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On Chevalley–Eilenberg and Cyclic Homologies

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INTRODUCTION

The computation of the Lie algebra homology of matrices due to Loday and Quillen [LQ] and independently Tsygan [T] establishes an isomorphism of graded commutative Hopf algebras

$$H_*^L(gl(A)) \cong S^*(HC_{*-1}(A)) \tag{*}$$

where $H_*^L(\_\_)$ denotes the Lie algebra homology, $HC_*(\_\_)$ cyclic homology, and A is an algebra with unit over a field of characteristic zero. In this paper we give an alternative proof of this theorem which does not involve Weyl's invariant theory for $GL(\mathbb{C})$. We use this approach to compute Chevalley–Eilenberg homology in some interesting new cases. In Section 1, we begin by proving (in characteristic 0) a Loday–Quillen–Tsygan theorem for complexes which occur as subcomplexes D_* of the Chevalley–Eilenberg complex $\bigwedge^* gl(A)$; the conditions we require of D_* are minimal. In particular, D_* need not be closed under the conjugation action of $GL(\mathbb{Q})$, but only $W(\mathbb{Q})$ = monomial matrices with entries in $\mathbb{Q}$. Under the condition that this action is the identity map on homology, together with 3 other basic conditions (P1-P4, section 1) we show that there is an isomorphism of graded commutative Hopf algebras

$$H_*(D_*) \cong S^*(H_{*-1}(C(D_*))), \tag{**}$$

where $C(D_*)$ denotes the image of the "primitive cyclic" subcomplex of D_* in $C_*(A)$ under the restriction of the cyclic trace. One way such subcomplexes arise is via rational homotopy theory. We show how (**) combines with the results of [OW] to give a "geodesic" proof of Goodwillie's computation

$$K_*(R, I) \otimes \mathbb{Q} \cong HC_{*-1}(R, I) \otimes \mathbb{Q}$$

for a nilpotent ideal I in a ring with unit R.

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We conclude section 1 with some results in characteristic $p \neq 0$. For each positive integer m and prime p we construct a stably non-trivial class $[x(m, p)]$ in $H^L_{2mp-1}(gl_N(\mathbb{F}_p))$ for $N \geqslant (mp+1)$, not accounted for by the cyclic subcomplex of $\bigwedge^* gl_N(\mathbb{F}_p)$. These elements show that the isomorphism of $(*)$ does not hold in characteristic p, or integrally.

In Section 2, we use the techniques of Section 1 to compute (in characteristic zero) the Lie algebra homology of $sl(A)$ and $st(A)$, the universal central extension of $sl(A)$. If

$$HC^{(i)}(A) = \begin{cases} HC_*(A) & \text{for} \quad * \geqslant i \\ 0 & \text{for} \quad * < i \end{cases} \quad (i = 1, 2)$$

we show that

$$\begin{aligned} H^L_*(sl(A)) &\cong S^*(HC^{(1)}_{*-1}(A)) \\ H^L_*(st(A)) &\cong S^*(HC^{(2)}_{*-1}(A)). \end{aligned} \qquad (***)$$

This is what one expects. If

$$K^{(i)}_*(A) = \begin{cases} K_*(A) & \text{for} \quad * > i \\ 0 & \text{for} \quad * \leqslant i, \end{cases}$$

where $K_*(A)$ are the algebraic K-theory groups of the discrete ring with unit A, one has the classical results

$$\begin{aligned} H_*(BE(A); \mathbb{Q}) &\cong S^*(K^{(1)}_*(A)) \otimes \mathbb{Q} \\ H_*(BSt(A); \mathbb{Q}) &\cong S^*(K^{(2)}_*(A)) \otimes \mathbb{Q}. \end{aligned}$$

The results for subcomplexes and prime characteristic apply as well for $\bigwedge^* sl(A)$ and $\bigwedge^* st(A)$. For example, the element $[x(m, p)]$ mentioned above lifts canonically to an element in $H^L_{2mp-1}(st_N(\mathbb{F}_p))$, $N \geqslant (mp+1)$.

Section 2.2 contains results on the Lie algebra homology of block-triangular matrices, among them an additive analogue of a result due to Quillen and Suslin in group homology. In Section 2.3 we prove a stability theorem for the homology of the above Lie algebras. Section 2.4 and 2.5 contain two further generalizations, in preliminary form. In Section 2.4 we show how the decomposition of $\bigwedge^* (gl(A))$ given in (1.1.6), together with the notion of primitive cyclic, generalizes to abstract Lie algebras $\mathscr{L}$ (over $\mathbb{C}$) with a given Cartan–Serre basis. This sum decomposition is none other than the eigen-space decomposition of the adjoint representation of the Cartan subalgebra H acting on $\bigwedge^* \mathscr{L}$. The Weyl group W of $\mathscr{L}$ (defined with respect to a fixed basis of simple roots of the root space) acts on $(\bigwedge^* \mathscr{L})^C$; when $(\bigwedge^* \mathscr{L})/W$ is a graded commutative Hopf algebra we conclude

$$H^L_*(\mathscr{L}) = S^*(H_*((\bigwedge^* \mathscr{L})^{PC}/W)), \qquad (****)$$

where $(\bigwedge^* \mathscr{L})^{PC}$ is the linear span in $\bigwedge^* \mathscr{L}$ of the primitive cyclic basis elements of $\bigwedge^* \mathscr{L}$. The complex $(\bigwedge^* \mathscr{L})^{PC}/W$ is the complex which plays the same role for $H^L_*(\mathscr{L})$ as the cyclic complex $C_{*-1}(A)$ does for $H^L_*(gl(A))$, and (****) is the analogue for $\mathscr{L}$ of the Loday–Quillen–Tsygan Theorem. Finally in Section 2.5 we briefly discuss which results carry over to Leibniz algebra homology. Basically everything works as before, the one exception being the construction of the torsion classes $[x(m, p)]$. It is not clear at this point whether analogues of these classes exist in Leibniz algebra homology.

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1. Lie Algebra Homology of Matrices (Revisited)

1.1. *A Loday–Quillen–Tsygan Theorem for Subcomplexes*

Throughout this paper, k will be a field. For a Lie algebra $\mathscr{L}$ over k and a left $\mathscr{L}$-module V, we recall that the Chevalley–Eilenberg complex $\bigwedge^* (\mathscr{L}; V)$ is the complex which in degree n is

$$\bigwedge^n (\mathscr{L}; V) = (\bigwedge^n \mathscr{L}) \otimes V, \qquad (1.1.1)$$

where $\bigwedge^n \mathscr{L}$ is the nth exterior power of $\mathscr{L}$ as a vector space over k. The differential d is given by

$$\begin{aligned} d(x_1 \wedge \cdots \wedge x_n \otimes v) &= \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \otimes v \\ &\quad + \sum_{i=1}^{n} (-1)^i x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes (x_i v). \end{aligned} \qquad (1.1.2)$$

When $V = k$ is the trivial representation, we denote $\bigwedge^* (\mathscr{L}; k)$ by $\bigwedge^* (\mathscr{L})$. The Lie algebra homology of $\mathscr{L}$ with coefficients in V over k can be computed as the homology of $(\bigwedge^* (\mathscr{L}; V), d_*)$ (c.f. [CE, K]). This homology will be denoted by $H^L(\mathscr{L}; V)$.

We will restrict ourselves in this section to the case $\mathscr{L} = gl_n(A)$ the Lie algebra of $n \times n$ matrices with entries in A, where A is an associative algebra with unit over k, and $V=k$. $m_{ij}(a)$ will denote the matrix with (i, j)-entry equal to a, all other entries being zero. We will call a basis of $\bigwedge^* (gl_n(A))$ *standard* if it is the extension of a basis of $gl_n(A)$ of the form $\{m_{ij}(x_\alpha)\}_{1 \leqslant i,j \leqslant n}$, where $\{x_\alpha\}$ is a basis of A over k.

DEFINITION 1.1.3. Let $x = m_{i_1 j_1}(a_1) \wedge \cdots \wedge m_{i_p j_p}(a_p)$ be a standard basis element of $\bigwedge^p (gl_n(A))$. The *support* of x, denoted $\operatorname{Supp}(x)$ is the set

$$\operatorname{Supp}(x) = \{i \mid {}^\exists k \text{ with } i = i_k \text{ or } i = j_k\}.$$

We also define numbers

$$E_{1i}(x) = \#\{i_k = i\}_{1 \leqslant k \leqslant n},\ E_{2i}(x) = \#\{j_k = i\}_{1 \leqslant k \leqslant n}.$$

The *excess* of x with respect to i is the difference

$$E_i(x) = E_{1i}(x) - E_{2i}(x).$$

DEFINITION 1.1.4. Let x be a basis element as in 1.1.3 above. x is called *cyclic* if $E_i(x) = 0$ for all $1 \leqslant i \leqslant n$, and *simple cyclic* if in addition to being cyclic one has $E_{1i}(x) \leqslant 1$ for all $1 \leqslant i \leqslant n$. A cyclic basis element x is *primitive cyclic* if it cannot be written as a wedge $x = x_1 \wedge \cdots \wedge x_m$, $m > 1$ where each x_i is cyclic. Note that if x is primitive cyclic, then it is simple cyclic.

A basis element x is *non-cyclic* if there exists an i with $E_i(x) \neq 0$.

DEFINITION 1.1.5. Define $\bigwedge^* (gl_n(A))^C$ to be the subspace of $\bigwedge^* (gl_n(A))$ generated by cyclic basis elements, and $\bigwedge^* (gl_n(A))^{NC}$ to be the subspace generated by non-cyclic basis elements.

Clearly $\bigwedge^* (gl_n(A)) \cong \bigwedge^* (gl_n(A))^C \oplus \bigwedge^* (gl_n(A))^{NC}$ as vector spaces. In fact we may further decompose $\bigwedge^* (gl_n(A))$ as

$$\bigwedge^* (gl_n(A)) \cong \bigwedge^* (gl_n(A))^C \oplus \bigoplus_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha \neq 0}} \bigwedge^* (gl_n(A))^{NC(\alpha)}, \tag{1.1.6}$$

where for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$, $\bigwedge^* (gl_n(A))^{NC(\alpha)}$ is the subspace generated by standard basis elements x with $E_i(x) = \alpha_i$ for each i, $1 \leqslant i \leqslant n$.

PROPOSITION 1.1.7. *The decomposition of* (1.1.6) *is a decomposition into subcomplexes.*

Proof. If x is a standard basis element, then an inspection of the differential d in (1.1.2) shows that $d(x) = \sum_i \pm \lambda_i x_i$, where for each i, x_i is a basis element in dimension one less with $E_j(x_i) = E_j(x)$. Thus d preserves excess with respect to each index integer j. The result follows. ∎

PROPOSITION 1.1.8.

(i) *If* char. $k=0$, *the inclusion*

$$\wedge^* (gl_n(A))^C \subsetneq \wedge^* (gl_n(A))$$

is a quasi-isomorphism (i.e., induces an isomorphism on homology).

(ii) *If* char. $k=p\neq 0$, *the inclusion*

$$\left(\wedge^* (gl_n(A))^C \oplus \bigoplus_{\substack{\alpha\in p\cdot\mathbb{Z}^n\\ \alpha\neq 0}} \wedge^* (gl_n(A))^{NC(\alpha)}\right) \subsetneq \wedge^* (gl_n(A))$$

is a quasi-isomorphism.

Proof. The adjoint action of the element $m_{ii}(1)\in(gl_n(k))$ (i an integer between 1 and n) preserves the decomposition of (1.1.6), for if x is a standard basis element of $\wedge^* (gl_n(A))^{NC(\alpha)}$ for some fixed n-tuple of integers α, then $\operatorname{ad}(m_{ii}(1))(x)=\alpha_i\cdot x$. As the action of $\operatorname{ad}(m_{ii}(1))$ is trivial on homology, this implies $\wedge^* (gl_n(A))^{NC(\alpha)}$ is acyclic for all $\alpha\neq 0$ when char. $k=0$ and for all $\alpha\notin p\cdot\mathbb{Z}^n$ when char. $k=p$. ▮

Remark 1.1.9. When $k\subseteq\mathbb{C}$, one can use the action of $GL_n(k)$ to give an alternative proof of 1.1.8(i). Recall that this action is given by

$$g\cdot(x_1\wedge\cdots\wedge x_n)=x_1^g\wedge x_2^g\wedge\cdots\wedge x_n^g,$$

the action on each term induced by the conjugation action of $GL_n(k)$ on $gl_n(k)$. A local multi-variable Taylor series expansion, together with the fact that after complexification the action in this case differentiates to the additive adjoint action of $gl_n(k)$ (compare [Kn, p. 302]) shows that the conjugation action on $GL_n(k)$ acts as the identity on homology. If $d_{ii}(r)$ is the diagonal matrix with

$$d_{ii}(r)_{jj}=\begin{cases}1 & \text{if } i\neq j\\ r\neq 0 & \text{if } i=j\end{cases}$$

then the action of $d_{ii}(r)$ preserves the decomposition of (1.1.6), since $d_{ii}(r)\cdot x=(r^{\alpha_i})x$ for $x\in\wedge^* (gl_n(A))^{NC(\alpha)}$, $\alpha=(\alpha_1,\ldots,\alpha_n)$. Choosing $r\neq 1$ and letting i vary, we conclude (as before) that $\wedge^* (gl_n(A))^{NC(\alpha)}$ is acyclic for $\alpha\neq 0$. This ends the remark.

The standard inclusion $gl_n(A)\subsetneq gl_{n+1}(A)$ which sends M to

$$M\oplus 0=\left[\begin{array}{c|c} M & 0\\ \hline 0 & 0\end{array}\right]$$

is a Lie algebra homomorphism; stabilizing with respect to n yields the Lie algebra $gl(A) = \varinjlim_n gl_n(A)$. The decomposition of (1.1.7) is preserved under stabilization, where on index sets $\mathbb{Z}^n$ maps to $\mathbb{Z}^{n+1}$ by $(\alpha_1, ..., \alpha_n) \mapsto (\alpha_1, ..., \alpha_n, 0)$. Let $\mathbb{Z}^\infty = \varinjlim_n \mathbb{Z}^n$. Then there is an isomorphism of complexes

$$\wedge^* (gl(A)) \cong \wedge^* (gl(A))^C \oplus \bigoplus_{\substack{\alpha \in \mathbb{Z}^\infty \\ \alpha \neq 0}} \wedge^* (gl(A))^{NC(\alpha)}, \tag{1.1.10}$$

where $\wedge^* (gl(A))^C = \varinjlim_n \wedge^* (gl_n(A))^C$ and $\wedge^* (gl(A))^{NC(\alpha)}$ consists of those x for which $E_i(x) = \alpha_i \; \forall i \in \mathbb{N}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., 0, 0, ...)$.

We will denote by $\wedge^* (gl_n(A))^{\text{sim}}$ $(n \leqslant \infty)$ the linear span of the simple cyclic basis elements and $\wedge^* (gl_n(A))^{PC}$ the linear span of the primitive cyclic basis elements. These are subcomplexes. If $D_* \subset \wedge^* (gl_n(A))$ is a subcomplex, let $D_*^C = D_* \cap \wedge^* (gl_n(A))^C$ and $D_*^{NC(\alpha)} = D_* \cap \wedge^* (gl_n(A))^{NC(\alpha)}$.

In what follows, we will assume that

P1. There exists a standard basis of $\wedge^* (gl(A))$ which restricts to a basis for D_* over k.

PROPOSITION 1.1.11. *Let* char. $k = 0$.

(1) *Suppose that for each* i, $\text{ad}(m_{ii}(1)) : D_* \to D_*$ *is trivial on homology. Then the inclusion*

$$D_*^C \subsetneq D_*$$

is a quasi-isomorphism.

(2) *If for each* $1 \leqslant i \leqslant n$, $r \in \mathbb{Q}$ *the action of* $d_{ii}(r)$ *restricts to a chain map* $d_{ii}(r) : D_* \to D_*$ *which acts as the identity on homology, then* $D_*^C \subsetneq D_*$ *is a quasi-isomorphism.*

Condition P1 implies the decomposition given in (1.1.6) holds with D_* in place of $\wedge^* (gl_n(A))$. The proof is the same as in Proposition 1.1.8.

Propositions 1.1.8 and 1.1.11 apply in the case $n = \infty$ just as they do for finite n. If $D_* \subset \wedge^* (gl_n(A))$ then we set $D_*^{\text{sim}} = D_* \cap \wedge^* (gl_n(A))^{\text{sim}}$ and $D_*^{PC} = D_* \cap \wedge^* (gl_n(A))^{PC}$.

THEOREM 1.1.12. *Assume* char. $k = 0$. *Let* D_* *be a subcomplex of* $\wedge^* (gl(A))$ *satisfying* P1 *as well as the following properties*:

P2. *The conjugation action of* $W(\mathbb{Q})$ *(infinite monomial matrices with entries in* $\mathbb{Q}$*) on* $\wedge^* (gl(A))$ *restricts to* D_* *to give an action which is trivial on* $H_*(D_*)$.

P3. *D_* is closed under the product induced by block-sum on $gl(A)$.*

P4. *D_* is a sub-co-algebra of $\bigwedge^*(gl(A))$ under Δ (c.f.* (1.1.C) *below).*

Then $H_*(D_*) \cong S^*(H_{*-1}(C(D_*)))$, where $C(D_*)$ is the image of $D^{PC} \subset \bigwedge^*(gl(A))$ under the cyclic trace map $Tr : \bigwedge^*(gl(A)) \to C_{*-1}(A)$ (see (1.1.13) *below).*

Proof. As $W(\mathbb{Q})$ contains the (infinite invertible) diagonal matrices with entries in $\mathbb{Q}$, we have a quasi-isomorphism $D^C_* \subsetneq D_*$ (Prop. 1.1.11(2) and property P2). Let $\bar{D}_* = D_*/D^{NC}_*$ be the quotient complex of D_* by $D^{NC}_* = \bigoplus_{0 \neq \alpha \in \mathbb{Z}^\infty} D^{NC(\alpha)}_*$. As Σ_∞ acts trivially on $H_*(D_*) = H_*(\bar{D}_*)$, there is a sequence of quasi-isomorphisms

$$D_* \to \bar{D}_* \to \bar{D}_*/\Sigma_\infty.$$

The latter complex is well-defined, because the decomposition $D_* \cong D^C_* \oplus D^{NC}$ is preserved under the action of $W(\mathbb{Q})$.

Now consider the coproduct $\Delta : \bigwedge^*(gl(A)) \to \bigwedge^*(gl(A)) \otimes \bigwedge^*(gl(A))$. Let x be a standard basis element $x_1 \wedge \cdots \wedge x_n$, S an ordered subset of $\underline{n} = (1, 2, ..., n)$; if $S = (i_1, ..., i_s)$, let $x(S) = x_{i_1} \wedge \cdots \wedge x_{i_s}$. Then $\Delta(x)$ is given by the formula (c.f. [MM])

$$\Delta(x) = \sum_{S \subseteq n} (-1)^{\mathrm{sgn}(\sigma(S))} x(S) \otimes x(S^c), \tag{1.1.C}$$

where S^c denotes the ordered complement of S in $\underline{n}$ and $\sigma(S)$ is the permutation which sends n to $S \amalg S^c$. According to P4, Δ restricts to a coproduct on D_*. If x is a standard basis element of $(\bigwedge^*(gl(A))^{NC})$ contained in D_*, then

$$\Delta(x) \in D^C \otimes D^{NC} + D^{NC} \otimes D^C + D^{NC} \otimes D^{NC}$$

as one sees from (1.1.C). It follows that on the quotient complex $\bar{D}_*$, Δ induces a coproduct $\bar{\Delta} : \bar{D}_* \to \bar{D}_* \otimes \bar{D}_*$. This coproduct descends to $\bar{D}/\Sigma_\infty$. On the other hand D^{NC}_*/Σ_∞ is an ideal in D_*/Σ_∞ with respect to the product structure induced by block sum. This product on D_*/Σ_∞ induces an associative product on $\bar{D}_*/\Sigma_\infty$; together with the previously defined coproduct structure, $\bar{D}_*/\Sigma_\infty$ is a graded commutative Hopf algebra. Now consider a cyclic basis element $x \in D^C_n$. The component $\Delta(x)^C$ of $\Delta(x)$ lying in $D^C_* \otimes D^C_*$ is exactly

$$\Delta(x)^C = \Sigma \pm x_\alpha \otimes x_\beta, \tag{1.1.CF}$$

where the sum is over all standard basis elements $x_\alpha \in D^C_k$, $x_\beta \in D^C_{n-k}$ $(0 \leqslant k \leqslant n)$ such that $x = x_\alpha \wedge x_\beta$. It follows that the image $\bar{x}$ of x in the Hopf algebra $\bar{D}_*/\Sigma_\infty$ is primitive if and only if x is primitive cyclic (in the sense of definition 1.1.4).

Let $\tilde{D}_* \subset \bar{D}_*/\Sigma_\infty$ denote the linear span of such elements. We claim that $\tilde{D}_*$ is the subcomplex of primitives (or said another way, that any primitive element is a linear combination of primitive cyclic basis elements). For suppose that $x = \sum_{i \in T} \lambda_i x_i$ is a sum of (scalar multiples of) linearly independent basis elements x_i which is primitive. We assume without loss of generality that the sum is minimal, in the sense that there is no partition $\{T_1, T_2\}$ of the indexing set T with $\sum_{i \in T_j} \lambda_i x_i$ primitive for both $j=1$ and 2. The coproduct for $\bar{D}$ on a standard basis element y satisfies the following property:

(R) if $y_\alpha \otimes y_\beta$ appears in the expansion of $\Delta(y)$, then $y = y_\alpha \wedge y_\beta$, up to a sign.

Now suppose that $i_1 \in T$ and $|T| > 1$. Then x_{i_1} is not primitive; let $x_\alpha \otimes x_\beta$ be a term appearing in the expansion of $\Delta(x_{i_1})$, where neither x_α or x_β is a scalar multiple of 1. Then, up to scalar multiplication, this term must cancel with a term in the expansion of $\Delta(x_{i_2})$, for some $i_2 \in T - \{i_1\}$. By (R) above, this implies that x_{i_2} is a scalar multiple of x_{i_1}, contradicting the assumption that the original set of elements $\{x_i\}$ is linearly independent. This argument descends to $\bar{D}$. We may therefore conclude that T has cardinality 1, verifying the claim.

Recall that there is a cyclic trace

$$Tr_n(M_1 \wedge M_2 \wedge \cdots \wedge M_n) = \sum_{i_1, i_2, \ldots, i_n} ((M_1)_{i_1 i_2}, (M_2)_{i_2 i_3}, \ldots, (M_n)_{i_n i_1}) \tag{1.1.13}$$

from the Chevalley–Eilenberg complex of $gl(A)$ to the cyclic complex of A (c.f. [LQ]). Tr_* is invariant under the action of Σ_∞, hence descends to define a map

$$\overline{Tr}_* : \bar{D}_*/\Sigma_\infty \to C_{*-1}(A).$$

Now $\tilde{D}_*$ is a subcomplex of $\bar{D}_*/\Sigma_\infty$ which is *isomorphic* as a complex to its image $C(D_*)$ under the cyclic trace. This isomorphism is seen by noting that a basis element $x \in D_n^C$ is primitive cyclic if and only if it is of the form

$$x = m_{i_1 i_2}(a_1) \wedge m_{i_2 i_3}(a_2) \wedge \cdots \wedge m_{i_n i_1}(a_n),$$

where $i_j \neq i_k$ if $j \neq k$. Appealing to the theorem of Milnor and Moore [MM] then completes the proof. ∎

Remark 1.1.14. Applying theorem 1.1.12 to $D_* = \bigwedge^* (gl(A))$ we get the Loday–Quillen–Tsygan result, which states that there is an isomorphism of graded commutative Hopf algebras

$$H_*^L(gl(A)) \cong S^*(HC_{*-1}(A)).$$

The conclusion of theorem 1.1.12 holds in other cases: adopt the convention that for $z \in gl(A)$, $z_1 \wedge z_2 \wedge \cdots \wedge z_k \in \bigwedge^k (gl(A))$,

$$[z, z_1 \wedge z_2 \wedge \cdots \wedge z_k]$$
$$= \left(\sum_{i=1}^{k} (-1)^{i+1} [z, z_i] \wedge z_1 \cdots \wedge \hat{z}_i \wedge \cdots \wedge z_k \right) \in \bigwedge^k (gl(A)),$$

where $\hat{z}_i$ means omit z_i.

LEMMA 1.1.15. *Let* char. $k=0$, *and* D_* *be a subcomplex of* $\bigwedge^* (gl(A))$ *closed under the conjugation action of* Σ_∞ *satisfying properties* P1 *and* P2. *Assume additionally that for each* n *and* $x \in D_*(n)$, $m_{iN}(1) \wedge [m_{Ni}(1), x] \in D_*$ *for all* $1 \leqslant i \leqslant n$, *for some* $N > n$. *Then*

$$H_*(D_*) \cong H_*(D_*^{\text{sim}}) \cong (H_{*-1}(C(D_*))),$$

where $C(D_*)$ *is as in* (1.1.12).

Proof. For a cyclic basis element x, let $\phi(x) = \sum_{i, E_{1i}(x)>0} (E_{1i}(x) - 1)$. For each integer $m \geqslant 0$, define $F_m D_*$ to be the linear span of those cyclic basis elements x in D_* with $\phi(x) \leqslant m$. $\{F_m D_*\}$ is an increasing filtration of D_* with $F_0 D_* = D_*^{\text{sim}}$ and $D_* = \varinjlim_m F_m D_*$. Let x be a cyclic basis element in $F_m D_*(n)$ with $m > 0$, and i an index such that $E_{1i}(x) > 1$. The element

$$\tilde{x} = m_{iN}(1) \wedge [m_{Ni}(1), x]$$

is in $F_m D_*$. If $d(x) \in F_{m-1} D_*$, a straighforward computation yields

$$d(\tilde{x}) = n_i x + x'$$

where $n_i = E_{1i}(x)$, and $x' \in F_{m-1} D_*$. This implies $H_*(F_m D_* / F_{m-1} D_*) = 0$ for $m \geqslant 1$, hence the inclusion $F_0 D_* \subsetneq D_*$ is a quasi-isomorphism.

A cyclic basis element $x \in D_*^{\text{sim}}$ can be written uniquely as

$$x = x_1 \wedge \cdots \wedge x_n,$$

where x_i is a primitive cyclic basis element and $\text{Supp}(x_i) \cap \text{Supp}(x_j) = \phi$ if $i \neq j$. From this we conclude that

$$H_*(D_*) = H_*(D_*^{\text{sim}}) = S^* H_*(D_*^{PC}).$$

By property P2, Σ_∞ acts trivially on $H_*(D_*)$ hence also on $H_*(D_*^{PC})$. The rest follows as in the proof of Theorem 1.1.12. ∎

1.2. *Application to Relative K-Theory*

Theorem 1.1.12 can be used to streamline the arguments of [OW] (we refer the reader to this paper for details and terminology).

Let $\underline{C}(1)$ denote the category with objects $(\underline{n}; \alpha)$ where $\underline{n} = (1, 2, ..., n)$ and α is a partial ordering of $\underline{n}$. A morphism $\varphi : (\underline{n}, \alpha) \to (\underline{m}, \beta)$ in $\underline{C}(1)$ is an inclusion of sets $\underline{n} \rightarrowtail \underline{m}$ such that $i <^{\alpha} j$ implies $\varphi(i) <^{\beta} \varphi(j)$. If R is a discrete ring with unit, I an ideal in R, define a functor F: $\underline{C}(1) \to (\text{spaces})_*$ by

$$F((\underline{n}, \alpha)) = BT_n^{\alpha}(R, I),$$

where $T_n^{\alpha}(R, I) \subset GL_n(R)$ is the subgroup consisting of those g with

$$\begin{aligned} &g_{ii} \in Id + I \\ &g_{ij} \in I \qquad \text{if} \quad i \not<^{\alpha} j \qquad (i \neq j)) \end{aligned}$$

If $K(R, I) =$ homotopy fibre $(BGL(R)^+ \to BGL(R/I)^+)$, then we have a homology isomorphism (Th. 6.1, [OW]):

$$\underset{\underline{C}(1)}{\text{hocolim}}\, F \simeq \varinjlim_{n} \bigcup_{\alpha} BT_n^{\alpha}(R, I) \to K(R, I), \tag{1.2.1}$$

where hocolim denotes the basepointed homotopy colimit construction of Bousfield and Kan ([BK], Chap. XII). When I is a nilpotent ideal in R, rational homotopy theory produces an explicit sequence of quasi-isomorphisms (c.f., Section 6 of [OW]):

$$\begin{aligned} &C_*(\underset{\underline{C}(1)}{\text{hocolim}}\, F; \mathbb{Q}) \\ &\quad \xrightarrow{\simeq} C_*(\underset{\underline{C}(1)}{\text{hocolim}}\, F_{\mathbb{Q}}; \mathbb{Q}) \\ &\quad \xleftarrow{\simeq} \underset{\underline{C}(1)}{\text{hocolim}}\, C_*(F_{\mathbb{Q}}; \mathbb{Q}) \\ &\quad \xleftarrow{\simeq} \underset{\underline{C}(1)}{\text{hocolim}}\, G_{\mathbb{Q}} \\ &\quad \xrightarrow{\simeq} \underset{\underline{C}(1)}{\text{colim}}\, G_{\mathbb{Q}}. \end{aligned} \tag{1.2.2}$$

Here $F_{\mathbb{Q}}$: $\underline{C}(1) \to (\text{spaces})_*$ is given by

$$F_{\mathbb{Q}}((\underline{n}, \alpha)) = BT_n^{\alpha}(R \otimes_{\mathbb{Z}} \mathbb{Q}, I \otimes_{\mathbb{Z}} \mathbb{Q})$$

and $G_{\mathbb{Q}}$ is the functor

$$G_{\mathbb{Q}}((\underline{n}, \alpha)) = \wedge^* (t_n^{\alpha}(R \otimes_{\mathbb{Z}} \mathbb{Q}, I \otimes_{\mathbb{Z}} \mathbb{Q})),$$

where $t_n^\alpha(S, J)$ is the Lie algebra corresponding to $T_n^\alpha(S, J)$; m is an element of $t_n^\alpha(S, J)$ iff $m_{ij} \in J$ whenever $i = j$ or $i \not<^\alpha j$. Each $\bigwedge^* (t_n^\alpha(R \otimes_{\mathbb{Z}} \mathbb{Q}, I \otimes_{\mathbb{Q}} \mathbb{Q}))$ is naturally a subcomplex of $\bigwedge^* (gl_n(R \otimes_{\mathbb{Z}} \mathbb{Q}))$. Finally, $\operatorname{colim}_{\underline{C}(1)} G_{\mathbb{Q}}$ is a subcomplex of $\bigwedge^* (gl(R \otimes \mathbb{Q}))$; we denote this subcomplex by $\Sigma_\alpha \bigwedge_* (t^\alpha(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}))$.

Claim 1.2.3. $\Sigma_\alpha \bigwedge^* (t^\alpha(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}))$ satisfies conditions P1–P4 of Theorem 1.1.12.

Proof. The sequence of quasi-isomorphisms in (1.2.2) starting with the second line are all equivariant with respect to the conjugation action of $W(\mathbb{Q})$. $\operatorname{hocolim}_{\underline{C}(1)} F_{\mathbb{Q}}$ is closed under this action, which must be trivial on homology in light of the homology isomorphism (1.2.1). This verifies property P2. P3 is obvious, and P1 follows by choosing a standard basis of $\bigwedge^*(gl(R \otimes \mathbb{Q}))$ induced by a basis for $R \otimes \mathbb{Q}$ which at the level of *vector spaces* is compatible with the vector space decomposition $R \otimes \mathbb{Q} \cong R/I \otimes \mathbb{Q} \oplus I \otimes \mathbb{Q}$. P4 is clear because $\Sigma_\alpha \bigwedge^* (t^\alpha(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}))$ is a sum of the coalgebras $\bigwedge^*(t^\alpha(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}))$. ∎

The image of the primitive cyclic subcomplex of $\Sigma_\alpha \bigwedge^* (t^\alpha(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}))$ under the cyclic trace is easily seen to be the complex

$$C_*(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}) \overset{\text{def}}{=} \ker(C_*(R \otimes \mathbb{Q}) \to C_*(R/I \otimes \mathbb{Q})).$$

This complex computes the relative cyclic homology of the pair $(R \otimes \mathbb{Q}, I \otimes \mathbb{Q})$. The quasi-isomorphisms of (1.2.2) together with (1.2.1) and Theorem 1.1.12 imply by Milnor and Moore that one has a rational isomorphism

$$\begin{aligned} K_*(R, I) \otimes \mathbb{Q} &\cong K_*(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}) \\ &\cong HC_{*-1}(R \otimes \mathbb{Q}, I \otimes \mathbb{Q}) \end{aligned} \tag{1.2.4}$$

when I is a nilpotent ideal in R. This result is due to Goodwillie (c.f. [G1, G2]).

1.3. *The Characteristic p and Integral Cases*

Let p be a non-zero prime, m a positive integer. Let

$$\begin{aligned} x(m, p) = m_{13}(1) \wedge m_{32}(1) \wedge m_{14}(1) \wedge m_{42}(1) \\ \wedge \cdots \wedge m_{1l}(1) \wedge m_{l2}(1) \wedge m_{12}(1) \end{aligned} \tag{1.3.1}$$

with $l = (mp + 1)$.

PROPOSITION 1.3.2. *The class of* $x(m, p)$ *(denoted* $[x(m, p)]$*) is non-trivial in* $H^L_{2mp-1}(gl_q(\mathbb{F}_p))$*, where* $q \geqslant (mp+1)$.

Proof. Consider the subcomplex $(\bigwedge^* gl_q(\mathbb{F}_p))^{NC(\alpha_{m,p})}$ where $\alpha_{m,p} = (mp, -mp, 0, 0, 0, ...)$. $x(m, p)$ lies in the subspace $(\bigwedge^{2mp-1}(gl_q(\mathbb{F}_p)))^{NC(\alpha_{m,p})}$. Let

$$\chi_p : (\bigwedge^{2mp-1}(gl_q(\mathbb{F}_p)))^{NC(\alpha_{m,p})} \to \mathbb{F}_p$$

be the linear map which assigns to a basis element x of the form

$$\begin{aligned} x = m_{1i_1}(1) \wedge m_{i_1 2}(1) \wedge m_{1i_2}(1) \wedge m_{i_2 2}(1) \\ \wedge \cdots \wedge m_{1i_{mp-1}}(1) \wedge m_{i_{mp-1}2}(1) \wedge m_{12}(1) \end{aligned}$$

the number

$$\#\{i \mid E_i(x) = 0\} \text{ reduced mod } p.$$

Then it is easy to see that

$$\chi(x(m, p)) = -1, \qquad \text{while} \quad \chi(dy) = 0 \text{ for all } y \in (\bigwedge^{2mp}(gl_q(\mathbb{F}_p)))^{NC(\alpha_{m,p})}.$$

This shows that the homology class of $x(m, p)$ is non-zero. ∎

COROLLARY 1.3.3. *The homology group* $H^L_{2mp-1}(\bigwedge^* gl_q(\mathbb{F}_p)^{NC(\alpha_{m,p})})$ *is non-zero whenever* $q \geqslant (mp+1)$.

This corollary illustrates the failure of Theorem 1.1.12 when char. $k \neq 0$. The Chevalley–Eilenberg complex computes the Lie algebra homology over a commutative ring R for Lie algebras $\mathscr{L}$ which are free as R-modules. The decomposition in (1.1.6) applies here; in particular it applies when $R = \mathbb{Z}$ and $\mathscr{L} = gl_q(\mathbb{Z})$. The element $x(m, p)$ as defined in (1.3.1) lifts to a cycle in $\bigwedge^{2mp-1}(gl_q(\mathbb{Z}))$. Thus we have

COROLLARY 1.3.4. *The homology* $H^L_{2mp-1}(gl_q(\mathbb{Z})) \neq 0$ *for* $q \geqslant (mp+1)$.

On $(\bigwedge^* gl_q(\mathbb{Z}))^{NC(\alpha)}$ $(\alpha = (\alpha_1, \alpha_2, ..., \alpha_q) \in \mathbb{Z}^q)$, we still have $\mathrm{ad}(m_{ii}(1))$ acting as multiplication by α_i for each i. Hence multiplication by c on $H^L_*((\bigwedge gl_q(\mathbb{Z}))^{NC(\alpha)})$ is zero where $c = \text{g.c.d.}(\alpha_1, ..., \alpha_q)$. Over $\mathbb{Z}$ the map χ_p lifts to

$$\begin{aligned} \chi \colon \bigwedge^{2mp-1}(gl_q(\mathbb{Z}))^{NC(\alpha_{m,p})} &\to \mathbb{Z}/mp\mathbb{Z} \\ v &\mapsto \chi(v) \text{ reduced mod } mp. \end{aligned}$$

This map is zero on $d(\bigwedge^{2mp}(gl_q(\mathbb{Z})))$, and -1 on $x(m, p)$. We conclude that $x(m, p)$ viewed as an element of $H^L_{2mp-1}(gl_q(\mathbb{Z}))$ has order mp.

In what follows, $k=\mathbb{F}_p$ (p prime), or $\mathbb{Z}$. The action of Σ_q permutes the summands $H^L_{2mp-1}(\bigwedge^* gl_q(k))^{NC(\alpha)} = H^L_{2mp-1}(\bigwedge^* gl_q(k)^{NC(\alpha)})$ in the obvious way:

$$\text{if} \quad x \in H^L_{2mp-1}(\textstyle\bigwedge^* gl_q(k))^{NC(\alpha)} \text{ and } \sigma \in \Sigma_q,$$

then

$$\sigma x \in H^L_{2mp-1}(\textstyle\bigwedge^* gl_q(k))^{NC(\sigma(\alpha))}.$$

Applying the action of Σ_q, we see from the above arguments that each of the summands $H^L_*(\bigwedge^* gl_q(k))^{NC(\alpha_{i,j})}$ is non-trivial, where $\alpha_{i,j}$ is the q-tuple with (mp) in the ith place, $(-mp)$ in the jth place, and zero elsewhere. So for all $q \geqslant mp+1$ and $\sigma \in \Sigma_q$, there exists an $x \in H^L_{2mp-1}(\bigwedge^* gl_q(k))$ such that $\sigma x \neq x$. We conclude that

COROLLARY 1.3.5. *The conjugation action of Σ_q on $H^L_{2mp-1}(\bigwedge^* gl_q(k))$ induces a faithful representation for all primes p and $q \geqslant mp+1$. In particular, the action of $GL_q(k)$ on $H^L_*(\bigwedge^* gl_q(k))$ is non-trivial for all $q \geqslant p+1$ when $k=\mathbb{F}_p$ and $q \geqslant 4$ when $k=\mathbb{Z}$.*

2. Homology of More General Lie Algebras

2.1. *Lie Algebra Homology of $sl(A)$ and $st(A)$*

Recall that for $n \geqslant 1$ $sl_n(A)$ is the commutator $[gl_n(A), gl_n(A)]$. As a vector space, $sl_n(A)$ is spanned by

$$\begin{aligned} &m_{ij}(a),\ a \in A, \qquad 1 \leqslant i \neq j \leqslant n \\ &m_{ii}(ab) - m_{jj}(ba) = h_{ij}(a,b), \qquad 1 \leqslant i, j \leqslant n. \end{aligned} \tag{2.1.1}$$

For A an algebra over a field k, a standard basis of $sl_n(A)$ is given by the elements in (2.1.1) where a and b are basis elements of A over k. When $n \geqslant 3$, $sl_n(A)$ admits a universal central extension $st_n(A)$ given as in [B] and [KL]. Recall that $st_n(A)$ is the Lie algebra spanned by

$$u_{ij}(a), \qquad a \in A \quad 1 \leqslant i \neq j \leqslant n \tag{2.1.2}$$

with relations

(i) $u_{ij}(\lambda a + \mu b) = \lambda u_{ij}(a) + \mu u_{ij}(b)$ for $\lambda, \mu \in k$; $a, b \in A$,

(ii) $[u_{ij}(a), u_{kl}(b)] = \begin{cases} 0 & \text{if } i \neq l, j \neq k \\ u_{il}(ab) & \text{if } i \neq l, j = k. \end{cases}$

The commutator $[u_{ij}(a), u_{ji}(b)]$ is denoted by $\tilde{h}_{ij}(a, b)$. Having fixed a basis for A, a standard basis element for $st_n(A)$ is of the form $u_{ij}(a)$ or $\tilde{h}_{ij}(a, b)$ where a and b are basis elements of A. These bases extend to give standard bases for $\bigwedge^* sl_n(A)$ and $\bigwedge^* st_n(A)$. As in Section 1.1 we define

$$\begin{aligned} E_{1i}(m_{jk}(a)) &= E_{1i}(u_{jk}(a)) = \delta^i_j \\ E_{2i}(m_{jk}(a)) &= E_{2i}(u_{jk}(a)) = \delta^i_k \\ E_{ij}(h_{kl}(a, b)) &= E_{ij}(\tilde{h}_{kl}(a, b)) = 0 \qquad i = 1, 2. \end{aligned} \tag{2.1.3}$$

If $x = x_1 \wedge \cdots \wedge x_m$ is a standard basis element of $\bigwedge^m sl_n(A)$ (resp. $\bigwedge^m st_n(A)$) we set

$$E_{ki}(x) = \sum_{j=1}^{m} E_{ki}(x_j) \qquad k = 1, 2$$

and define the excess of x with respect to i as

$$E_i(x) = E_{1i}(x) - E_{2i}(x).$$

The functions $E_{1i}(\text{---})$, $E_{2i}(\text{---})$ are compatible under the maps $st_n(A) \twoheadrightarrow sl_n(A)$ and $sl_n(A) \rightarrowtail gl_n(A)$. As with $gl_n(A)$, we may define for an n-tuple $\alpha \in \mathbb{Z}^n$ subspaces

$$\bigwedge^* (sl_n(A))^{NC(\alpha)}, \qquad \bigwedge^* (st_n(A))^{NC(\alpha)} \tag{2.1.4}$$

consisting of those x with $E_i(x) = \alpha_i \; \forall 1 \leqslant i \leqslant n$.

For $\alpha = 0$ we write the superscript as C instead of $NC(0)$. The same arguments as in the case of $gl_n(A)$ apply to show

PROPOSITION 2.1.5. *The decompositions*

$$\bigwedge^* (sl_n(A)) \cong \bigwedge^* (sl_n(A))^C \bigoplus_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha \neq 0}} \bigwedge^* (sl_n(A))^{NC(\alpha)}$$

$$\bigwedge^* (st_n(A)) \cong \bigwedge^* (st_n(A))^C \bigoplus_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha \neq 0}} \bigwedge^* (st_n(A))^{NC(\alpha)}$$

are decompositions of complexes.

We also have

PROPOSITION 2.1.6. *If* char. $k = 0$ *the inclusions*

$$\bigwedge^* (sl_n(A))^C \subsetneq \bigwedge^* (sl_n(A)) \qquad n \geqslant 1$$

$$\bigwedge^* (st_n(A))^C \subsetneq \bigwedge^* (st_n(A)) \qquad n \geqslant 3$$

are quasi-isomorphisms.

Proof. The element $h_{ij}(1, 1) = m_{ii}(1) - m_{jj}(1)$ lies in $sl_n(k) \subseteq sl_n(A)$, and $\operatorname{ad}(h_{ij}(1, 1))$ preserves the decomposition of Proposition 2.1.5. Precisely, if $x \in \bigwedge^* (sl_n(A))^{NC(\alpha)}$, then

$$\operatorname{ad}(h_{ij}(1, 1))(x) = (E_i(x) - E_j(x))x = (\alpha_i - \alpha_j)x.$$

On the other hand, from the definition of $E_i(x)$ one has the additional relation $\sum_{i=1}^n E_i(x) = 0$. As char. $k = 0$, the fact that $\operatorname{ad}(h_{ij}(1, 1))$ acts trivially on homology for all $1 \leqslant i, j \leqslant n$; $i \neq j$ implies $\bigwedge^* (sl_n(A))^{NC(\alpha)}$ is acyclic. The same argument applies to $st_n(A)$ in both cases once we know that the action of $\operatorname{ad}(\tilde{h}_{ij}(1, 1))$ on a basis element x of $\bigwedge^* (st_n(A))$ is given by

$$\operatorname{ad}(\tilde{h}_{ij}(1, 1))(x) = (E_i(x) - E_j(x))x = (\alpha_i - \alpha_j)x.$$

This result follows by the Steinberg identities in (2.1.2), from which one derives the identity $\tilde{h}_{ij}(1, 1) + \tilde{h}_{jk}(1, 1) + \tilde{h}_{ki}(1, 1) = 0$, $i \neq j \neq k$ (see also [KL]). ∎

Following Section 1, we call a basis element $x \in \bigwedge^* sl_n(A)$ or $\bigwedge^* st_n(A)$ *cyclic* if $E_i(x) = 0$ for all $1 \leqslant i \leqslant n$ $(n \leqslant \infty)$, *simple cyclic* if it is cyclic with $E_{i1}(x) \leqslant 1$ for all i, and *primitive cyclic* if it cannot be written as a wedge product $x = x_1 \wedge x_2 \wedge \cdots \wedge x_k$ of cyclic elements x_i with $k > 1$.

$$\text{Let } HC_*^{(i)}(A) = \begin{cases} HC_*(A) & \text{if } \; * \geqslant i \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2. \tag{2.1.7}$$

THEOREM 2.1.8. *If* char. $k = 0$, *we have isomorphisms*

$$H_*^L(sl_n(A)) \cong S^*(HC_{*-1}^{(1)}(A))$$
$$H_*^L(st(A) \cong S^*(HC_{*-1}^{(2)}(A)).$$

Proof. The alternating subgroup $A_\infty = [\Sigma_\infty, \Sigma_\infty]$ embeds as a subgroup of $SL(A)$. Therefore conjugation by an element of A_∞ on $\bigwedge^* sl(A)$ acts as the identity map on homology. Following Theorem 1.1.12, we have quasi-isomorphisms

$$\textstyle\bigwedge^* sl(A) \supsetneq (\bigwedge^* sl(A))^C \twoheadrightarrow (\bigwedge^* sl(A))^C/A_\infty.$$

Block-sum and the coalgebra structure of $\bigwedge^* sl(A)$ induce a Hopf algebra structure on $(\bigwedge^* sl(A))^C/A_\infty$. The coproduct formula in this complex is the same as before (see (1.1.CF)), identifying the subcomplex of primitives as

$$(\textstyle\bigwedge^* sl(A))^{PC}/A_\infty.$$

If a basis element $x \in \bigwedge^p sl(A)$ is primitive cyclic and $p > 1$, then $[x]$ is of the form

$$[x] = [m_{i_1 i_2}(a_1) \wedge m_{i_2 i_3}(a_2) \wedge \cdots \wedge m_{i_{p-1} i_p}(a_{p-1}) \wedge m_{i_p i_1}(a_p)],$$
$$i_j \neq i_k \text{ if } j \neq k. \tag{2.1.9}$$

Therefore above dimension 1, $(\bigwedge^* sl(A))^{PC}/A_\infty$ is isomorphic under the cyclic trace to the subcomplex $C_*^{(1)}(A)$ defined as the kernel of the augmentation map

$$C_*(A) \overset{\varepsilon}{\twoheadrightarrow} HC_0(A). \tag{2.1.10}$$

We have $H_*(C_*^{(1)}(A)) = HC_*^{(1)}(A)$. Together with

$$H_1^L(sl(A)) = 0$$
$$H_2^L(sl(A)) = HC_1(A)$$

(c.f. [KL]), this identifies the primitives of $H_*^L(sl(A))$ with $HC_*^{(1)}(A)$.

To deal with $st(A)$, we first observe that one can define a block-sum operation

$$st_n(A) \oplus st_m(A) \to st_{n+m}(A)$$

using the sum in $st_{n+m}(A)$ and the embedding of $st_m(A)$ in $st_{n+m}(A)$ which uses the last m coordinates. Similarly, one can define an infinite block-sum. Furthermore, the group A_∞ acts by conjugation on $\bigwedge^* st(A)$. For $n \geqslant 3$, the equality $sl_n(\mathbb{C}) = st_n(\mathbb{C})$ follows from [B]. This implies

$$SL_n(\mathbb{C}) = \exp(sl_n(\mathbb{C})) = \exp(st_n(\mathbb{C})).$$

Thus $SL_n(\mathbb{C})$ acts on $\bigwedge^* (st_n(A))$ by conjugation when A is an algebra over $\mathbb{C}$. This action is trivial on homology because its derivative is the zero map (c.f. remark 1.1.9). From this we may conclude that the conjugation action of A_∞ on $\bigwedge^* (st(A))$ induces the identity map in homology for algebras A over k.

We may proceed as before, identifying $H_*^L(st(A))$ with the graded symmetric algebra on the primitives, which are computed as the homology of

$$\bigwedge^* (st(A))^{PC}/A_\infty. \tag{2.1.11}$$

Above dimension 1 a basis element in this complex is of the form

$$x = [u_{i_1 i_2}(a_1) \wedge u_{i_2 i_3}(a_2) \wedge \cdots \wedge u_{i_{p-1} i_p}(a_{p-1}) \wedge u_{i_p i_1}(a_p)]$$
$$i_j \neq i_k \text{ if } j \neq k. \tag{2.1.12}$$

We have $H_*(st(A))=0$ for $*=1, 2$. Above dimension 2, the cyclic trace defines an isomorphism of complexes between (2. 1. 11) and $C_{*-1}(A)$, whence

$$H_*((\wedge^* st(A))^{PC}/A_\infty)\cong HC_{*-1}(A), \qquad *\geqslant 3.$$

We conclude Prim $H_*^L(st(A))\cong HC_*^{(2)}(A)$.

Remarks 2.1.13. (i) The techniques of Theorem 1.1.12 apply equally well to subcomplexes of $\wedge^*(sl(A))$ and $\wedge^*(st(A))$ satisfying analogous properties to those listed in P1–P4.

(ii) The elements $x(m, p)$ defined in (1.3.1) lift to cycles in $\wedge^{2mp-1} st_N(\mathbb{Z})$. Corollaries 1.3.3 and 1.3.4 therefore apply with gl_N replaced by sl_N or st_N.

(iii) The methods of this section also generalize to central extensions of $sl(A)$ which fit into the diagram

$$\begin{tikzcd} HC_1(A) \arrow[r] \arrow[d, tail] & st(A) \arrow[r, two heads] \arrow[d] & sl(A) \arrow[d, equal] \\ \mathscr{K}_2^\varepsilon(A) \arrow[r] & \varepsilon(A) \arrow[r, two heads] & sl(A). \end{tikzcd}$$

Let $K_\varepsilon(A)=\operatorname{Ker}(HC_1(A)\twoheadrightarrow \mathscr{K}_2^\varepsilon(A))$. Define a graded vector space $V_*^\varepsilon(A)$ by

$$V_*^\varepsilon(A)=\begin{cases} HC_{*-1}(A) & \text{if } \ *\geqslant 3 \\ K_\varepsilon(A) & \text{if } \ *=2 \\ 0 & \text{for } \ *\leqslant 1. \end{cases}$$

Then there is an isomorphism of graded commutative Hopf algebras

$$H_*^L(\varepsilon(A))\cong S^*(V_*^\varepsilon(A)).$$

2.2. *Block-Triangular Matrices*

Let $t_n(A)$ denote the Lie subalgebra of $gl_n(A)$ generated by $\{m_{ij}(a)\mid a\in A,\ i\leqslant j\}$. Theorem 1.1.12 does not apply to the complex $\wedge^*(t(A))$ $(t(A)=\varinjlim t_n(A))$ because it is not closed under the action of Σ_∞. However, Proposition 1.1.11 does apply, as $t_n(A)$ contains the elements $\{m_{ii}(1)\}_{1\leqslant i\leqslant n}$.

Denote by $d_n(A)$ the Lie subalgebra generated by $\{m_{ii}(a)\mid a\in A, 1\leqslant i\leqslant n\}$.

PROPOSITION 2.2.1. *Let A be an algebra over k with* char. $k=0$. *Then the inclusion*

$$d_n(A) \subsetneq t_n(A)$$

induces an isomorphism in homology for all $n \geqslant 1$.

Proof. By (1.1.11), the inclusion

$$\bigwedge^* (t_n(A))^C \subsetneq \bigwedge^* (t_n(A))$$

is a quasi-isomorphism. But there is an isomorphism of complexes $\bigwedge^* (t_n(A))^C \cong \bigwedge^* (d_n(A))$ for all $n \geqslant 1$. ∎

Similar results hold for the Lie algebra homology of $t_n(A)$ with non-trivial coefficients in a $t_n(A)$-module V. Decomposing V according to the action of $d_n(k)$ we see that the cyclic complex of $\bigwedge^* (t_n(A); V)$ is

$$\bigwedge^* (t_n(A))^C \otimes V^{(0)} \oplus \bigoplus_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha \neq 0}} \bigwedge^* (t_n(A))^{NC(\alpha)} \otimes V^{(-\alpha)}.$$

If $V^{(-\alpha)}$ is zero whenever $\bigwedge^*(t_n(A))^{NC(\alpha)}$ is non-zero ($\alpha \neq 0$), then the inclusion

$$\bigwedge^* (t_n(A))^C \otimes V^{(0)} = \bigwedge^* (d_n(A); V^{(0)}) \subsetneq \bigwedge^*(t_n(A); V) \qquad (2.2.2)$$

is a quasi-isomorphism. This occurs, for example, where $V = t_n(M)$ where M is a bi-module over A and $t_n(A)$ acts on V by the adjoint representation.

Another example is the following. Let A, B be algebras over k, L_n a Lie subalgebra of $gl_n(A)$ containing $d_n(k)$, L_m a lie subalgebra of $gl_m(B)$ containing $d_m(k)$ and $\mathscr{M}$ an $A-B$ bi-module. Let $\mathscr{L}$ be the Lie algebra consisting of matrices of the form

$$\left[\begin{array}{c|c} a & c \\ \hline 0 & b \end{array}\right] \qquad a \in L_n,\ b \in L_m,\ c \in M_{nm}(\mathscr{M}). \qquad (2.2.3)$$

This is a Lie algebra, with Lie bracket given by $[x, y] = xy - yx$. Let $\mathscr{L}_0$ be the Lie subalgebra consisting of those matrices for which $c = 0$. There is an evident split-extension

$$M_{nm}(\mathscr{M}) \rightarrowtail \mathscr{L} \twoheadrightarrow \mathscr{L}_0$$

$$c \mapsto \left[\begin{array}{c|c} 0 & c \\ \hline 0 & 0 \end{array}\right], \qquad \left[\begin{array}{c|c} a & c \\ \hline 0 & b \end{array}\right] \mapsto \left[\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right] \qquad (2.2.4)$$

It is clear that $d_{n+m}(k) \subseteq \mathscr{L}_0$, and that there is an isomorphism of cyclic complexes

$$\wedge^* (\mathscr{L})^C \cong \wedge^* (\mathscr{L}_0)^C. \tag{2.2.5}$$

Proposition 1.1.11 then implies

PROPOSITION 2.2.6. *One has an isomorphism* $H^L_*(\mathscr{L}_0) \xrightarrow{\cong} H^L_*(\mathscr{L})$.

This is an analogue of a theorem due to Suslin and Quillen concerning the homology of certain extensions of linear groups. As before, one may easily formulate generalizations of this to homology with non-trivial coefficients.

PROPOSITION 2.2.7. *Let A be an algebra over* k, char. $k=0$, *and M a bi-module over A. Then there are isomorphisms in Hochschild homology*:

$$HH_*(t_n(A), t_n(M)) \cong HH_*(d_n(A), d_n(M)) \cong \bigoplus_n HH_*(A, M).$$

Proof. As before, this homology is computed by the cyclic part of the complex $\wedge^* (gl(A); gl(M))$, where the differential is given in (1.1.2). The techniques of Lemma 1.1.15 lead to the isomorphism

$$H^L_*(gl(A), gl(M)) \cong S^*(HC_{*-1}(A)) \otimes HH_*(A, M).$$

Applying this to $t_n(A)$ and $t_n(M)$ we have on the one hand

$$H^L_*(gl(t_n(A)), gl(t_n(M))) \cong S^*(HC_{*-1}(t_n(A))) \otimes HH_*(t_n(A), t_n(M));$$

on the other hand from a direct inspection we have

$$H^L_*(gl(t_n(A)), gl(t_n(M))) \cong S^*\left(\bigoplus_n HC_{*-1}(A)\right) \otimes \left(\bigoplus_n HH_*(A, M)\right).$$

By the isomorphism $HC_*(t_n(A)) \cong \bigoplus_n HC_*(A)$ the result follows. ∎

2.3. *Stability*

Let $\mathscr{L}_n = gl_n(A)$, $sl_n(A)$ or $st_n(A)$ if $n \geqslant 3$. Let $C(n)_* = \wedge^*(\mathscr{L}_n(A))^C/G_n$ where $G_n = \Sigma_n$ in the first case, A_n in the last two cases. Stabilization induces a chain map

$$C(n)_* \xrightarrow{i_n} C(n+1)_*.$$

It is easy to see that i_n is an injection which is an *isomorphism* through dimension n. The homology of $C(n+1)_*/i_n(C(n)_*)$ measures the failure of the stabilization map from being a homology isomorphism. So

PROPOSITION 2.3.1. *Let A be an algebra over k with* char. $k=0$. *Then the stabilization map in homology*

$$H_*(C(n)) \xrightarrow{(i_n)_*} H_*(C(n+1))$$

is an isomorphism for $*<n$ and an epimorphism for $*=n$.

Proof. This follows from the connectivity of $C(n+1)_*/i_n(C(n)_*)$. ∎

This is due to Loday and Quillen for $gl_n(A)$([LQ]). The complex $C(n+1)_*/i_n(C(n)_*)$ may be used to construct a spectral sequence converging to the relative homology $H_*(C(n+1)_*, C(n)_*)$. These types of stabilization theorems also apply to subcomplexes which satisfy some obvious (but mild) conditions.

2.4. *Homology of Lie Algebras with Cartan–Serre Presentation*

We assume give a Lie algebra $\mathscr{L}$ over $\mathbb{C}$ together with a Cartan–Serre basis (c.f. [M, p. 260]). Thus we have generators

$$e_\alpha, f_\alpha, h_\alpha$$

indexed on (a chosen) set of simple roots of $\mathscr{L}$ with relations (normalized according to the convention in [M])

$$\begin{aligned}
&[e_\alpha, f_\alpha] = h_\alpha \\
&[e_\alpha, f_\beta] = 0 \qquad \text{if} \quad \alpha \neq \beta \\
&[h_\alpha, e_\beta] = a_{\alpha\beta} e_\beta \\
&[h_\alpha, f_\beta] = -a_{\alpha\beta} f_\beta \\
(\mathrm{ad}\, e_\alpha)^{1-a_{\alpha\beta}}&(e_\beta) = 0 \qquad \text{if} \quad \alpha \neq \beta \\
(\mathrm{ad}\, f_\alpha)^{1-a_{\alpha\beta}}&(f_\beta) = 0 \qquad \text{if} \quad \alpha \neq \beta,
\end{aligned}$$

where $\{a_{\alpha\beta}\}$ is the Cartan matrix. For a finite set of generators (corresponding to a finite set of simple roots) and a Cartan matrix of finite type, the resulting Lie algebra is finite-dimensional and semi-simple. Moreover, the above constitues a complete set of relations and any finite dimensional semi-simple Lie algebra over $\mathbb{C}$ admits such a presentation. The Lie algebra homology of $\mathscr{L}$ in this case is well-known, and determined completely in terms of the roots and weights of $\mathscr{L}$. Given $\mathscr{L}$ as above, we consider $\mathrm{ad}(h_\alpha)$

acting on a basis element $x = x_1 \wedge \cdots \wedge x_n \in \bigwedge^n (\mathscr{L})$, where x_i is a basis element of $\mathscr{L}$ for each i. From the above relations we get

$$\mathrm{ad}(h_\alpha)(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} \left(\sum_{\beta} (\delta^{x_i}_{e_\beta} - \delta^{x_i}_{f_\beta})\, a_{\alpha\beta} \right) x.$$

The decomposition of (1.1.6) now has an obvious analogy. The adjoint action of the Cartan subalgebra H of L (generated by the h_α) decomposes $\bigwedge^* \mathscr{L}$ as a complex into eigen-spaces:

$$\bigwedge^* \mathscr{L} \cong (\bigwedge^* \mathscr{L})^{(0)} \oplus \bigoplus_{\substack{\alpha \in \tilde{T} \\ \alpha \neq 0}} (\bigwedge^* \mathscr{L})^{(\alpha)}. \tag{2.4.1}$$

$(\bigwedge^* \mathscr{L})^{(0)}$ is the intersection of the null-spaces, which corresponds to the cyclic complex, and the second sum is over all non-zero roots α in the root lattice of the eigen-space corresponding to α. The adjoint action of H is trivial on $H^L_*(\mathscr{L})$, and so as we are in characteristic zero we have a quasi-isomorphism $(\bigwedge^* \mathscr{L})^{(0)} \subsetneq \bigwedge^* \mathscr{L}$ (c.f. (1.1.8)). Call a basis element $x_1 \wedge \cdots \wedge x_n$ in $(\bigwedge^n \mathscr{L})^{(0)}$ *primitive cyclic* if it cannot be written as $x = y \wedge z$ for $y \in (\bigwedge^p \mathscr{L})^{(0)}$, $z \in (\bigwedge^{n-p} \mathscr{L})^{(0)}$, $p > 0$. The Weyl group W of $\mathscr{L}$ permutes the simple roots of $\mathscr{L}$. In fact, there is a natural representation $W \to \mathrm{Aut}_k((\bigwedge^* \mathscr{L})^{(0)})$ induced by the action of W on the root space. The linear span of the primitive cyclic basis elements forms a subcomplex $(\bigwedge^* \mathscr{L})^{PC}$ closed under the action of W.

We now suppose

HYPOTHESIS 2.4.2. *$(\bigwedge^* \mathscr{L})/W$ admits the structure of a graded commutative Hopf algebra, where the coalgebra structure is induced by the coalgebra structure on $\bigwedge^* \mathscr{L}$.*

THEOREM 2.4.3. *Under this hypothesis, there is an isomorphism*

$$H^L_*(\mathscr{L}) \cong S^*(H_*((\bigwedge^* \mathscr{L})^{PC}/W)).$$

The proof follows as in Section 1. The complex here that plays the role of Connes' cyclic complex is $(\bigwedge^* \mathscr{L})^{PC}/W$.

2.5. *Leibniz Algebra Homology*

A is called a *Leibniz algebra* over k if it admits a k-bilinear pairing

$$[\ ,\ ]: A \times A \to A$$

satisfying the Jacobi identity but which is not necessarily anti-symmetric. For such an algebra one defines its homology with coefficients in V as the homology of the complex

$$\left(\left(\overset{n}{\otimes} A\right)\otimes V, d\right)_{n\geqslant 0}, \qquad \text{with}$$

$$\begin{aligned} d(a_1\otimes \cdots \otimes a_n\otimes v) &= \sum_{1\leqslant i<j\leqslant n} (-1)^i a_1\otimes \cdots \otimes \hat{a}_i\otimes \cdots \otimes a_{j-1}\otimes [a_i, a_j]\otimes \cdots \otimes v \\ &\quad + \sum (-1)^i a_1\otimes \cdots \otimes \hat{a}_i\otimes \cdots \otimes a_n\otimes (a_i v). \end{aligned}$$

Denote the resulting homology by $H^{Le}_*(A; V)$ (resp. $H^{Le}_*(A)$ if $V=k$ with trivial A module structure). A Lie algebra $\mathscr{L}$ is a Leibniz algebra by forgetting the antisymmetry of the bracket. In [C], it was shown that

$$H^{Le}_*(gl(A))\cong T^*(HH_{*-1}(A))$$

for a k-algebra with unit A when char. $k=0$. This easily follows by the techniques of section 1, the primitive part being computed as the homology of $(\otimes^* gl(A), d)^{PC}/\Sigma_\infty$ which is isomorphic as a complex to the Hochschild complex $(\otimes^{*+1} A, b)$. One has a similar statement for subcomplexes, as well as for $sl(A)$ and $st(A)$. The results are exactly what one expects: define

$$HH^{(i)}_*(A)=\begin{cases} HH_*(A) & \text{if } *\geqslant i \\ 0 & \text{if } *<i. \end{cases}$$

We then have (for char. $k=0$)

THEOREM 2.5.1. *These are isomorphism of graded Hopf algebras*

$$H^{Le}_*(sl(A))\cong T^*(HH^{(1)}_{*-1}(A))$$
$$H^{Le}_*(st(A))\cong T^*(HH^{(2)}_{*-1}(A)).$$

In fact, the only results that doesn't seem to have an analogue in Leibniz algebra homology are the constructions of the torsion classes given in (1.3.2)–(1.3.4).

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