 4.4 by η. Then we have homotopy commutative diagrams

$$\begin{array}{ccc} \mathscr{V}\mathscr{M}_{n+1}(X) & \xrightarrow{\eta} & \mathscr{V}\mathscr{M}_{n+1}(X) \\ \downarrow \tilde{e}_{n+1,i}(-u^*) & & \downarrow \tilde{e}_{i',n+1}(u) \\ \mathscr{V}\mathscr{M}_{n+1}(X) & \xrightarrow{\eta} & \mathscr{V}\mathscr{M}_{n+1}(X) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{V}\mathscr{M}_n(X) & \xrightarrow{\eta} & \mathscr{V}\mathscr{M}_n(X) \\ \downarrow{\scriptstyle \tilde{i}_n} & & \downarrow{\scriptstyle \tilde{i}_n} \\ \mathscr{V}\mathscr{M}_{n+1}(X) & \xrightarrow{\eta} & \mathscr{V}\mathscr{M}_{n+1}(X) \end{array}$$

which together with (4.3) imply

$$\tilde{e}_{i,n+1}(u)\circ\tilde{i}_n\circ\eta \simeq \tilde{e}_{i,n+1}(u)\circ\eta\circ\tilde{i}_n \simeq \eta\circ\tilde{e}_{n+1,i}(-u^*)\circ\tilde{i}_n \simeq \eta\circ\tilde{i}_n \simeq \tilde{i}_n\circ\eta.$$

Hence the action by elements of the form $e_{i,n+1}(u)$ is also trivial. Since $E(\pi_0 X)$ is generated by elements of the form $e_{i,n+1}(u)$ and $e_{n+1,i}(u)$, this proves that $E(\pi_0 X)$ acts trivially on $\mathscr{V}\mathscr{M}(X)$ in the case when X has an A_∞ involution.

The general case follows by passage from an A_∞ ring X to the A_∞ ring with involution $X \times X^{op}$, which is described by the obvious functor

$$\Theta_{r,i} \to \Theta_r, \qquad n \mapsto 2n.$$

For clearly, if $E(\pi_0(X \times X^{op})) \cong E(\pi_0 X) \times E(\pi_0 X^{op})$ acts trivially on $\mathscr{V}\mathscr{M}(X \times X^{op})$ then $E(\pi_0 X)$ acts trivially on $\mathscr{V}\mathscr{M}(X)$, which is a homotopy retract of $\mathscr{V}\mathscr{M}(X \times X^{op})$ by (2.11).

4.5 *Remark.* We note here that the proof of Lemma A given here actually proves a stronger statement. Given any element $g \in \pi_1 BE(X) = E(\pi_0 X)$, we have shown that the action of g on $\mathscr{V}\mathscr{M}(X)$ is homotopic to the identity map via an unbased homotopy, when restricted to any finite subcomplex of $\mathscr{V}\mathscr{M}(X)$. (The last restriction is due to the fact that the trivialization homotopy of Lemma 4.2 is defined on $\mathscr{V}\mathscr{M}_n(X)$ individually for each n.) We shall use this stronger formulation in §6 to show that $\mathscr{V}\mathscr{M}(X)$ is nilpotent.

5. PROOF OF LEMMA *C*

To prove that $\mathscr{V}\mathscr{S}(X)$ is acyclic, we follow closely Suslin's argument for discrete rings, rewording it in the language of homotopical universal algebra.

We first expand our indexing category $\mathscr{P}$ to a bigger category $\tilde{\mathscr{P}}$. The objects of $\tilde{\mathscr{P}}$ are the same as those of $\mathscr{P}$, pairs $(\underline{n}, \alpha)$ with α a partial order of $\underline{n}$. However, we enlarge the set of morphisms: the morphisms $(\underline{n}, \alpha) \to (\underline{m}, \beta)$ in $\tilde{\mathscr{P}}$ are injections $f: \underline{n} \to \underline{m}$ such that $i <_\alpha j$ implies $f(i) <_\beta f(j)$. Now for each $(\underline{n}, \alpha)$ construct a category object $\mathscr{T}_n^\alpha$ in $\mathscr{C}at(\Sigma \int \Theta_r)$ with $ob(\mathscr{T}_n^\alpha) = 0 \in \Theta_r$ and $\mathrm{mor}(\mathscr{T}_n^\alpha) = \langle\alpha\rangle$. The source and target maps are trivially defined, the unit map is given by inclusion of the identity matrix and composition is multiplication of triangular matrices phrased in the language of §2. A morphism $f: (\underline{n}, \alpha) \to (\underline{m}, \beta)$ in $\tilde{\mathscr{P}}$ induces an obvious functor $\mathscr{T}_n^\alpha \to \mathscr{T}_n^\beta$. We thus obtain a functor

$$\rho: \tilde{\mathscr{P}} \to \mathscr{C}at(\Sigma \int \Theta_r)$$

with $\rho(\underline{n}, \alpha) = \mathscr{T}_n^\alpha$. For any A_∞ ring X, we obtain, by taking nerves and pullbacks, a diagram

$$\begin{array}{ccccc} \mathscr{Q} & \xrightarrow{H\rho} & \Sigma \int \Theta_r & \xrightarrow{\amalg X} & \mathscr{T}op \\ \downarrow{\scriptstyle \tilde{\varepsilon}} & & \downarrow{\scriptstyle \Sigma \int \varepsilon} & & \\ \tilde{\mathscr{P}} \times \Delta^{op} & \longrightarrow & \Sigma \int \Theta_r & & \end{array} \tag{5.1}$$

By applying the Segal pushdown to the composite functor $(\amalg X) \circ H\rho$ we obtain a functor

$$S(\rho): \tilde{\mathscr{P}} \times \Delta^{op} \to \mathscr{T}op$$

whose value at $(\underline{n}, \alpha)$ is a simplicial space whose k-simplices have the homotopy type of $T_n^\alpha(X)^k$. Let $B\mathscr{T}_n^\alpha(X)$ denote its geometric realization. This produces another functor

$$\mathscr{E}(\rho): \tilde{\mathscr{P}} \to \mathscr{T}op$$

whose value at $(\underline{n}, \alpha)$ is $B\mathscr{T}_n^\alpha(X)$. By standard arguments it can be shown that

$$\mathscr{V}\mathscr{S}(X) \simeq \operatorname{hocolim}_{\mathscr{P}} B\mathscr{T}_n^\alpha(X).$$

5.2 *Note.* We do **not** claim $\mathscr{V}\mathscr{S}(X) \simeq \operatorname{hocolim}_{\mathscr{P}} B\mathscr{T}_n^\alpha(X)$. Indeed we will show in §7 that $\operatorname{hocolim}_{\mathscr{P}} B\mathscr{T}_n^\alpha(X)$ is contractible.

5.3. *Remark.* As previously noted in 2.4, it follows from general properties of the Segal pushdown that

$$\mathscr{V}\mathscr{S}(X) \simeq \operatorname{colim}_{\mathscr{P}} B\mathscr{T}_n^\alpha(X) = \bigcup_{n,\alpha} B\mathscr{T}_n^\alpha(X).$$

We denote by $\tilde{B}\mathscr{T}_n^\alpha(X)$ the simplicial space obtained by restricting diagram 5.1 to the object $(\underline{n}, \alpha) \in \tilde{\mathscr{P}}$. As in §3, there is an equivalence $\tilde{B}\mathscr{T}_n^\alpha(X) \simeq B\mathscr{T}_n^\alpha(X)$ well-defined and natural up to homotopy.

5.4 Lemma. *Let $f: (\underline{n}, \alpha) \to (\underline{m}, \beta)$ be a morphism in $\tilde{\mathscr{P}}$ having the following properties:*

(i) *For each $i \in \underline{n}$ which is α-nonminimal, there exist elements $f_1(i)$, $f_2(i), \ldots, f_k(i) \in \underline{m} - f(\underline{n})$ with*

$$f(i) >_\beta f_1(i) >_\beta f_2(i) >_\beta \cdots >_\beta f_k(i)$$

(ii) *If $i >_\alpha j$ then $f_k(i) >_\beta f(j)$*

(iii) *The sets $S_i = \{f_l(i)\}_{l=1}^k$ are pairwise disjoint.*

Then the induced map

$$H_l(B\mathscr{T}_n^\alpha(X); F) \to H_l(B\mathscr{T}_n^\beta(X); F)$$

is zero for $1 \leq l \leq k$, for any field F and any A_∞ ring X.

Proof. We proceed by induction on k and $\langle\alpha\rangle$. For $k = 0$ the statement is vacuous, while for $\langle\alpha\rangle = 0$ it is obvious.

Clearly we may suppose that $\underline{m} = f(\underline{n}) \cup (\bigcup_{i=1}^k f_i(\underline{n} - \min(\underline{n}, \alpha)))$. In particular $\min(\underline{m}, \beta) = f(\min(\underline{n}, \alpha))$. Let $D_1 = \underline{m} - f_1(\underline{n} - \min(\underline{n}, \alpha))$, $D_2 = f(\min(\underline{n}, \alpha) \cup f_1(\underline{n} - \min(\underline{n}, \alpha))$. Then $D_1 \cap D_2 = \min(\underline{m}, \beta)$. Let $\beta_1 = \beta|_{D_1}$ and $\beta_2 = \beta|_{D_2}$. Then for any discrete semiring R, the groups $T_m^{\beta_1}(R)$ and $T_m^{\beta_2}(R)$ commute with each other in $T_m^\beta(R)$. Denote by $g: \underline{n} \to \underline{m}$ the embedding defined by

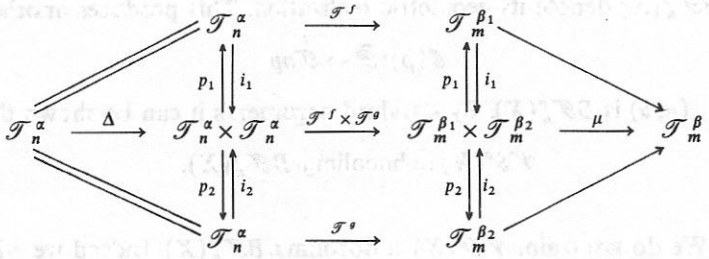
$$g(i) = \begin{cases} f(i) & \text{if } i \in \min(\underline{n}, \alpha) \\ f_1(i) & \text{otherwise} \end{cases}$$

Then $g(\underline{n}) \subseteq D_2, f(\underline{n}) \subseteq D_1$, so the induced homomorphisms $T^f, T^g: T_n^\alpha(R) \to T_m^\alpha(R)$ have commuting images and thus define a product homomorphism.

$$T_n^\alpha(R) \xrightarrow{\Delta} T_n^\alpha(R) \times T_n^\alpha(R) \xrightarrow{T^f \times T^g} T_m^{\beta_1}(R) \times T_m^{\beta_2}(R) \xrightarrow{\mu} T_m^\beta(R).$$

Reinterpreting this in the language of universal algebra we obtain the following diagram in

$\mathscr{C}at(\Sigma\int\Theta_r)$



(5.5)

As in the previous sections this diagram induces a map well defined and natural up to homotopy

$$\tilde{B}\mathcal{T}_n^{\alpha}(X)\xrightarrow{\Delta}\tilde{B}\mathcal{T}_n^{\alpha}(X)\times\tilde{B}\mathcal{T}_n^{\alpha}(X)\xrightarrow{\tilde{B}\mathcal{T}^f\times\tilde{B}\mathcal{T}^g}\tilde{B}\mathcal{T}_m^{\beta_1}X\times\tilde{B}\mathcal{T}_m^{\beta_2}X\xrightarrow{\tilde{B}\mu}\tilde{B}\mathcal{T}_m^{\beta}X \quad (5.6)$$

Using the equivalences $\tilde{B}\mathcal{T}_n^{\alpha}(X)\simeq B\mathcal{T}_n^{\alpha}(X)$, passing to homology, noting that ϕ: $(\underline{n},\alpha)\to(\underline{m},\beta_1)$ satisfies the conditions of the lemma with k replaced by $k-1$, one uses the induction hypothesis as in Suslin to conclude that

$$H_k(B(\mathcal{T}^f\cdot\mathcal{T}^g))=H_k(B\mathcal{T}^f)+H_k(B\mathcal{T}^g) \quad (5.7)$$

where $B(\mathcal{T}^f\cdot\mathcal{T}^g)$ corresponds to the composite along 5.6.

Next we note that for a discrete ring R we have a matrix identity

$$uT^f(A)\cdot T^g(A)=T^{f'}(\pi(A))\cdot T^g(A)u,\qquad \forall A\in T_n^{\alpha}(R)$$

where $u=\prod_{i\in\underline{n}-\min(\underline{n},\alpha)}e_{f_1(i)f(i)}(1)$ (product of elementary matrices) and where $\pi: T_n^{\alpha}(R)\to T_n^{\alpha'}(R)$ is the projection obtained by taking α' to be the restriction of α to $\underline{n}-\min(\underline{n},\alpha)$, and taking f' to be the corresponding restriction of f. In our language this can be interpreted as a natural transformation between $\mathcal{T}^f\cdot\mathcal{T}^g$ and the following composite of functors in $\mathscr{C}at(\Sigma\int\Theta_r)$

$$\mathcal{T}_n^{\alpha}\xrightarrow{\Delta}\mathcal{T}_n^{\alpha}\times\mathcal{T}_n^{\alpha}\xrightarrow{\pi\times id}\mathcal{T}_n^{\alpha'}\times\mathcal{T}_n^{\alpha}\xrightarrow{\mathcal{T}^{f'}\times\mathcal{T}^g}\mathcal{T}_m^{\beta_1}\times\mathcal{T}_m^{\beta_2}\xrightarrow{\mu}\mathcal{T}_m^{\beta}.$$

Arguing as above we obtain that

$$H_k(B(\mathcal{T}^f\cdot\mathcal{T}^g))=H_k(B((\mathcal{T}^{f'}\circ\pi)\cdot\mathcal{T}^g))=H_k(B\mathcal{T}^{f'})\circ H_k(B\pi)+H_k(B\mathcal{T}^g).$$

Combining with (5.7) we obtain that

$$H_k(B\mathcal{T}^f)=H_k(B\mathcal{T}^{f'})\circ H_k(B\pi).$$

Since $\langle\alpha'\rangle<\langle\alpha\rangle$, we have by induction hypothesis that $H_k(B\mathcal{T}^f)=0$. □

For each integer m we now construct a functor $\lambda_m:\mathcal{P}\to\mathcal{P}$ together with a natural transformation $Id_{\mathcal{P}}\to\lambda_m$. For an object $(\underline{n},\alpha)$ in $\mathcal{P}$ let $\underline{m}\times\underline{n}$ be the product set identified with $\underline{mn}$ using the natural lexicographic ordering. Let $\alpha\times m^r$ denote the partial order on $\underline{m}\times\underline{n}=\underline{mn}$ given by taking **reverse** lexicographic ordering with respect to α on $\underline{n}$ and the **reverse** ordering $1>2>\cdots>m$ on $\underline{m}$. Clearly $\lambda_m(\underline{n},\alpha)=(\underline{mn},\alpha\times m^r)$ defines a functor and $i\mapsto(1,i)$ defines a natural transformation $Id_{\mathcal{P}}\to\lambda_m$.

5.8 *Proof of Lemma C.* To prove the acyclicity of $\mathcal{V}\mathcal{S}(X)$ one uses Remark 5.3 and argues as in [15, (2.3)], proving by induction on k and Lemma 5.4 that given any $p>0$, the

induced homomorphism

$$\lambda_{m*}: \tilde{H}_j\left(\bigcup_{i=1}^{k} B\mathscr{T}_n^{\alpha_i}(X)\right) \to \tilde{H}_j\left(\bigcup_{i=1}^{k} B\mathscr{T}_{mn}^{\alpha_i \times m^r}(X)\right)$$

is zero for $j < p$ for n large enough.

To prove that $\pi_1 \mathscr{V}\mathscr{S}(X) = St(\pi_0 X)$ one uses a similar induction argument, using van Kampen's theorem to conclude that the natural map $X \to \pi_0 X$ induces an isomorphism $\pi_1 \mathscr{V}\mathscr{S}(X) \xrightarrow{\cong} \pi_1 \mathscr{V}\mathscr{S}(\pi_0 X) = St(\pi_0 X)$.

6. PROOF OF LEMMA *B*

To prove the nilpotency $\mathscr{V}\mathscr{M}(X)$ we need to investigate Volodin models corresponding to two different versions of the Steinberg group for A_∞ rings. The first version $\widetilde{St}(X)$ is defined by the pullback

$$\begin{array}{ccc} \widetilde{St}_n(X) & \to & \tilde{E}_n(X) \\ \downarrow & & \downarrow \\ St_n(\pi_0 X) & \to & E_n(\pi_0 X) \end{array} \tag{6.1}$$

By the methods of §4 we have a fibration sequence

$$\mathscr{W}\mathscr{M}(X) \to \mathscr{V}\mathscr{S}(X) \to B\widetilde{St}(X)$$

with $\pi_1 B\widetilde{St}(X) = St(\pi_0 X)$ acting trivially on the $H_*(\mathscr{W}\mathscr{M}(X))$. (Actually a stronger condition holds, cf. Remark 4.5, which we will use below.) Here we denote $\mathscr{W}\mathscr{M}(X) = \operatorname{hocolim}_{\mathscr{P}} B(\widetilde{St}_n(X), \tilde{T}_n^{\alpha}(X), *)$, the Volodin model associated to $\widetilde{St}(X)$. Since $K_2(\pi_0 X)$ is the kernel of $St(\pi_0 X) \to E(\pi_0 X)$ we obtain a diagram of fibration sequences

$$\begin{array}{ccccccc} K_2(\pi_0 X) & \longrightarrow & \mathscr{W}\mathscr{M}(X) & \longrightarrow & \mathscr{V}\mathscr{M}(X) & \xrightarrow{f} & BK_2(\pi_0 X) \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & \mathscr{V}\mathscr{S}(X) & = & \mathscr{V}\mathscr{S}(X) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ BK_2(\pi_0 X) & \longrightarrow & B\widetilde{St}(X) & \longrightarrow & B\tilde{E}(X) & & \end{array} \tag{6.2}$$

To prove that $\mathscr{V}\mathscr{M}(X)$ is nilpotent, it suffices to show that $\mathscr{W}\mathscr{M}(X)$ is simple. Here is the argument: By the methods of §4 (cf. Remark 4.5) we have that in the fibration sequence

$$\mathscr{W}\mathscr{M}(X) \to \mathscr{V}\mathscr{S}(X) \to B\widetilde{St}(X)$$

the action of any element of $\pi_1 BSt(X) = St(\pi_0 X)$ on $\mathscr{W}\mathscr{M}(X)$ restricted to an finite subcomplex is trivial through an unbased homotopy. Once we show that $\mathscr{W}\mathscr{M}(X)$ is simple, the same holds through a based homotopy. It follows that in the fibration

$$\mathscr{W}\mathscr{M}(X) \to \mathscr{V}\mathscr{M}(X) \to BK_2(\pi_0 X)$$

the same holds for elements of

$$\pi_1 BK_2(\pi_0 X) = K_2(\pi_0 X) \subset St(\pi_0 X) = \pi_1 B\widetilde{St}(X)$$

acting on $\mathscr{WM}(X)$. It now follows that $\mathscr{VM}(X)$ is nilpotent. To prove $\mathscr{WM}(X)$ is simple, we need a Volodin model of the universal cover of $\mathscr{WM}(X)$. We begin by noting that a comparison of homology spectral sequences associated to the diagrams of fibrations

$$\begin{array}{ccccc} \mathscr{WM}(X) & \longrightarrow & \mathscr{VS}(X) & \longrightarrow & B\widetilde{St}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega B\widetilde{St}(X)^+ & \longrightarrow & fiber & \longrightarrow & B\widetilde{St}(X) \longrightarrow B\widetilde{St}(X)^+ \end{array}$$

shows that $\mathscr{WM}(X) \to \Omega B\widetilde{St}(X)^+$ is a homology equivalence and that $\pi_1\mathscr{WM}(X)$ is abelian (being the image of $\pi_2 B\widetilde{St}(X)$). Hence

$$\pi_1\mathscr{WM}(X) = H_1\Omega B\widetilde{St}(X)^+ = \pi_1\Omega B\widetilde{St}(X)^+ = \pi_2\Omega B\widetilde{St}(X)^+$$

$$= \mathrm{Ker}(K_2(X) \to K_2(\pi_0 X)),$$

which group we will denote by A. Let

$$B\tilde{E}(X) \to K(K_2(X), 2) \tag{6.3}$$

be a map which induces an isomorphism

$$H_2 B\tilde{E}(X) \xrightarrow{\cong} K_2(X)$$

and let $F(X)$ denote the homotopy fiber of (6.3). (Homotopically $\Omega F(X)$, rather than $\widetilde{St}(X)$, is the correct analog of the Steinberg group of an A_∞ ring.) Since $\mathscr{VS}(X)$ is acyclic, the map $\mathscr{VS}(X) \to B\tilde{E}(X)$ lifts to a map $\mathscr{VS}(X) \to F(X)$. Since $B\widetilde{St}(X)$ can be identified with the fiber of

$$B\tilde{E}(X) \to K(K_2(\pi_0 X), 2)$$

there is an induced map

$$F(X) \to B\widetilde{St}(X)$$

which induces an isomorphism

$$\pi_1 F(X) \xrightarrow{\cong} \pi_1 B\widetilde{St}(X) = St(\pi_0 X)$$

and whose homotopy fiber can be identified with A. Since (cf. 2.4)

$$\mathscr{VS}(X) \simeq \mathrm{colim}\, BT_n^\alpha(X)$$

we can construct a diagram

$$\eta : \mathscr{P} \cup \{\omega\} \to \mathscr{T}op,$$

where ω denotes a terminal object adjoined to $\mathscr{P}$, with

$$\eta(\underline{n}, \alpha) = BT_n^\alpha(X)$$

$$\eta(\omega) = F(X).$$

By composing with a group valued loop functor G we obtain a diagram

$$G\eta : \mathscr{P} \cup \{\omega\} \to \mathscr{T}op$$

from which we can construct the space

$$\operatorname{hocolim}_{\mathscr{D}} B(GF(X), GBT_n^{\alpha}(X), *)$$

which can be identified with the universal cover of $\mathscr{W}\mathscr{M}(X)$. The action of $\pi_1 \mathscr{W}\mathscr{M}(X) = A$ can be identified with the left action of the homotopy fiber of

$$GF(X) \to GB\widetilde{St}(X) \tag{6.4}$$

on $\mathrm{GF}(X)$. (Here as previously G denotes a loop group functor.) Since (6.4) is an isomorphism on π_0 this action can be homotoped to the trivial action by choosing paths to the unit element of $GF(X)$.

7. INFINITE LOOP VOLODIN MODELS AND GROUP COMPLETION THEOREM

Let $\mathscr{J}$ denote the category whose objects are $\underline{n}$, $n \geq 0$ and whose morphisms are all injections. Let $\Sigma = \coprod_{n \geq 0} \Sigma_n$ be the subcategory of isomorphisms in $\mathscr{J}$. If $M: \mathscr{J} \to \mathscr{T}op$ is any functor, notice that $\operatorname{hocolim}_{\Sigma} M = \coprod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} M(n)$ and that the various order preserving inclusions $[n] \to [n+1]$ induce homotopic maps

$$t: E\Sigma_n \times_{\Sigma_n} M(n) \to E\Sigma_{n+1} \times_{\Sigma_{n+1}} M(n+1).$$

Our first result is

7.1 PROPOSITION. *There is a natural map* $\operatorname{hocolim}_{\Sigma} M \to \operatorname{hocolim}_{\mathscr{J}} M \times QS^0$ *which induces an isomorphism*

$$H_*(\operatorname{hocolim}_{\Sigma} M; F)[t^{-1}] \xrightarrow{\cong} H_*(\operatorname{hocolim}_{\mathscr{J}} M \times QS^0; F) \tag{7.2}$$

with any field coefficients F.

Proof. The first component of the map is induced by the inclusion $\Sigma \subseteq \mathscr{J}$. The second component is the composite of the natural map $\operatorname{hocolim}_{\Sigma} M \to B\Sigma$ followed by the group completion map $B\Sigma \to QS^0$. The proof that (7.2) holds in general proceeds by reduction to a special case via the following observations.

(i) If $M_*: \mathscr{J} \times \Delta^{op} \to \mathscr{T}op$ is a simplicial $\mathscr{J}$-functor and if (7.2) holds for all M_k, then it holds for the geometric realization $|M_*|: \mathscr{J} \to \mathscr{T}op$ as well.

(ii) Given a collection $M_\alpha: \mathscr{J} \to \mathscr{T}op$ *of* $\mathscr{J}$-functors, $\alpha \in \mathscr{A}$, if (7.2) holds for each M_α, then it holds for $\coprod_\alpha M_\alpha: \mathscr{J} \to \mathscr{T}op$ as well.

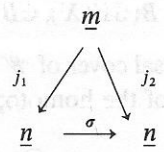
(iii) Given a $\mathscr{J}$-functor $M: \mathscr{J} \to \mathscr{T}op$ and a topological space X, if (7.2) holds for M, then it holds for $X \times M$ as well.

Point (i) follows from the fact that homotopy colimits commute with geometric realization, the spectral sequence for the homology of a simplicial space, and the fact that localization is exact. Points (ii) and (iii) are trivial. Now applying these observations to the resolution

$$B(\mathscr{J}, \mathscr{J}, M) \xrightarrow{\simeq} M$$

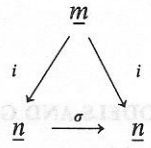
(cf. [7]) we reduce to showing that (7.2) holds in the particular case $M(-) = \mathscr{J}(\underline{m}, -)$ where $\underline{m}$ is fixed. In this case $\operatorname{hocolim}_{\mathscr{J}} \mathscr{J}([m], -) \simeq *$, while $\operatorname{hocolim}_{\Sigma} \mathscr{J}(\underline{m}, -) = B\mathscr{C}_m$ where $\mathscr{C}_m$ is the category whose objects are morphisms $\underline{m} \to \underline{n}$ in $\mathscr{J}$ and whose morphisms

are commutative triangles



(7.3)

with $\sigma \in \Sigma_n$. The natural functor $\operatorname{hocolim}_{\Sigma} \mathscr{J}(\underline{m}, -) \to B\Sigma$ is induced by the functor $\mathscr{C}_m \to \Sigma$ which sends the object $\underline{m} \to \underline{n}$ to $\underline{n}$ and the morphism (7.3) to $\underline{n} \xrightarrow{\sigma} \underline{n}$. Now $\mathscr{C}_m$ is equivalent to the skeletal subcategory $\tilde{\mathscr{C}}_m$ whose objects are the **standard** inclusions $\underline{m} \xrightarrow{i} \underline{n}$ and whose morphisms are diagrams



$\tilde{\mathscr{C}}_m$ can be naturally identified with Σ_{n-m}. The composite functor $\tilde{\mathscr{C}}_m \xrightarrow{\simeq} \mathscr{C}_m \to \Sigma$ can then be identified with the functor

$$\coprod_{k \geq 0} \Sigma_k \longrightarrow \coprod_{n \geq 0} \Sigma_n$$

given by the inclusions $\Sigma_k \cong id_m \times \Sigma_k \subseteq \Sigma_{m+k}$. In homology this map induces

$$t^m : H_*(B\Sigma) \to H_*(B\Sigma).$$

Now it follows from the Barrett–Quillen–Priddy theorem that the composite

$$B\tilde{\mathscr{C}}_m \xrightarrow{\simeq} B\mathscr{C}_m \longrightarrow B\Sigma \longrightarrow QS^0$$

induces an isomorphism

$$H_*(B\tilde{\mathscr{C}}_m; F)[t^{-1}] \xrightarrow{\cong} H_*(QS^0; F). \qquad \square$$

7.4 **The wreath product construction**

Before we can proceed to our main result, we need to reformulate the constructions of [4] in homotopy invariant form. To do this we require several constructions of [17].

First recall that $\mathscr{F}$ is the category of finite based sets $[\eta] = \{0, 1, 2, \ldots, n\}$ with 0 as basepoint. Given a symmetric monoidal category S, Thomason constructs an associated wreath product category $\mathscr{F} \int S$ whose objects are $([m]; a_1, a_2, \ldots, a_m)$, where the a_i are objects of S (including the null object $([0], \phi)$). A morphism from $([m]; a_1, a_2, \ldots, a_m)$ to $([n]; b_1, b_2, \ldots b_n)$ has the form $(\varphi; f_1, f_2, \ldots, f_n)$ where $\varphi: [m] \to [n]$ is a morphism in $\mathscr{F}$ and for each $1 \leq i \leq n$, $f_i: \bigoplus_{s \in \varphi^{-1}(i)} a_s \to b_i$ is a morphism in S. Composition in $\mathscr{F} \int S$ is given by composition in $\mathscr{F}$ in the first slot and by a combination of composition in S, applying $\oplus$, and rearranging summands using the coherence isomorphisms of S in the other slots. (cf. [17, §3, §4] for details.)

Recall that an additive functor $F: S \to T$ between symmetric monoidal categories is a functor together with a natural transformation $F(a) \oplus F(b) \to F(a \oplus b)$ which is suitably compatible with the coherence isomorphisms on both sides. To any such F, there is associated a functor $\mathscr{F} \int F: \mathscr{F} \int S \to \mathscr{F} \int T$, (cf. [17, §4]).

We will need one additional construction involving wreath product categories: Suppose T is a permutative category in which the object 0 is a terminal object. Then we can define

a functor $\mathscr{F} \int T \to T$ as follows. We send the object $([m]; a_1, \ldots a_m)$ to $a_1 \oplus a_2 \oplus \ldots \oplus a_m$. Given a morphism $(\varphi, f_1, f_2, \ldots f_n)$ as above, we send it to the morphism in T given by the following composite

$$a_1 \oplus a_2 \oplus \cdots \oplus a_m \xrightarrow[\cong]{(1)} \bigoplus_{s \in \varphi^{-1}(0)} a_s \oplus \bigoplus_{s \in \varphi^{-1}([n]-0)} a_s \xrightarrow{(2)} \bigoplus_{S \in \varphi^{-1}(0)} 0 \oplus \bigoplus_{s \in \varphi^{-1}([n]-0)} a_s$$

$$\xrightarrow[\cong]{(3)} \bigoplus_{i=1}^{n} \bigoplus_{s \in \varphi^{-1}(i)} a_s \xrightarrow{f_1 \oplus f_2 \oplus \cdots \oplus f_n} b_1 \oplus b_2 \oplus \cdots \oplus b_n$$

where the isomorphisms labelled (1) and (3) are combinations of coherency isomorphisms in T, while (2) is induced by the unique maps into the terminal object 0.

We are interested in one special case of the last construction: when $T = \mathscr{T}op$ with symmetric monoidal structure given by taking products. We thus obtain a functor $F \int \mathscr{T}op \to \mathscr{T}op$. Now suppose we are given an additive functor $F : S \to \mathscr{T}op$. We then obtain a composite functor

$$\mathscr{F} \int S \xrightarrow{\mathscr{F} \int F} \mathscr{F} \int \mathscr{T}op \longrightarrow \mathscr{T}op.$$

We will abusively also denote this composite as $\mathscr{F} \int F$. Alternatively we can take $T = \mathscr{S}ets^{\Delta^{op}}$ instead of $\mathscr{T}op$.

7.5 *Definition.* An E_∞ *permutative monoid is a functor* $M : \mathscr{F} \int \mathscr{J} \longrightarrow \mathscr{T}op$ such that there are equivalences

$$(\delta_1, \delta_2, \ldots, \delta_n) : M([n], \underline{k}_1, \underline{k}_2, \ldots \underline{k}_n) \xrightarrow{\simeq} \prod_{i=1}^{n} M([1]; \underline{k}_i) \tag{7.6}$$

where $\delta_i : [n] \to [1]$ in $\mathscr{F}$ is given by

$$\delta_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Note that for fixed $[n]$ there is a functor $M^{(n)} : \mathscr{J}^n \to \mathscr{T}op$ given by

$$M^{(n)}(\underline{k}_1, \ldots, \underline{k}_n) = M([n]; \underline{k}_1, \ldots, \underline{k}_n),$$

which in turn determine functors $\mathscr{F} \longrightarrow \mathscr{T}op$ given by

$$[n] \mapsto \operatorname{hocolim}_{\mathscr{J}^n} M^{(n)}$$

$$[n] \mapsto \operatorname{hocolim}_{\Sigma^n} M^{(n)}$$

$$[n] \mapsto \coprod_{k_1, k_2, \ldots, k_n} M^{(n)}(\underline{k}_1, \ldots, \underline{k}_n)$$

and it follows from (7.5) that these functors are Segal Γ-spaces (cf. [13]). We will abusively denote these Γ-spaces as $\operatorname{hocolim}_{\mathscr{J}} M$, $\operatorname{hocolim}_{\Sigma} M$, and M_* respectively and we will denote by $\mathbf{S}$ $(\operatorname{hocolim}_{\mathscr{J}} M)$, $\mathbf{S}(\operatorname{hocolim}_{\Sigma} M)$, $\mathbf{S}(M_*)$ the infinite loop spaces which result from applying Segal's infinite loop machine to them.

It is an immediate consequence of Proposition 7.1 and the homology group completion property of Segal's infinite loop machine that

7.7 Corollary. *Let M be an E_∞ permutative monoid. Then there is a natural equivalences*

$$\mathbf{S}(\operatorname{hocolim}_{\Sigma} M) \xrightarrow{\simeq} \mathbf{S}(\operatorname{hocolim}_{\mathscr{J}} M) \times QS^0. \qquad \square$$

In the following result $M_\infty = \operatorname{hocolim}_{\mathbb{N}} M$ where $\mathbb{N}$ is the poset of natural numbers regarded as a subcategory of $\mathscr{J}$.

7.8 THEOREM. *Let M be an E_∞ permutative monoid. Assume that Σ_∞ acts trivially on $H_*(M_\infty)$. Then there is a natural fibration sequence*

$$\mathbf{S}(\operatorname{hocolim}_{\mathscr{J}} M) \to \mathbf{S}(M_*) \to \mathbb{Z}.$$

Proof. This follows directly by the arguments of [4]. One first shows that there is a splitting

$$\mathbf{S}(\operatorname{hocolim}_{\Sigma} M) \simeq F \times QS^0$$

where F denotes the fiber of $\mathbf{S}(M_*) \to \mathbb{Z}$. Comparing this with 7.6 one obtains the result. □

7.9 *Remark.* (i) The theorem also holds if we assume that Σ_∞ acts trivially on $H_*(M_\infty; F)$ for any field F, using the same argument. (ii) In the applications we consider below, $\operatorname{hocolim}_{\mathscr{J}} M$ is grouplike so $\mathbf{S}(\operatorname{hocolim}_{\mathscr{J}} M) \simeq \operatorname{hocolim}_{\mathscr{J}} M$. Moreover, $\operatorname{hocolim}_{\mathscr{J}} M$ may be identified with a "configuration space model" in the sense of [4]. For it follows from definition that

$$\operatorname{hocolim}_{\mathscr{J}} M = B(-//\mathscr{J}) \otimes_{\mathscr{J}} M = \coprod_{n \geq 0} B(\underline{n}//\mathscr{J}) \times_{\Sigma_n} M(n)/\text{basepoint relations}$$

where $\underline{n}//\mathscr{J}$ denotes the under category. Assuming the functor M is sufficiently free (which can be arranged by replacing M by the equivalent functor $B(\mathscr{J}, \mathscr{J}, M)$) we get a chain of equivalences

$$\begin{array}{ccc} \coprod_{n\geq 0} C(n, \mathbb{R}^\infty) \times_{\Sigma_n} M(n)/\approx & \xleftarrow{\simeq} & \coprod_{n\geq 0} (C(n, \mathbb{R}^\infty) \times B(\underline{n}//\mathscr{J})) \times_{\Sigma_n} M(n)/\approx \\ & & \downarrow \simeq \\ & & \coprod_{n\geq 0} B(\underline{n}//\mathscr{J}) \times_{\Sigma_n} M(n)/\approx \\ & & \| \\ & & \operatorname{hocolim}_{\mathscr{J}} M. \end{array}$$

The rest of this section will be devoted to the proof of the following results.

7.10 THEOREM. *For any A_∞ ring X there is an E_∞-permutative monoid $\underline{\underline{BGl}}(X): \mathscr{F} \int \mathscr{J} \to \mathscr{T}op$ such that there are natural equivalences*

$$\underline{\underline{BGl}}(X)([n]; \underline{k}_1, \ldots, \underline{k}_n) \simeq \prod_{i=1}^{n} \widehat{BGl}_{k_i}(X)$$

and such that there is an equivalence of infinite loop spaces

$$\operatorname{hocolim}_{\mathscr{J}} \underline{\underline{BGl}}(X) \simeq K(X)_0$$

where $K(X)_0$ is the path component of 0 in $K(X)$.

Remark. For A_∞ rings of the form $Q(G_+)$, where G is a topological group (or grouplike topological monoid), one can easily show using Theorem 7.10 that

$$\operatorname{hocolim}_{\mathscr{J}} \mathscr{H}_k(G) \simeq K(Q(G_+))_0,$$

where $\mathscr{H}_k(G) = \operatorname{colim}_n \mathscr{H}_k^n(G)$ and $\mathscr{H}_k^n(G)$ is the topological monoid of G-equivariant pointed self-equivalences of $\bigvee^k S^n \wedge G_+$ (cf. [20]).

7.11 THEOREM. *For any A_∞ ring X there is an E_∞-permutative monoid $\underline{\underline{VM}}^{Gl}(X): \mathscr{F} \int \mathscr{J} \to \mathscr{T}op$ such that there are natural equivalences*

$$\underline{\underline{VM}}^{Gl}(X)([n]; \underline{k}_1, \ldots, \underline{k}_n) \simeq \prod_{i=1}^{n} \mathscr{V}\mathscr{M}_{k_i}^{Gl}(X)$$

(*cf.* 2.4) *and such that there is an equivalence of infinite loop spaces*

$$\operatorname{hocolim}_{\mathscr{J}} \underline{\underline{VM}}^{Gl}(X) \simeq \Omega K(X).$$

Theorem 7.11 may be thought of as an infinite loop version of Theorem 2.7 and $\operatorname{hocolim}_{\mathscr{J}} \underline{\underline{VM}}^{Gl}(X)$ as a infinite loop space Volodin model.

Now recall that in §5 we introduced a category $\tilde{\mathscr{P}}$ with objects $(\underline{m}, \alpha)$ and morphisms all order preserving injections. Taking disjoint unions endows $\tilde{\mathscr{P}}$ with the structure of a permutative category.

Let R be a semiring. We consider the functors

$$\lambda_i: \tilde{\mathscr{P}} \to \mathscr{S}ets^{\Delta^{op}} \qquad i = 1, 2, 3$$

given by

$$\lambda_1(\underline{n}, \alpha) = B_*(M_n(R), T_n^\alpha(R), *)$$

$$\lambda_2(\underline{n}, \alpha) = B_* T_n^\alpha(R)$$

$$\lambda_3(\underline{n}, \alpha) = B_* M_n(R).$$

These functors are clearly additive and there are evident additive natural transformations

$$\lambda_1 \to \lambda_2 \to \lambda_3 \tag{7.12}$$

By the discussion 7.4 above we obtain a sequence of functors $\mathscr{F} \int \lambda_i: \mathscr{F} \int \tilde{\mathscr{P}} \to \mathscr{S}ets^{\Delta^{op}}$ and natural transformations

$$\mathscr{F} \int \lambda_1 \to \mathscr{F} \int \lambda_2 \to \mathscr{F} \int \lambda_3.$$

Moreover it is readily seen that these functors factor as follows

$$\begin{array}{ccc} \mathscr{F} \int \tilde{\mathscr{P}} \times \Delta^{op} & \xrightarrow{\mathscr{F} \int \lambda_i} & \mathscr{S}ets \\ & {\scriptstyle \hat{\lambda}_i} \searrow \quad \nearrow {\scriptstyle R} & \\ & \Theta_r & \end{array}$$

with $\hat{\lambda}_i$ meeting only simple morphisms and that the natural transformations (7.12) give rise to natural transformations

$$\hat{\lambda}_i \to \hat{\lambda}_2 \to \hat{\lambda}_3 \tag{7.13}$$

which can be consolidated into a single functor

$$\hat{\lambda}: \mathscr{F} \int \tilde{\mathscr{P}} \times \Delta^{op} \times \mathscr{D} \to \Theta_r$$

where $\mathscr{D}$ stands for the category $1 \to 2 \to 3$. Now let X be an A_∞ ring given by a diagram

$$\begin{array}{ccc} \Theta & \xrightarrow{X} & \mathscr{T}op \\ \downarrow{\scriptstyle \varepsilon} & & \\ \Theta_r & & \end{array}$$

We can form the pullback diagram

$$\begin{array}{ccccc} \mathscr{H} & \xrightarrow{H\hat{\lambda}} & \Theta & \xrightarrow{X} & \mathscr{T}op \\ \downarrow{\scriptstyle \hat{\varepsilon}} & & \downarrow{\scriptstyle \varepsilon} & & \\ \mathscr{F}\int \tilde{\mathscr{P}} \times \Delta^{op} \times \mathscr{D} & \xrightarrow{\hat{\lambda}} & \Theta_r & & \end{array}$$

We can then apply the Segal pushdown to the restriction of $X \circ H\hat{\lambda}$ to invertible components and obtain a functor

$$\hat{\lambda}(X)_* : \mathscr{F}\int \tilde{\mathscr{P}} \times \Delta^{op} \times \mathscr{D} \longrightarrow \mathscr{T}op$$

which we geometrically realize and rewrite as a sequence of functors

$$\hat{\lambda}_1(X) \longrightarrow \hat{\lambda}_2(X) \longrightarrow \hat{\lambda}_3(X) : \mathscr{F}\int \tilde{\mathscr{P}} \longrightarrow \mathscr{T}op. \tag{7.14}$$

Up to homotopy

$$\begin{aligned} \hat{\lambda}_1(X)([m]; (\underline{k}_1, \alpha_1), \ldots, (\underline{k}_m, \alpha_m)) &\simeq \prod_{i=1}^{m} B(\widetilde{Gl}_{k_i}(X), \tilde{T}_{k_i}^{\alpha_i}(X), *) \\ \hat{\lambda}_2(X)([m]; (\underline{k}_1, \alpha_1), \ldots, (\underline{k}_m, \alpha_m)) &\simeq \prod_{i=1}^{m} B\tilde{T}_{k_i}^{\alpha_i}(X) \\ \hat{\lambda}_3(X)([m]; (\underline{k}_1, \alpha_1), \ldots, (\underline{k}_m, \alpha_m)) &\simeq \prod_{i=1}^{m} B\widetilde{Gl}_{k_i}(X) \end{aligned} \tag{7.15}$$

and (7.14) can be considered as an extension of the fibration sequence (2.2). Next we note that there is a functor

$$\mathscr{J} \xrightarrow{F} \mathscr{C}at \qquad \underline{n} \mapsto \mathscr{P}_n$$

and that $\mathscr{J}\int F = \tilde{\mathscr{P}}$. We have induced functors $\bar{\lambda}_i(X) : \mathscr{F}\int \mathscr{J} \to \mathscr{T}op$

$$\bar{\lambda}_i(X)([m]; \underline{k}_1, \underline{k}_2, \ldots, \underline{k}_m) = \operatorname{hocolim}_{\mathscr{P}_{k_1} \times \mathscr{P}_{k_2} \times \cdots \times \mathscr{P}_{k_m}} \hat{\lambda}_i(X)$$

and induced natural transformations

$$\bar{\lambda}_1(X) \longrightarrow \bar{\lambda}_2(X) \longrightarrow \bar{\lambda}_3(X) : \mathscr{F}\int \mathscr{J} \longrightarrow \mathscr{T}op. \tag{7.15}$$

It follows from (7.12) that the $\bar{\lambda}_i(X)$ are E_∞ permutative monoids. We now apply Theorem 7.8 to each of these:

7.16 *Analysis of* $\bar{\lambda}_3(X)$. We have

$$\bar{\lambda}_3(X)([1], \underline{k}) \simeq B\widehat{Gl}_k(X).$$

By classical arguments, Σ_∞ acts trivially on $H_*(B\widehat{Gl}(X))$ so we conclude that

$$\operatorname{hocolim}_{\mathscr{J}} \bar{\lambda}_3(X) = \operatorname{hocolim}_{\tilde{\mathscr{P}}} \bar{\lambda}_3(X) \simeq B\widehat{Gl}(X)^+.$$

7.17 *Analysis of* $\bar{\lambda}_2(X)$. We have

$$\bar{\lambda}_2(X)([1];\underline{k}) \simeq \mathscr{V}\mathscr{S}_k(X)$$

in the notation of §3. By §5, $\tilde{H}_*(\text{hocolim}_k \mathscr{V}\mathscr{S}_k(X)) = 0$, so Σ_∞ necessarily acts trivially. It follows that

$$\text{hocolim}_{\mathscr{J}} \bar{\lambda}_2(X) = \text{hocolim}_{\tilde{\mathscr{J}}} \bar{\lambda}_2(X) \simeq \mathscr{V}\mathscr{S}(X)^+ \simeq *.$$

7.18 *Analysis of* $\bar{\lambda}_1(X)$. We have

$$\bar{\lambda}_1(X)([1]; k) \simeq \mathscr{V}\mathscr{M}_k^{Gl}(X)$$

To see that Σ_∞ acts trivially on $H_*(\mathscr{V}\mathscr{M}^{Gl}(X); F)$ when F is a field, we note first that because of the equivalence

$$\mathscr{V}\mathscr{M}^{Gl}(X) \simeq \mathscr{V}\mathscr{M}^{E}(X) \times K_1(\pi_0 X),$$

the fact that Σ_∞ acts trivially on $K_1(\pi_0 X)$, and (2.8), it suffices to prove that Σ_∞ acts trivially on $H_*(\mathscr{V}\mathscr{M}^{E}(X); F)$. We appeal to the fibration sequence

$$\mathscr{V}\mathscr{M}(X) \to \mathscr{V}\mathscr{S}(X) \to B\tilde{E}(X) \tag{7.19}$$

We first note that $\pi_1 B\tilde{E}(X) = E(\pi_0 X)$ acts trivially on $H_*(\mathscr{V}\mathscr{M}^{E}(X); F)$ by (2.11), and that Σ_∞ acts trivially on $H_*(\mathscr{V}\mathscr{S}(X); F)$ by (7.17) above and on $H_*(B\tilde{E}(X); F)$ by (7.16). Note also that Σ_∞ acts compatibly on the fibration sequence (7.19) and thus Σ_∞ acts on the Serre spectral sequence (with F coefficients) of (7.19).

If char $F \neq 2$ we note that Σ_∞ is generated by 2-cycles, so it suffices to show any 2-cycle α acts trivially on $H_*(\mathscr{V}\mathscr{M}^{E}(X); F)$. Since taking invariants of an involution is exact over F, we can take the invariants of α of the Serre spectral sequence of (7.19) and obtain another spectral sequence. By the comparison theorem for spectral sequences we obtain that $H_*(\mathscr{V}\mathscr{M}^{E}(X); F)^\alpha \to H_*(\mathscr{V}\mathscr{M}^{E}(X); F)$ is an isomorphism.

If char $F = 2$ we note that the alternating group A_∞ is generated by 3-cycles and use a similar argument to conclude that A_∞ acts trivially on $H_*(\mathscr{V}\mathscr{M}^{E}(X); F)$. It easily follows that Σ_∞ acts trivially too.

We also note that $\pi_0(\text{hocolim}_{\mathscr{J}} \bar{\lambda}_1) = K_1(X)$ is a group, by an argument similar to (2.5). Hence $\mathbf{S}(\text{hocolim}_{\mathscr{J}} \bar{\lambda}_1(X)) \simeq \text{hocolim}_{\mathscr{J}} \bar{\lambda}_1(X)$. Also $\mathscr{V}\mathscr{M}^{Gl}(X) \simeq \Omega K(X)$ by Theorem 2.7 and hence is a simple space. It follows from Theorem 7.8 that

$$\text{hocolim}_{\mathscr{J}} \bar{\lambda}_1(X) = \text{hocolim}_{\tilde{\mathscr{J}}} \bar{\lambda}_1(X) \simeq \mathscr{V}\mathscr{M}^{E}(X).$$

7.20 *Proofs of* 7.10 *and* 7.11. Taking $\underline{\underline{BGl}}(X) = \bar{\lambda}_3(X)$, (7.10) follows immediately from (7.16). By (7.16)–(7.18), the sequence of $\underline{\underline{E}}_\infty$-spaces

$$\text{hocolim}_{\mathscr{J}} \bar{\lambda}_1(X) \to \text{hocolim}_{\mathscr{J}} \bar{\lambda}_2(X) \to \text{hocolim}_{\mathscr{J}} \bar{\lambda}_2(X) \tag{7.21}$$

may be identified with the fibration sequence of spaces

$$\mathscr{V}\mathscr{M}^{Gl}(X) \to \mathscr{V}\mathscr{M}(X)^+ \to \widetilde{BGl}(X)^+$$

(cf. proof of Theorem 2.7). We conclude that (7.21) is a fibration sequence of infinite loop spaces. Since $\text{hocolim}_{\mathscr{J}} \bar{\lambda}_2(X) \simeq *$ by (7.17), we further conclude that there is an equivalence of infinite loop spaces

$$\text{hocolim}_{\mathscr{J}} \bar{\lambda}_1(X) \simeq \Omega(\text{hocolim}_{\mathscr{J}} \bar{\lambda}_3(X)),$$

and thus by (7.16) we get the infinite loop equivalence

$$\operatorname{hocolim}_{\mathcal{J}} \bar{\lambda}_1(X) \simeq \Omega K(X).$$

Thus we take $\underline{\underline{VM}}^{Gl}(X) = \bar{\lambda}_1(X)$. □

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