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Polynomially bounded cohomology and discrete groups

C. Ogle

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

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Abstract

We establish the homological foundations for studying polynomially bounded group cohomology, and show that the natural map from $PH^*(G;\mathbb{Q})$ to $H^*(G;\mathbb{Q})$ is an isomorphism for a certain class of groups.

MSC: 18G10; 18G40; 13D99; 16E30; 20J05; 20F05; 20F65

0. Introduction

The cohomology of a discrete group G with coefficients in a G-module A can be defined in various equivalent ways. Typically one first constructs a cocomplex, which for now we will label $(C^*(G;A),\ \delta^*)$; the cohomology of G with coefficients in A is then the cohomology of this complex.

Suppose G is a countable group equipped with word-length function L. Given the pair (G, L), one can consider various refinements of this cocomplex which involve a growth condition on the level of cochains. The most restrictive is a uniform bound. This condition defines a subcomplex of bounded cochains which is already quite interesting and has been extensively studied over the last 30 years [9–11,13,14,16,17,19]. Less restrictive (and also less studied) is the case when the growth rate on the level of cochains is *polynomial*. This growth condition is related to the Novikov conjecture, as shown in [3]. For suitable A (as defined in Section 1.1), one has a natural subcocomplex $PC^*_L(G;A) \subset C^*(G;A)$

E-mail address: ogle@math.ohio-state.edu (C. Ogle).

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consisting of cochains of polynomial growth with respect to L, and the inclusion is functorial with respect to polynomially bounded group homomorphisms in the first coordinate, and polynomially bounded module homomorphisms in the second coordinate. The resulting cohomology groups of $PC_L^*(G;A)$ are denoted by $PH_L^*(G;A)$; in general the cocomplex $PC_L^*(G;A)$ and therefore also its cohomology groups depend on the choice of word-length function L. The inclusion of cocomplexes induces a transformation $\eta(G,L;A)^* : PH_L^*(G;A) \to H^*(G;A)$. Of most interest to us is the case when $A = \mathbb{Q}$. We consider three successively weaker conditions one could ask of the group G:

(PC1) The map $\eta(G,L;\mathbb{Q})^*$ is an isomorphism.
(PC2) The map $\eta(G,L;\mathbb{Q})^*$ is an epimorphism.
(PC3) For every $0 \neq x \in H_*(G;\mathbb{Q})$ there is a $y \in PH_L^*(G;\mathbb{Q})$ with $\langle \eta(G,L;\mathbb{Q})(y), x\rangle \neq 0$.

The dual of $\eta(G,L;\mathbb{Q})^*$ is a $\mathbb{Q}$-vector space map $(\eta(G,L;\mathbb{Q})^*)^* : (H^*(G;\mathbb{Q}))^* \to (PH_L^*(G;\mathbb{Q}))^*$, and we can form the composition

$$\alpha_*(G,L) : H_*(G;\mathbb{Q}) \rightarrowtail (H^*(G;\mathbb{Q}))^* \to (PH_L^*(G;\mathbb{Q}))^* \tag{0.1}$$

where the first map is induced by the Universal Coefficient Theorem, and the second is $(\eta(G,L;\mathbb{Q})^*)^*$. The above 3 conditions can be rephrased as

(PC1) $(\eta(G,L;\mathbb{Q}^*)^*)$ is an isomorphism.
(PC2) $(\eta(G,L;\mathbb{Q}^*)^*)$ is a monomorphism.
(PC3) $\alpha_*(G,L)$ is a monomorphism.

For certain geometric groups it is feasible to verify property (PC1), which we do in this paper. A weaker condition is (PC2); this is the condition (PC) of [3] and equivalent to (PC3) when the rational homology groups of G are degreewise finitely generated. However when the rational homology of G is not finitely generated in each degree, (PC2) is more restrictive than (PC3). For example, if G is a free group on a countably infinite set of generators and L a word-length metric on G (see below), then (PC3) holds but (PC2) fails. Also, injectivity of the map in (PC3) is sensitive to the choice of word-length, and injectivity may hold for some choices of word-length but not for others. To illustrate, we see that the condition is obviously satisfied for $G = \mathbb{Z}$ with the standard word-length. However, if we use instead a word-length which depends logarithmically on the standard one, then with respect to this word-length $PH^1(\mathbb{Z};\mathbb{Q}) = 0$ and so (PC3) fails. The issue of injectivity of $\alpha_i(G,L)$ is related to the *Dehn function* f_G of G. This function, introduced and studied by Gersten [4,5] is defined in terms of the presentation of the group. Given a word whose image in G is trivial, the Dehn function measures the increase in word-length when one writes this word as a minimal product of conjugates of relators occurring in the relator set of the presentation. Although the Dehn function itself depends on the presentation, the linear equivalence class in which it lies does not [4]. Thus up to such equivalence, one may simply refer to the Dehn function of G. The word-problem for G is solvable iff G has a recursively enumerable Dehn function, and solvable in polynomial time iff f_G is polynomial. All known computations support the following conjecture

Conjecture A. *If $\mathscr{P} = \langle \mathscr{S} \mid \mathscr{W} \rangle$ is a finite presentation of G with polynomial Dehn function f_G, then $\alpha_i(G, L_G^{\mathrm{st}})$ is an injection for all $i \geqslant 0$.*

In this paper we establish a framework for proving Conjecture A. First, in Section 1.1 we establish some basic results in p-bounded homological algebra. Primarily, we construct the Serre spectral sequence associated to a short-exact sequence of groups with word-length (as defined in that section); the existence of the proper E^1_{pq}-term for $q > 1$ requires an additional hypothesis, but for $q = 0, 1$ the spectral sequence takes the usual form, which leads to a five-term exact sequence analogous to the one in ordinary cohomology (cf. [16,19] for the corresponding spectral sequence in bounded cohomology). We also prove a Comparison Theorem, which tells us under what conditions a resolution can be used to compute p-bounded cohomology. Section 1.2 uses the five-term sequence to identify the obstruction to injectivity of $\alpha_i(G, L)$ in even dimensions. In Section 1.3, we show that for groups with polynomial Dehn function, a related obstruction vanishes. The results of this section are in preliminary form; a detailed account will appear in a sequel to this paper. In Section 1.4 we verify the injectivity of $\alpha_1(G, L)$ for a suitable choice of L when the $H_1(G)$ is finitely-generated.

Section 2 contains various results related to Dehn functions. In Section 2.1, we show how type P resolutions (Appendix A) can be used to define the higher Dehn functions for groups of type FP^∞. The constructions in this section are then used in Section 2.2 to prove

Theorem B. *If $\mathscr{P} = \langle \mathscr{S} \mid \mathscr{W} \rangle$ is a finite presentation of G with polynomial Dehn function f_G, then $\eta(G, L; A)^i$ is an isomorphism for any p-semi-normed* (*p.s.*) *G-module A* (*defined in Section* 1.1) *and $i = 1, 2$.*

In general $\eta(G, L; \mathbb{Q})^2$ fails to be surjective when f_G is non-polynomial [6]. In fact, $\alpha_2(G, L_G^{\mathrm{st}})$ is not injective for the example Gersten constructs in that paper. In Section 2.2, using [7] we show

Theorem C. *If G admits a bounded combining* (*in particular, if G is automatic*), *then $\eta(G, L; A)^*$ is an isomorphism for all p.s. G-modules A.*

This map is also an isomorphism when G is nilpotent. In Section 2.3, we define linearly bounded (or Lipschitz) cohomology $LH^*(G; A)$ for appropriate coefficient modules A. As with p-bounded cohomology the inclusion map on the cochain level induces a natural homomorphism

$$\eta_{\mathrm{lin}}(G, L; A)^* : LH^*(G; A) \to H^*(G; A).$$

It is a theorem due to Gromov that f_G is linear iff G is word-hyperbolic. Recently a complete cohomological characterization of word-hyperbolic groups has been obtained by Mineyev in [13]. Using the result of [15] we show

Theorem D. *If G is word-hyperbolic, then $\eta_{\mathrm{lin}}(G, L; A)^*$ is an isomorphism for all l.s. G-modules A.*

In fact, the results in [13] suggest the stronger statement that $\eta_{\mathrm{lin}}(G, L; A)^*$ is an isomorphism for all A iff G is word-hyperbolic.

In the appendix we cover the definition and formal properties of type P resolutions as developed in [18].

A remark on notation: throughout the paper we write $PC^*_L(\ ;A)$ resp. $PH^*_L(\ ;A)$ as $PC^*(\ ;A)$ resp. $PH^*(\ ;A)$, and $\alpha(G, L)$ as $\alpha(G)$ unless we need to emphasize a particular word-length function.

1. Polynomially bounded group cohomology

1.1. *Basic results in polynomially bounded cohomology and the Leray-Serre spectral sequence*

If S is a generating set for a free group F and $f : S \to \mathbb{N}^+$ a function, then S and f determine a word-length function L_F on F given by

$$L_F(id) = 0$$

$$L_F(x) = f(x) \quad \text{if } x \text{ or } x^{-1} \text{ is in } S$$

$$L_F(g) = \sum_{i=1}^{r} f(x_i)$$

where $x_1x_2\ldots x_r$ is the unique reduced word representing g. Such a word-length function on F is referred to as a *word-length metric*. If $F' \subset F$ is a subgroup of F equipped with a word-length metric L_F, then the restriction of L_F to F' defines an *induced metric* $L_{F'}$ on F'. Finally, if $p : F' \twoheadrightarrow G$ is a surjection of F' to G, then $L_{F'}$ determines a word-length function L_G on G by $L_G(g) = \min\{L_{F'}(f) \mid p(f) = g\}$. Any non-degenerate word-length function on G may be realized in this fashion for an appropriate choice of F, f and p. When the set S is finite and $f(x) = 1$ for each $x \in S$, L_G is referred to as the standard word-length function L_G^{st} associated with the set of generators S. Note that L_G depends only on the pair (S, f), so that if $\langle S|W\rangle$ and $\langle S|W'\rangle$ are two presentations of G which have the same set of generators and weight function f, then the induced word-length functions will also be the same.

We will use the notation $A[S]$ to denote the free A-module with basis S for a countable set S. In particular, if $\mathbb{Q}[G]$ is the rational group algebra of G, $\mathbb{Q}[G][S]$ is the free $\mathbb{Q}[G]$-module with basis S.

A *weighted set* is a pair (S, f_S) where $f_S : S \to \mathbb{R}^+$ is a function, referred to as the weight function. When the weight function is understood, we write (S, f_S) simply as S.

A homomorphism $(G, L) \to (G', L')$ of groups with word-length will mean a homomorphism $f : G \to G'$ for which $L'(f(g)) = \min\{L(h) \mid f(h) = f(g)\}$. Thus, if f is a monomorphism it preserves word-length, and if f is an epimorphism, L' is the word-length function induced by f and L. A short-exact sequence of groups with word-length is a sequence of morphisms of groups with word-length

$$(K, L_K) \rightarrowtail (G, L_G) \twoheadrightarrow (N, L_N)$$

where the underlying sequence of groups and group homomorphisms is short-exact. Note that if $K\rightarrowtail G\twoheadrightarrow N$ is a short-exact sequence of groups and L_G a word-length function on G, then there exist unique word-length functions L_K resp. L_N on K resp. N making $(K, L_K)\rightarrowtail(G, L_G)\twoheadrightarrow(N, L_N)$ a short-exact sequence of groups with word-length.

A *semi-norm* η on a k vector space V ($k \subset \mathbb{R}$) is a map $\eta : V \to \mathbb{R}_+$ satisfying (i) $\eta(a+b)\leqslant\eta(a)+\eta(b)$ and (ii) $\eta(\lambda a)\leqslant|\lambda|\eta(a)$ for all $a, b \in V$ and $\lambda \in \mathbb{Q}$.

Before proceeding, we illustrate an essential homological difference between the notions of "bounded" and "p-bounded". If (S, f_S) is a weighted set and $(V, \| \|)$ a normed vector space, one may define $BHom(S, V)$ the set of bounded morphisms from S to V, and a larger space *PHom(S,V)*, the set of p-bounded morphisms from S to V. $\phi : S \to V$ is p-bounded if there is a polynomial p such that $\| \phi(s) \| \leqslant p(f_S(s))$ for all $s \in S$. Then ϕ is bounded if we can take the polynomial to be a constant function. So $BHom(S, V)$ is again a normed vector space with respect to the sup norm, allowing one to construct spaces such as $BHom(S', BHom(S, V))$. The larger space $PHom(S, V)$ has no natural norm. However, it does have an obvious collection of semi-norms given by $\eta_s(\phi)=\| \phi(s) \|$. This suggests that in the p-bounded setting, one needs to work with semi-normed modules of a sufficiently general type in order to define iterated *Hom* spaces such as $PHom(S', PHom(S, V))$ (a necessary construction for the development of the Serre spectral sequence in p-bounded cohomology).

Definition 1.1.1. A p-*semi-normed G-module*, or p.s. G-module, is a $\mathbb{Q}[G]$-module M equipped with a collection of semi-norms $\{\eta_x\}_{x\in\Lambda}$ indexed on a countable G-set $\Lambda=\Lambda_M$. The semi-norms satisfy the following properties:

(i) If $\eta_{x_1}, \ldots, \eta_{x_n} \in \Lambda$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}^+$, then there is a constant C and an $\eta_y \in \Lambda$ with

$$C_{\eta_y}\geqslant\lambda_1\eta_{x_1}+\cdots+\lambda_n\eta_{x_n}$$

(ii) there exist constants $C, n>0$ such that for all $x' \in \Lambda$ there is an $x \in \Lambda$ with

$$\eta_{gx'}(hm)\leqslant C(1+L(h))^n\eta_{ghx}(m)$$

for all $g, h \in G$ and $m \in M$, where L is the word-length function on G.

When $G=\{id\}$ we will refer to M as a p.s. module. Any p.s. G-module is a p.s. module by forgetting the G-module structure.

Definition 1.1.2. A *homomorphism* $f : M \to M'$ *of p.s. G-modules* is a $\mathbb{Q}[G]$-module homomorphism which is p-*bounded*; i.e. there exists C_1, C_2 and $n>0$ such that for all $x' \in \Lambda_{M'}$ there exists $x \in \Lambda_M$ with

$$\eta_{gx'}(f(a))\leqslant C_1C_2(1+\eta_{gx}(a))^n(1+L(g))^n$$

for all $g \in G$ and $a \in M$. In this inequality, the constants C_1 and n may vary with f but are independent of x', while C_2 depends only on x'.

Two p.s. G-modules M, M' are *isomorphic* if there exist homomorphisms of p.s. G-modules $f: M \to M'$, $f': M' \to M$ with $f \circ f' = id_{M'}$, $f' \circ f = id_M$. By (1.1.1)(i), the collection of all p.s. G-module homomorphisms from M to M' forms a vector space over $\mathbb{Q}$ which we denote $PHom_G(M, M')$. Dropping the requirement that the maps be G-equivariant, we get the $\mathbb{Q}$-vector space of p-bounded maps from M to M' which we denote simply as $PHom(M, M')$. The same conventions apply for *Hom* in place of *PHom*. If M' is a sub-G-module of M and $M'' = M/M'$, we may define a p.s. G-module structure on M'' by setting $\Lambda_{M''} = \Lambda_M$ and for all $x \in \Lambda_{M''}$ defining

$$\eta_x(\bar{m}) = \min\{\eta_x(m) \mid p(m) = \bar{m}\} \tag{1.1.3}$$

where $p: M \twoheadrightarrow M/M'$ is the projection. The reader may verify that this defines a p.s. G-module structure on M''. We refer to this as the *quotient p.s. G-module structure* induced by M and the projection p.

It will sometimes be the case that a G-module M comes equipped with two p.s. G-module structures, which we may denote $\mathscr{S}_1$ and $\mathscr{S}_2$. Let $M_{\mathscr{S}_i}$ denote M equipped with structure $\mathscr{S}_i$. We say that the two structures are *equivalent* if the identity map on M induces an isomorphism of p.s. G-modules $M_{\mathscr{S}_1} \stackrel{\text{id}}{\to} M_{\mathscr{S}_2}$.

Definition 1.1.4. A *free* p-*bounded G-module*, or p.f. G-module, is a free $\mathbb{Q}[G]$-module $P = \mathbb{Q}[G][S]$ with countable basis $S \neq \emptyset$ equipped with a weight function $w_S: S \to \mathbb{R}^+$. The indexing set for the semi-norms on P is $\Lambda_P = \{*\}$ equipped with trivial G action. The unique semi-norm on P is

$$\left|\sum \lambda_i g_i s_i\right| = \sum |\lambda_i|(1 + |g_i s_i|)$$

where $|gs| = L(g) + w_S(s)$.

In the special case $S = \{*\}$, we adopt the convention that $|*| = 1$. In particular, when $S = \{*\}$ and $G = \{id\}$, this defines a semi-norm on $P = \mathbb{Q}$ given by $|q|_* = 2|q|$. A p.f. G-module is a p.s. G-module. To see this, note that (1.1.1)(i) is trivially satisfied because the indexing set has only one element, and (1.1.1)(ii) follows from the inequality

$$\begin{aligned}\left|h\left(\sum \lambda_i g_i s_i\right)\right| &= \sum |\lambda_i|(1 + |hg_i s_i|)\\ &\leqslant (1 + L(h)) \sum |\lambda_i|(1 + |g_i s_i|).\end{aligned}$$

Suppose now that $P = \mathbb{Q}[G][S]$ is a p.f. G-module and M a p.s. G-module. Let Λ_M be the indexing set for the semi-norms on M and let $T = \{gs \mid g \in G, s \in S\}$. Associated to a finite subset $U \subseteq T$ and an element $x \in \Lambda_M$ is the semi-norm on $PHom(P, M)$ given by

$$\eta_{(x,U)}(f) = \sum_{t \in U} \eta_x(f(t)).$$

The indexing set for this collection of semi-norms is $\Lambda_M \times \mathscr{P}(T)$, where $\mathscr{P}(T)$ denotes the set of finite subsets of T. The G-action on $PHom(P, M)$ is given by $g \cdot f(p) = gf(g^{-1}p)$, and the G action on the index set $\Lambda_M \times \mathscr{P}(T)$ is given by $g(x, U) = (gx, g^{-1}U)$.

Proposition 1.1.5. *The above defines a p.s. G-module structure on PHom*(P, M).

Proof. For $g, h \in G$, the series of inequalities

$$\begin{aligned}\eta_{h(x'U)}(g\cdot f) &= \eta_{(hx',h^{-1}U)}(g\cdot f)\\ &= \sum_{t_i\in U}\eta_{hx'}(gf(g^{-1}h^{-1}t_i))\\ &\leqslant C(1+L(g))^n\left(\sum_{t_i\in U}\eta_{hgx}(f(g^{-1}h^{-1}t_i))\right)\\ &= C(1+L(g))^n\eta_{(hgx,g^{-1}h^{-1}U)}(f)\\ &= C(1+L(g))^n\eta_{hg(x,U)}(f)\end{aligned}$$

implies the G-action is p-bounded in the sense of Definition 1.1.1(ii). To verify (1.1.1)(i) we suppose given numbers $\lambda_i > 0$ and semi-norms $\eta_{(x_j,U_j)}$, $1 \leqslant j \leqslant N$. Let $U = \bigcup_j U_j$ and choose $y \in \Lambda_M$ and a constant C' with $\sum_{j=1}^N \lambda_j\eta_{x_j} \leqslant C\eta_y$. Letting $C = NC'$, one has the inequality

$$\sum_{j=1}^N \lambda_j\eta_{(x_j,U_j)} \leqslant C\eta_{(y,U)}. \qquad \square$$

Proposition 1.1.6. *If* $P = \mathbb{Q}[G][S]$ *is a p.f. G-module and M a p.s. G-module, then there is an isomorphism of vector spaces over* $\mathbb{Q}$

$$PHom_G(P, M) \cong PHom(\mathbb{Q}[S], M).$$

Proof. This is the p-bounded analogue of a standard fact from homological algebra. The map from left to right is given by restriction to the subspace $\mathbb{Q}[S]$; this restriction map is obviously p-bounded. The map in the other direction is the inflation map. A p-bounded map $f : \mathbb{Q}[S] \to M$ defines a G-module homomorphism $\tilde{f} : P \to M$ given on basis elements by $\tilde{f}(gs) = gf(s)$. The inequalities

$$\begin{aligned}\eta_{hx'}\left(\tilde{f}\left(\sum\lambda_i g_i s_i\right)\right) &= \eta_{hx'}\left(\sum\lambda_i g_i f(s_i)\right)\\ &\leqslant \sum|\lambda_i|C(1+L(g_i))^n\eta_{hg_ix}(f(s_i))\\ &\leqslant \sum|\lambda_i|C(1+L(g_i))^n C_1C_2(1+|s_i|)^{n'}\\ &\quad\times(1+L(h))^{n'}(1+L(g_i))^{n'}\\ &\leqslant (CC_1)C_2\left(1+\left|\sum\lambda_i g_i s_i\right|\right)^{n+n'}(1+L(h))^{n+n'}\end{aligned}$$

imply $\tilde{f}$ is p-bounded. $\square$

Notes: (i) By the last proposition, the p.s. module structure on $PHom(\mathbb{Q}[S], M)$ determines one on $PHom_G(P, M)$. There is also the induced p.s. module structure coming from the inclusion of $PHom_G(P, M)$ into $PHom(P, M)$ as the fixed-point set under the action of G. The indexing set for the first is $\Lambda_M \times \mathscr{P}(S)$, which includes into that of the second, which is $\Lambda_M \times \mathscr{P}(T)$. It is an easy exercise to verify that these two structures are equivalent in the sense defined above.

(ii) Inspection of the proofs of the previous two propositions show that $f : P \to M$ is p-bounded precisely when it is p-bounded on the weighted set T, where T is as in Proposition 1.1.5. If f is G-equivariant, then it lies in $PHom_G(P, M)$ precisely when it is a G-module homomorphism which is p-bounded on the weighted set S.

We next discuss short-exact sequences.

Definition 1.1.7. An *admissible monomorphism* $i : M' \rightarrowtail M$ of p.s. G-modules is a G-module monomorphism where

(i) $\Lambda_{M'} = \Lambda_M$;
(ii) the semi-norm η_x on M' is given by the restriction of η_x on M to $im(i)$.

An *admissible epimorphism* $M \twoheadrightarrow M''$ is an epimorphism which

(i) is a p.s. G-module homomorphism, and
(ii) admits a section of p.s. modules (i.e., a p-bounded homomorphism which is not necessarily equivariant).

In particular, the semi-norms on M'' may be given separately and are not necessarily induced by the semi-norms on M. A *short-exact sequence* of p.s. G-modules

$$M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M''$$

is then a short-exact sequence of $\mathbb{Q}[G]$-modules consisting of an admissible monomorphism followed by an admissible epimorphism.

Lemma 1.1.8. *If P is a p.f. G-module and* $M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M''$ *a short-exact sequence of p.s. G-modules, then i and j induce a short-exact sequence*

$$PHom_G(P, M') \overset{i_*}{\rightarrowtail} PHom_G(P, M) \overset{j_*}{\twoheadrightarrow} PHom_G(P, M'')$$

of $\mathbb{Q}$*-modules.*

Proof. Write P as $\mathbb{Q}[G][S]$. By Proposition 1.1.6, the above sequence is isomorphic to the sequence

$$PHom(\mathbb{Q}[S], M') \overset{i_*}{\rightarrowtail} PHom(\mathbb{Q}[S], M) \overset{j_*}{\twoheadrightarrow} PHom(\mathbb{Q}[S], M'')$$

obviously the first map is injective. The existence of a p-bounded section from M'' to M implies the surjectivity of the second map. Lastly, if $f \in PHom(\mathbb{Q}[S], M)$ maps to zero in $PHom(\mathbb{Q}[S], M'')$, its image lies in $i(M')$. Because the semi-norms on M' are induced by those on M via the inclusion, the unique map $f' : \mathbb{Q}[S] \to M'$ for which $f = i \circ f'$ is also p-bounded, and therefore an element of $PHom(\mathbb{Q}[S], M')$. $\square$

[Addendum to Lemma 1.1.8: Although it will not be needed for what follows, we note that the short-exact sequence in the above Lemma is actually a short-exact sequence of p.s. modules, where the p.s. module structure of each term is that described in the note following Proposition 1.1.6.]

A p.s. *G-complex* is a $\mathbb{Q}[G]$-complex $M_* = (M_*, d_*)$ where each M_n is a p.s. G-module and each boundary map $d_n : M_n \to M_{n-1}$ is a p.s. G-module homomorphism. A p.s. G-cocomplex $M^* = (M^*, \delta^*)$ is defined in exactly the same manner, with $\delta^n : M^n \to M^{n+1}$. Given a p.s. G-complex $M_* = (M_*, d_*)$ and a p.s. G-module M', we have a well-defined cocomplex

$$PHom_G(M_*, M')$$

with corresponding cohomology groups $PH^*_G(M_*; M')$, which are the p-bounded G-equivariant cohomology groups of M_* with coefficients in M'.

Definition 1.1.9. A p.f. *resolution* of $\mathbb{Q}$ over $\mathbb{Q}[G]$ is a resolution (R_*, d_*) of $\mathbb{Q}$ over $\mathbb{Q}[G]$ where each R_n is a p.f. G-module and each $d_n : R_n \to R_{n-1}$ a p.s. G-module homomorphism. In addition, we require that (R_*, d_*) admits a p-bounded chain contraction $s_* = \{s_n : R_n \to R_{n+1}\}_{n \geqslant 0}$ as a p.s. complex.

The standard non-homogeneous bar resolution over $\mathbb{Q}$, which we write as $(EG)_* = C_*(EG.; \mathbb{Q})$, provides an example of such a resolution. Precisely, for each $n \geqslant 0$ we identify $C_n(EG.; \mathbb{Q})$ as the free p.s. G-module on the set S_n

$$C_n(EG.; \mathbb{Q}) = \mathbb{Q}[G][S_n]$$

where $S_n = \{(1, g_1, g_2, \ldots, g_n) \in EG_n = (G)^{n+1}\}$ and G acts by left multiplication in the left-most coordinate:

$$g(g_0, g_1, g_2, \ldots, g_n) = (gg_0, g_1, g_2, \ldots, g_n)$$

the weight function on S_n is given by

$$f_{S_n}((1, g_1, g_2, \ldots, g_n)) = 1 + \sum_{i=1}^{n} L_G(g_i).$$

The differential d_n defined on basis elements by

$$d_n(g_0, g_1, g_2, \ldots, g_n) = \left(\sum_{i=0}^{n-1} (-1)^i (g_0, \ldots, g_i g_{i+1}, \ldots, g_n) \right) + (-1)^n (g_0, g_1, \ldots, g_{n-1})$$

is p-bounded for each n, and G-equivariant. The standard section s_* is defined on basis elements by

$$s_n((g_0, g_1, \ldots, g_n)) = (1, g_0, g_1, \ldots, g_n)$$

and is a p.s. module homomorphism for each $n \geqslant 0$. The groups

$$PH^*(G;M) \overset{\text{def}}{=} PH^*_G(EG_*;M)$$

are the p-bounded group cohomology groups of G with coefficients in a p.s. G-module M. The following result extends the Comparison Theorem of [18]. When there is no confusion, we write EG_* for $(EG)_*$. However, for each n, EG_n is the set G^{n+1} while $(EG)_n = C_n(EG;\mathbb{Q})$ is the free $\mathbb{Q}$-module on EG_n.

Theorem 1.1.10 (*Comparison Theorem*). *Let (R_*, d_*) be a p.f. resolution of $\mathbb{Q}$ over $\mathbb{Q}[G]$ and M a p.s. G-module. Then there is an isomorphism*

$$PH^*_G(EG_*;M) \cong PH^*_G(R_*;M) = H^*(PHom_G(R_*,M),\delta^*)$$

Proof. As in [18], one forms the bi-complex $EG_* \otimes R_*$. Write $(EG)_p$ as $\mathbb{Q}[G][S_p]$ and R_q as $\mathbb{Q}[G][T_q]$. The method of proof of Proposition 1.1.6 provides isomorphisms

$$\begin{aligned} PHom(\mathbb{Q}[S_p]\otimes\mathbb{Q}[G][T_q],M) &\cong PHom_G(\mathbb{Q}[G][S_p]\otimes\mathbb{Q}[G][T_q],M)\\ &\cong PHom(\mathbb{Q}[G][S_p]\otimes\mathbb{Q}[T_q],M) \end{aligned} \tag{1.1.11}$$

where the G-action on $\mathbb{Q}[G][S_p]\otimes\mathbb{Q}[G][T_q]$ is the diagonal one. Now consider the bicocomplex formed by applying $PHom_G(\ ,M)$ to $EG_*\otimes R_*$. The qth row is

$$PHom_G(EG_*\otimes R_q,M) = PHom_G(EG_*\otimes\mathbb{Q}[G][T_q],M)$$

where the differential is the identity on the second coordinate. By the second isomorphism in (1.1.11), the cohomology of this cocomplex is equal to the cohomology of the cocomplex

$$PHom(EG_*\otimes\mathbb{Q}[T_q],M).$$

The standard p-bounded chain contraction on EG_* yields a cocontraction of this cocomplex above dimension zero. The resulting cohomology groups are zero in positive dimensions, with

$$\begin{aligned} PH^0_G(EG_*\otimes\mathbb{Q}[T_q],M) &= PHom(\mathbb{Q}[T_q],M)\\ &\cong PHom_G(\mathbb{Q}[G][T_q],M)\\ &= PHom_G(R_q,M). \end{aligned}$$

Hence filtration by rows produces a spectral sequence with $E_1^{0,*} = PHom_G(R_*,M)$, $E_1^{p,*}=0$ for $p>0$, and so $E_2^{0,*} = PH^*_G(R_*;M)$, $E_2^{p,*}=0$ for $p>0$. Filtering by columns instead of rows reverses the roles of EG_* and R_*, resulting in a spectral sequence with $E_2^{*,0} = PH^*_G(EG_*;M)$, $E_2^{*,q}=0$ for $q>0$. □

The next result will provide our main technical tool for studying the p-bounded group cohomology of a group G with coefficients in a p.s. G-module.

Theorem 1.1.12 (*Serre Spectral Sequence*). *Let $(K,L_K)\rightarrowtail(G,L_G)\twoheadrightarrow(N,L_N)$ be a short-exact sequence of groups with word-length and M' a p.s. N-module (M' is then also a p.s.*

G-module via the surjection $G \twoheadrightarrow N$). In addition, M' is required to satisfy the hypothesis (1.1.H) *stated below. Then there is a first quadrant spectral sequence with*

$$E_2^{*,*} = \{PH_N^p(EN_*;\, PH^q(BK_*;\, M'))\}_{p,q \geqslant 0}$$

converging to $PH_G^*(EG_*;\, M')$, with the natural transformation $PH^*() \to H^*()$ inducing a map of Serre spectral sequences in cohomology.

Proof. As above $(EG)_n = \mathbb{Q}[G]^{\otimes n+1}$ with the $\mathbb{Q}[G]$-module structure induced by left multiplication by G on the left-most coordinate. Tensoring over $\mathbb{Q}[K]$ with $\mathbb{Q}$ yields the complex

$$\mathbb{Q}[N] \leftarrow \mathbb{Q}[N] \otimes \mathbb{Q}[G] \leftarrow \mathbb{Q}[N] \otimes \mathbb{Q}[G]^{\otimes 2} \leftarrow \cdots. \tag{1.1.13}$$

By the Comparison Theorem above, there are isomorphisms

$$\begin{aligned} PH^*(BK_*;\, M') &\cong PH_K^*((EK)_*;\, M') \cong PH_K^*((EG)_*;\, M') \\ &\cong PH^*(K\backslash(EG)_*;\, M') \end{aligned}$$

where $BK_* = K\backslash(EK)_*$ and $K\backslash(EG)_*$ is the complex in (1.1.13). Form the bicomplex $B_{*,*} = EN_* \otimes K\backslash(EG)_*$. We will abbreviate the (p,q)th term of this bicomplex as $N_p \otimes M_q$ where $N_p = \mathbb{Q}[N]^{\otimes p+1},\ M_q = \mathbb{Q}[N] \otimes \mathbb{Q}[G]^{\otimes q}$. Applying $PHom_N(,\, M')$ and filtering by rows produces a spectral sequence which, by the isomorphisms of (1.1.11), collapses at the $E_1^{*,*}$-term, with the only non-zero groups being

$$E_1^{0,*} = PHom_N^*(K\backslash(EG)_*,\, M') = PHom_G^*((EG)_*,\, M').$$

Computing $E_2^{*,*}$ yields $E_2^{0,*} = PH_G^*(EG_*;\, M'),\ E_2^{p,*} = 0$ for $* > 0$. We now consider the spectral sequence arising from filtration by columns. To compute the $E_1^{*,*}$-term, we observe that the pth column is the cocomplex

$$\begin{aligned} \cdots \to PHom_N(N_p \otimes M_{q-1},\, M') &\xrightarrow{1\otimes\delta^{q-1}} PHom_N(N_p \otimes M_q,\, M') \\ &\xrightarrow{1\otimes\delta^{q}} PHom_N(N_p \otimes M_{q+1},\, M') \to \cdots. \end{aligned} \tag{1.1.14}$$

The $\mathbb{Q}[N]$-module structure on N_p and M_q is given by left multiplication by N in the left-most coordinate, and the $\mathbb{Q}[N]$-module structure on the tensor product is the diagonal one. In order to properly identify the cohomology of the sequence in (1.1.14), we will want to take partial adjoints.

Lemma 1.1.15. *For p.f. N-modules $P,\ P'$, and p.s. N-module M' there are natural isomorphisms of p.s. G-modules resp. p.s. modules*

$$\begin{aligned} &PHom(P \otimes P',\, M') \cong PHom(P, PHom(P',\, M')) \\ &PHom_N(P \otimes P',\, M') \cong PHom_N(P, PHom(P',\, M')). \end{aligned}$$

Proof. Again, these isomorphisms are well-known in the non-p-bounded case. We write T resp. T' for the orbit of S resp. S' under G. As vector spaces, $P = \mathbb{Q}[T],\ P' = \mathbb{Q}[T']$. Then $P \otimes P' = \mathbb{Q}[T \times T']$ with weight function determined by setting $|(t,t')| = |t| + |t'|$.

The G action on $T \times T''$ is the diagonal one. For an element $f \in PHom(P \otimes P', M')$, denote its partial adjoint on the right by $\tilde{f}$. Thus $\tilde{f}(gs)(g's') = f(gs, g's')$. Now suppose $\tilde{f}: P \to PHom(P', M')$ is p-bounded. Then there exists $C_1, n > 0$ depending only on $\tilde{f}$ and C_2 depending only on (x', U') such that

$$\eta_{g(x',U')}(\tilde{f}(a)) \leqslant C_1 C_2 (1 + |a|)^n (1 + L(g))^n$$

for all $g \in G$ and $a \in P$. Given $(t, t') \in T \times T'$, set $a = t$ and $U' = \{gt'\}$. Then

$$\eta_{gx'}(f(t, t')) = \eta_{g(x',\{gt'\})}(\tilde{f}(t)) \leqslant C_1 C_2 (1 + |t|)^n (1 + L(g))^n$$

implies f is p-bounded on $T \times T'$, hence p-bounded. In the other direction, suppose f is p-bounded. As before, there are $C_1', \ n' > 0$ depending only on f and C_2' depending only on x' such that

$$\eta_{gx'}(f(b)) \leqslant C_1' C_2' (1 + |b|)^{n'} (1 + L(g))^{n'}$$

for all $g \in G$ and $b \in P \otimes P'$. Then

$$\begin{aligned} \eta_{g(x',U')}(\tilde{f}(t)) &= \sum_{t_i' \in U'} \eta_{gx'}(f(t, g^{-1}t_i')) \\ &\leqslant \sum_{t_i' \in U'} C_1' C_2' (1 + |(t, g^{-1}t_i')|)^{n'} (1 + L(g))^{n'} \\ &\leqslant D_1 D_2 (1 + |t|)^n (1 + L(g))^{2n'} \end{aligned}$$

where $D_1 = C_1'(\sum_{t_i' \in U'} (1 + |t_i'|))^{n'}$ is independent of g and x' and $D_2 = C_2'$. This implies f is p-bounded, which verifies the first isomorphism. The second follows from the first by the fact that the two adjoint maps preserve the G-action, hence induce isomorphisms on fixed-point sets. $\square$

Accordingly we may rewrite (1.1.14) as

$$\begin{aligned} \cdots \to PHom_N(N_p, PHom(M_{q-1}, M')) &\xrightarrow{(\delta^{q-1})^*} PHom_N(N_p, PHom(M_q, M')) \\ &\xrightarrow{(\delta^q)^*} PHom_N(N_p, PHom(M_{q+1}, M')) \\ &\xrightarrow{(\delta^{q+1})^*} \cdots . \end{aligned} \tag{1.1.16}$$

where $(\delta^k)^*$ is the map induced by $\delta^k : PHom(M_k, M') \to PHom(M_{k+1}, M')$. Both $im(\delta^{k-1})$ and $ker(\delta^k)$ are submodules of $PHom(M_k, M')$ closed under the action of N, and so inherit a p.s. N-module structure via restriction (the p.s. N-module structure on $PHom(M_k, M')$ is that given by Proposition 1.1.5).

Proposition 1.1.17. *There are equalities of p.s. N-modules*

$$\begin{aligned} &ker((\delta^k)^*) = PHom(N_p, ker(\delta^k)) \\ &im((\delta^{k-1})^*) = PHom(N_p, im(\delta^{k-1})). \end{aligned} \tag{1.1.18}$$

Proof. First, $f : N_p \to PHom(M_k, M')$ maps to zero under $(\delta^k)^*$ exactly when $im(f)$ lies in $ker(\delta^k)$. Secondly, $f \in im((\delta^{k-1})^*)$ iff there exists $f' \in PHom(N_p, PHom(M_{k-1}, M'))$

with $f = f' \circ \delta^{k-1}$. But $f' \circ \delta^{k-1}$ is a map from N_p to $im(\delta^{k-1})$. This verifies the two equalities on the level of N-modules. They are equalities of p.s. N-modules because the p.s. N-module structure on both sides is induced by the restriction of a single p.s. N-module structure on $PHom(N_p, PHom(M_k, M'))$. $\square$

In order to have an identifiable E_1^{**}-term, we need an additional hypothesis. For applications below, we will state it in terms of a collection of hypotheses indexed on the non-negative integers.

Hypothesis 1.1.H(k). For fixed $k \geqslant 0$, $ker(\delta^k)/im(\delta^{k-1}) = PH^k(K; M')$ admits a p.s. N-module structure for which

$$ker(\delta^k) \twoheadrightarrow PH^k(K; M')$$

is an admissible epimorphism.

Hypothesis 1.1.H. Hypothesis 1.1.H(k) is true for all $k \geqslant 0$.

Given that $im(\delta^{k-1}) \rightarrowtail ker(\delta^k)$ is an admissible monomorphism with the p.s. N-module structures as given above, this hypothesis is equivalent to the statement that

$$im(\delta^{k-1}) \rightarrowtail ker(\delta^k) \twoheadrightarrow PH^k(K; M')$$

is a short-exact sequence of p.s. N-modules. Under these conditions, Lemma 1.1.8 implies there is a corresponding short-exact sequence

$$PHom_N(N_p, im(\delta^{k-1})) \rightarrowtail PHom_N(N_p, ker(\delta^k)) \twoheadrightarrow PHom_N(N_p, PH^k(K; M'))$$

which together with (1.1.18) imply the $E_1^{*,*}$ is given by

$$E_1^{p,q} = PHom_N(N_p, PH^q(K; M')).$$

The E_2^{**}-term indicated in the statement of the theorem then follows as in the standard Serre spectral sequence. $\square$

A discrete group with word-length function G has p-*bounded A-cohomology* (where A is a p.s. G-module) if the natural transformation of cohomology theories $PH_G^*(EG_*; A) \to H_G^*(EG_*; A)$ is an isomorphism. It is natural to suppose that the class of groups with p-bounded cohomology is closed under arbitrary extensions. The following corollary gives a partial result in this direction.

Corollary 1.1.19. *Let $K \rightarrowtail G \twoheadrightarrow N$ be a short-exact sequence of groups equipped with word-length function, and let A be a p.s. N-module, such that hypothesis* (1.1.H) *is satisfied. If K has p-bounded cohomology with coefficients in A, and N has p-bounded cohomology with coefficients in $PH^i(K; A) = H^i(K; A)$ for all i, then G has p-bounded cohomology with coefficients in A.*

Proof. The natural transformation from p-bounded cohomology to cohomology with coefficients in A induces a map of Leray–Serre spectral sequences. With the given hypothesis, there is an isomorphism of $E_2^{*,*}$-terms, where the $E_2^{*,*}$-term for p-bounded cohomology is given in Theorem 1.1.12. By spectral sequence comparison, the result follows. □

Before giving the main application of this spectral sequence, we will need a technical lemma.

Lemma 1.1.20. *Hypothesis* 1.1.H(k) *is satisfied for* $k = 0$ *and* $k = 1$.

Proof. When $k = 0$, $\delta^{k-1} = \delta^{-1} = 0$ and $ker(\delta^k) \twoheadrightarrow PH^k(K; M')$ is an isomorphism. So Hypothesis 1.1.H(0) is trivially satisfied. To handle the case $k=1$, we recall that the inclusion $K \rightarrowtail G$ induces a p-bounded inclusion of complexes

$$(BK)_* = K\backslash(EK)_* \hookrightarrow K\backslash(EG)_*. \tag{1.1.21}$$

Claim 1.1.22. *The inclusion of* (1.1.21) *induces an admissible epimorphism of p.s. cocomplexes*

$$(M^*, \delta^*) = (PHom(K\backslash(EG)_*, M'), \delta^*) \twoheadrightarrow (PHom(K\backslash(EK)_*, M'), \delta^*_K). \tag{1.1.23}$$

Proof. Let $\iota : S \hookrightarrow S'$ be an inclusion of sets. We also assume given $\mathbb{R}^+$-valued maps $f_S, f_{S'}$ with $f_S = f_{S'} \circ \iota$. Let P resp. P' be the p.f. module generated by (S, f_S) resp. $(S', f_{S'})$. We also suppose given a p-bounded surjection $p : S' \twoheadrightarrow S$ with $p \circ \iota = id$. This surjection induces an admissible epimorphism $P' \twoheadrightarrow P$ which we also denote by p. Then for any p.s. module M', ι induces an admissible epimorphism $\iota^* : PHom(P', M') \twoheadrightarrow PHom(P, M')$ with section equal to p^*. In fact, if $\phi : P \to M'$ is p-bounded, then so is $\phi' = \phi \circ p : P' \to M'$, and $\phi \in PHom(P', M')$ maps to ϕ under ι^*, proving the surjectivity of ι^*. For $\alpha \in PHom(P', M')$ the equality $\eta_{(x,U)}(\iota^*(\alpha)) = \eta_{(x,\iota(U))}(\alpha)$ implies the p-boundedness of ι^*. In the other direction the sequence

$$\begin{aligned}\eta_{(x,U')}(p^*(\beta)) &= \sum_{s'_i \in U'} \eta_x(p^*(\beta)(s'_i)) \\ &= \sum_{s'_i \in U'} \eta_x(\beta(p(s'_i)) \leqslant C_2 \eta_{(x,U)}(\beta)\end{aligned}$$

where $U = p(U')$ and $C_2 = |U| < \infty$ implies the p-boundedness of p^*. Thus ι is an admissible epimorphism.

Returning to the short-exact sequence $K \overset{i}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} N$, we fix a bounded section of sets $s : N \rightarrowtail G$; given $g \in G$ we denote the product $g(s(p(g)))^{-1}$ by $\lambda(g)$. For each $q \geqslant 0$ we have an inclusion of sets

$$i_q : EK_q \rightarrowtail EG_q$$

induced by i and a projection of sets

$$p_q : EG_q \twoheadrightarrow EK_q$$

$$(g_0, g_1, \ldots, g_q) \longmapsto (\lambda(g_0), \lambda(g_0)^{-1}\lambda(g_0 g_1), \ldots, \lambda(g_0 g_1, \ldots, g_{q-1})^{-1}\lambda(g_0 g_1 \ldots g_q)).$$

The following properties are easily verified:

(i) i_q and p_q are equivariant w.r.t. left multiplication by K in the left-most coordinate, hence descend to maps

$$\bar{i}_q : K\backslash EK_q = BK_q \hookrightarrow K\backslash EG_q$$

$$\bar{p}_q : K\backslash EG_q \twoheadrightarrow BK_q$$

(ii) $p_q \circ i_q = id$ for each q, implying $\bar{p}_q \circ \bar{i}_q = id$ for each q.

(iii) $\bar{p}_* = \{\bar{p}_q\}$ and $\bar{i}_* = \{\bar{i}_q\}$ are chain maps.

Now $(BK)_q = \mathbb{Q}[BK_q]$ and $(K\backslash EG)_q = \mathbb{Q}[K\backslash EG_q]$ are the p.f. modules generated respectively by the weighted sets $(K^q, L(K\backslash K)_q)$ and $(K\backslash G \times G^q, L(K\backslash G)_q)$, where

$$L(K\backslash K)_q(k_1, \ldots, k_q) = 1 + \sum_{i=1}^{q} L_K(k_i),$$

$$L(K\backslash G)_q(\bar{g}_0, g_1, \ldots, g_q) = 1 + \bar{L}_G(g_0) + \sum_{i=1}^{q} L_G(g_i),$$

$$\bar{L}_G(g_0) = \min_{k\in K}\{L_G(kg_0)\}.$$

By what we have shown above, we conclude that for any p.s. N-module M', the map

$$(\bar{i}_q)^* : PHom((K\backslash EG)_q, M') \twoheadrightarrow PHom((BK)_q, M')$$

is an admissible epimorphism of p.s. modules for each q with section given by p_q^*. As both $\bar{i}_*$ and $\bar{p}_*$ are chain maps, their duals $(\bar{i}_*)^*$ and $(\bar{p}_*)^*$ are cochain maps, which proves the claim. $\square$

We consider the following diagram:

$$\begin{tikzcd}
ker(\delta^1_K) \arrow[r, shift right, tail, "p(1)"'] \arrow[d, equal] & ker(\delta^1) \arrow[l, shift right, two heads, "i(1)"'] \arrow[d] \\
ker(\delta^1_K)/im(\delta^0_K) \arrow[r, shift right, tail, "\bar{p}(1)"'] & ker(\delta^1)/im(\delta^0) \arrow[l, shift right, two heads, "\bar{i}(1)"']
\end{tikzcd} \tag{1.1.24}$$

where δ^*_K resp. δ^* are the coboundary maps appearing in (1.1.23), and the surjections resp. injections in the diagram are those induced by $(\bar{i}_*)^*$ resp. $(\bar{p}_*)^*$. The epimorphism $(\bar{i}_*)^*$ induces an isomorphism in cohomology by the Comparison Theorem, and $(\bar{i}_*)^* \circ (\bar{p}_*)^* = id$, implying $\bar{i}(1)$ and $\bar{p}(1)$ are isomorphisms, and inverses of each other. Also $\delta^0_K = 0$ since the K-module structure on M' is trivial, implying the left vertical map is the identity as indicated. Denote the composition $p(1) \circ i(1)$ by Pr. Then for $f \in ker(\delta^1)$, $Pr(f)$ is given by the formula

$$Pr(f)(\overline{g_0}, g_1) = f(1, \lambda(g_0)^{-1}\lambda(g_0 g_1))$$

where $\overline{g_0}$ denotes the equivalence class of g_0 in $K\backslash G = N$. Now $PHom(\mathbb{Q}[N \times G], M')$ is a p.s. N-module with semi-norms indexed on the set $\Lambda_{M'} \times \mathscr{P}(N \times G)$. Both $ker(\delta^1)$ and $im(\delta^0)$ inherit a p.s. N-module structure via the inclusion into this p.s. N-module, inducing a quotient p.s. N-module structure on $ker(\delta^1)/im(\delta^0)$. Denoting the equivalence class of f in this quotient by $[f]$, we have

$$\eta_{(x,U)}([f]) = \min\{\eta_{(x,U)}(f') \mid [f'] = [f]\}$$

From the commutativity of the above diagram, we see that $[f'] = [f]$ implies $Pr(f') = Pr(f)$. We claim that the map $[f] \mapsto Pr(f)$ is a monomorphism of p.s. modules. In fact, we have an inequality

$$\eta_{(x,U)}(Pr(f)) \leqslant |U| \eta_{(x,Pr(U))}(f) = |U| \eta_{(x,Pr(U))}([f])$$

where $Pr(U)$ denotes the image of $U \subset N \times G$ under the composition $N \times G \overset{\bar{p}_1}{\twoheadrightarrow} \{id\} \times K \rightarrowtail N \times G$; this implies the result. Since we have already shown it is a splitting of the surjection $ker(\delta^1) \twoheadrightarrow ker(\delta^1)/im(\delta^0)$, we conclude that this surjection is an admissible epimorphism of p.s. N-modules, completing the proof of Lemma 1.1.20. □

An immediate consequence of this Lemma is the following 5-term sequence in p-bounded group cohomology.

Theorem 1.1.25 (*5-term sequence*). *Let $(K, L_K) \rightarrowtail (G, L_G) \twoheadrightarrow (N, L_N)$ be a short-exact sequence of groups with word-length, and M' a p.s. N-module. Then there is a short-exact sequence*

$$\begin{aligned} 0 &\to PH^1(N; M') \to PH^1(G; M') \to PH^0(N; PH^1(K; M')) \\ &\to PH^2(N; M') \to PH^2(G; M'). \end{aligned} \tag{1.1.26}$$

Proof. The proof follows exactly as in ordinary group cohomology by Lemma 1.1.20 and Theorem 1.1.12. □

1.2. The obstruction to injectivity

Recall that the natural transformation $PH^*(G; \mathbb{Q}) \to H^*(G; \mathbb{Q})$ induces a map of dual vector spaces $(H^*(G; \mathbb{Q}))^* \to (PH^*(G; \mathbb{Q}))^*$, and that a group satisfies property (PC3) if the composition

$$\alpha_*(G) : H_*(G; \mathbb{Q}) \to (H^*(G; \mathbb{Q}))^* \to (PH^*(G; \mathbb{Q}))^* \tag{1.2.1}$$

is injective for all $* \geqslant 0$.

In this section we will work with a short-exact sequence of groups with word-length

$$(F', L_{F'}) \rightarrowtail (F, L_F) \twoheadrightarrow (G, L_G) \tag{1.2.2}$$

where $F = F_S$ is a free group on a generating set S, L_F is the word-length metric induced by a function $f_S : S \to \mathbb{N}^+$ and (1.2.2) is the short-exact sequence of groups with word-length associated to the short-exact sequence of groups $F' \rightarrowtail F \twoheadrightarrow G$ and the word-length metric L_F. The Serre spectral sequence in homology with coefficients in a module M has $E^2_{*,*}$-term

$$E^2_{p,q} = H_p(G; H_q(F'; M))$$

and converges to $H_*(F; M)$. In general, M is a non-trivial F-module; we use the same notation for the $E^2_{*,*}$-term whether or not the corresponding action of G on $H_q(F'; M)$ is trivial. Now $H_*(F; M) = H_*(F'; M) = 0$ for $* > 1$. From this vanishing we conclude

Proposition 1.2.3. *For all coefficient modules M, there are isomorphisms*

$$H_p(G; M) \overset{\cong}{\to} H_{p-2}(G; H_1(F'; M)) \quad p \geqslant 3$$

and an injection

$$H_2(G; M) \rightarrowtail H_0(G; H_1(F'; M)).$$

We define G-modules inductively as follows:

(1.2.4.i) $B_0 = \mathbb{Q}$ with trivial G-module structure,
(1.2.4.ii) $B_m = H_1(F'; B_{m-1})$, with diagonal G-module structure.

Note that $B_1 = H_1(F'; \mathbb{Q})$, and for $m > 1$ there is a natural isomorphism

$$B_m \cong H_m((F')^m; \mathbb{Q}) \cong \otimes^m H_1(F'; \mathbb{Q})$$

equipped with the diagonal conjugation action of F. The action of F on B_m induced by this action of F on $\otimes^m H_1(F'; \mathbb{Q})$ factors by the projection to G.

Proposition 1.2.5. *There are isomorphisms*

$$H_{2m}(G; B_n) \overset{\cong}{\to} H_{2m-2}(G; B_{n+1}) \quad m \geqslant 2$$

and an injection

$$H_2(G; B_n) \rightarrowtail H_0(G; B_{n+1}).$$

Proof. This is a direct application of the previous proposition with $M = B_n$, as $B_{n+1} = H_1(F'; B_n)$. $\square$

Starting at $H_{2m}(G; \mathbb{Q})$, this proposition produces a sequence

$$H_{2m}(G; \mathbb{Q}) \cong H_{2m-2}(G; B_1) \cong \cdots \cong H_2(G; B_{m-1}) \rightarrowtail H_0(G; B_m) \tag{1.2.6}$$

where the maps in the sequence arise as differentials in the $E^2_{*,*}$-term of the appropriate Serre spectral sequence.

A similar result holds for cohomology.

Proposition 1.2.7. *For all coefficient modules M, there are isomorphisms*

$$H^{p-2}(G;\, H^1(F';\, M)) \overset{\cong}{\to} H^p(G;\, M) \quad p \geqslant 3$$

and a surjection

$$H^0(G;\, H^1(F';\, M)) \twoheadrightarrow H^2(G;\, M).$$

Let $B_m^* = Hom_{\mathbb{Q}}(B_m, \mathbb{Q})$ denote the dual of B_m, with G-module structure given by $gh(x) = h(g^{-1}x)$. The dual of Proposition 1.2.5 is

Proposition 1.2.8. *There are isomorphisms*

$$H^{2m-2}(G;\, B_{n+1}^*) \overset{\cong}{\to} H^{2m}(G;\, B_n^*) \quad m \geqslant 2$$

and a surjection

$$H^0(G;\, B_{n+1}^*) \twoheadrightarrow H^2(G;\, B_n^*).$$

This yields a sequence

$$H^0(G, B_m^*) \twoheadrightarrow H^2(G;\, B_{m-1}^*) \cong H^4(G;\, B_{m-2}^*) \cong \cdots \cong H^{2m}(G;\, \mathbb{Q}). \tag{1.2.9}$$

Again, the maps in the sequence occur as differentials in the $E_2^{*,*}$-term of the appropriate Serre spectral sequence.

Denote the composition in (1.2.6) by i_m and the composition in (1.2.9) by j_m. The following commuting diagram derives from standard properties of the Serre spectral sequence.

$$\begin{tikzcd}
H_{2m}(G;\mathbb{Q}) \arrow[r, tail, "i_m"] \arrow[d] & H_0(G; B_m) \arrow[d] \\
(H^{2m}(G;\mathbb{Q}))^* \arrow[r, tail, "(j_m)^*"] & (H^0(G; B_m^*))^*
\end{tikzcd} \tag{1.2.10}$$

Let $A_0^* = \mathbb{Q}$, and inductively set

$$A_m^* = PH^1(F';\, A_{m-1}^*) \tag{1.2.11}$$

for $m \geqslant 1$. For each m, a p.s. G-module structure on A_{m-1}^* induces a p.s. G-module structure on A_m^*, as shown in Lemma 1.1.20. This gives it a p.s. F-module structure via the projection $F \twoheadrightarrow G$, which when restricted to F' produces a p.s. F'-module structure where the action of F' on both the module and indexing set is trivial. We will examine this structure in more detail later on, noting for now only its existence. Thus starting with the trivial p.f. G-module structure on A_0^* as indicated following (1.1.4), we get a p.s. G-module structure on each A_m^*.

Lemma 1.2.12. *For all $m \geqslant 0$ there is a Serre spectral sequence*

$$E_2^{p,q} = PH^p(G; PH^q(F'; A_m^*))$$

converging to $PH^{p+q}(F; A_m^*)$. Moreover, $PH^q(F'; A_m^*) = 0$ for $q > 1$.

Proof. Recall F is a free group with basis S where it is assumed that $S \cap S^{-1} = \emptyset$. There is a $\mathbb{Q}[F]$-free resolution of $\mathbb{Q}$

$$M_S \xrightarrow{\eta} \mathbb{Q}[F] \to \mathbb{Q} \tag{1.2.13}$$

where $M_S = \mathbb{Q}[F][S]$ is the free $\mathbb{Q}[F]$-module on S, $\mathbb{Q}[F]$ the free $\mathbb{Q}[F]$-module on the single generator [1], and $M_S \to \mathbb{Q}[F]$ is the $\mathbb{Q}[F]$-module map defined on basis elements by $\eta([x]) = (x - 1)$. The map η induces an isomorphism between M_S and the augmentation ideal $I[F]$. We denote this "short complex" in (1.2.13) by $R_*(F)$. For $x \in S \coprod S^{-1}$, set

$$\psi(x) = \begin{cases} [x] & \text{if } x \in S, \\ -x[x^{-1}] & \text{if } x^{-1} \in S. \end{cases}$$

Each $g \in F$ admits a unique reduced word representation $g = x_1 x_2 \dots x_n$ where $x_i \in S \coprod S^{-1}$ for each i. Define $s_0 : \mathbb{Q}[F] \to M_S$ as the $\mathbb{Q}$-vector space map given on basis elements by

$$s_0(g[1]) = s_0(x_1 x_2 \dots x_n[1]) = \sum_{j=1}^{n} x_1 x_2 \dots x_{j-1} \psi(x_j).$$

The fact that L_F is a word-length metric implies $|x_1 x_2 \dots x_{j-1} \psi(x_j)| \leqslant 2L(g)$ for each j, and also that $L(g) \geqslant n$. From this, one concludes that for each $m \in \mathbb{N}$ there is a sequence of inequalities

$$|s_0(g)| \leqslant n(1 + 2L(g)) \leqslant (1 + 2L(g))^2$$

which in turn implies s_0 is p.s. module homomorphism with respect to the p.f. F-module semi-norms on $\mathbb{Q}[F]$ resp. $\mathbb{Q}[F][S]$. Since η is easily seen to be a p.s. F-module homomorphism, we conclude that the short complex described above is a p.f. resolution of $\mathbb{Q}$ over $\mathbb{Q}[F]$.

Proposition 1.2.14. *Let F' be a subgroup of F with induced metric. Then*

$$PH^*(F'; A) \cong PH^*_{F'}(R_*(F); A)$$

for any p.s. F'-module A.

Proof. As the function f_S is $\mathbb{N}^+$-valued, so is the word-length metric L_F. Let $T = F' \backslash F$ denote the right coset space, and $p : F \twoheadrightarrow T$ the natural projection. Define $f_T : T \to \mathbb{R}^+$ by $f_T(F'g) = \min\{L_F(f'g) \mid f' \in F'\}$. Note that as L_F is $\mathbb{N}^+$-valued, so is f_T. Thus for all $g \in F$, there exists an $f'_g \in F'$ with $f_T(F'g) = L_F(f'_g g)$. Choosing such an f'_g for each g and writing $F'g \in T$ as $\bar{g}$, we set $s(\bar{g}) = f'_g g$. Also, we will write $f_{T \times S}$ for the function

$T \times S \ni (t, s) \mapsto f_T(t) + f_S(s)$. Finally for $g \in F$ we denote $g(s(\bar{g}))^{-1} \in F'$ by $\lambda(g)$, as before. By construction, $L_F(\lambda(g)) \leqslant 2L_F(g)$.We now consider the morphism of complexes

$$\begin{array}{ccc} \mathbb{Q}[F][S] & \underset{s_0}{\overset{d_1}{\rightleftarrows}} & \mathbb{Q}[F] \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ \mathbb{Q}[F'][T \times S] & \underset{s_0'}{\overset{d_1'}{\rightleftarrows}} & \mathbb{Q}[F'][T] \end{array} \tag{1.2.15}$$

The top row is $R_*(F)$. Denote the bottom row by $\tilde{R}_*(F')$. The p.f. F'-module structure on $\mathbb{Q}[F'][T \times S]$ resp. $\mathbb{Q}[F'][T]$ is that induced by the function $f_{T \times S}$ resp. f_T, and the left action of F' on $\mathbb{Q}[F']$. The maps ϕ_i and their inverses are defined as

$$\begin{aligned} &\phi_1(g[s]) = \lambda(g)[\bar{g}, s], \quad \phi_1^{-1}(g'[\bar{g}, s]) = g's(\bar{g})[s], \\ &\phi_0(g) = \lambda(g)[\bar{g}], \quad \phi_0^{-1}(g'[\bar{g}]) = g's(\bar{g}). \end{aligned} \tag{1.2.16}$$

It is easily verified that ϕ_i^{-1} is non-increasing in norm, while ϕ_i increases the norm by no more than a factor of three. Defining

$$d_1' = \phi_0 \circ d_1 \circ \phi_1^{-1}$$

$$s_0' = \phi_1 \circ s_0 \circ \phi_0^{-1}$$

in diagram (1.2.15) makes ϕ_* a p-bounded $\mathbb{Q}[F']$-module isomorphism of complexes with p-bounded inverse. Moreover, the p-boundedness of ϕ_* and ϕ_*^{-1} make the contraction s_0' p-bounded. Thus $\tilde{R}_*(F')$ satisfies the hypothesis of the Comparison Theorem. We then have isomorphisms

$$PH^*(F'; A) \cong PH^*_{F'}(\tilde{R}_*(F'); A) \cong PH^*_{F'}(R_*(F); A)$$

by the Comparison Theorem, together with the p-boundedness of ϕ_* and its inverse. $\square$

The complex $R_*(F)$ is zero above dimension one. Thus

Corollary 1.2.17. *For all (free) subgroups with word-length function $(F', L_{F'})$ of (F, L_F), $(L_{F'} = (L_F)|_{F'})$ and p.s. F'-modules M, $PH^*(F'; M) = 0$ for $* \geqslant 2$.*

We now return to the bicomplex used in the proof of Theorem 1.1.12. Referring to (1.1.18), we see that Corollary 1.2.17 implies $ker(\delta^k) = im(\delta^{k-1})$ for all $k \geqslant 2$, so that Hypothesis 1.1.H(k) is trivially satisfied for $k \geqslant 2$. By Lemma 1.1.20, Hypothesis 1.1.H(k) is always satisfied for $k = 0$ and $k = 1$. Consequently Hypothesis 1.1.H is satisfied as stated, and Theorem 1.1.12 applies, completing the proof of Lemma 1.2.12. $\square$

Corollary 1.2.18. *There is a sequence*

$$PH^0(G;\, A_m^*) \twoheadrightarrow PH^2(G;\, A_{m-1}^*) \cong PH^4(G;\, A_{m-2}^*) \cong \cdots \cong PH^{2m}(G;\, \mathbb{Q}) \tag{1.2.19}$$

where the maps in the sequence occur as differentials in the $E_2^{*,*}$-term of the appropriate Serre spectral sequence for p-bounded cohomology.

The proof is exactly as before, given the previous lemma. The natural transformation $PH^*(G;\, A) \to H^*(G;\, A)$ induces an equally natural transformation

$$(H^*(G;\, A))^* \to (PH^*(G;\, A))^*.$$

Together with (1.2.6) and the duals of (1.2.9) and (1.2.19) we arrive at the following commuting diagram which is an extension of (1.2.10):

$$\begin{tikzcd}
H_{2m}(G;\mathbb{Q}) \arrow[r, tail] \arrow[d, tail] & H_0(G;B_m) \arrow[d, tail] \\
(H^{2m}(G;\mathbb{Q}))^* \arrow[r, tail] \arrow[d] & (H^0(G;B_m^*))^* \arrow[d] \\
(PH^{2m}(G;\mathbb{Q}))^* \arrow[r, tail] & (PH^0(G;A_m^*))^*
\end{tikzcd} \tag{1.2.20}$$

The injectivity of the horizontal arrows follows from what we have already shown. Recall property (PC3) is the statement

$$\alpha_*(G): H_*(G;\, \mathbb{Q}) \to (H^*(G;\, \mathbb{Q}))^* \to (PH^*(G;\, \mathbb{Q}))^* \tag{1.2.21}$$

is injective for all $*\geqslant 0$.

Proposition 1.2.22. *If the composition*

$$\beta_m(G): H_0(G;\, B_m) \rightarrowtail (H^0(G;\, B_m^*))^* \to (PH^0(G;\, A_m^*))^*$$

is injective for all $m \geqslant 0$, then $\alpha_{2m}(G)$ is injective for all $m \geqslant 0$.

Proof. This is an immediate consequence of (1.2.20). $\square$

1.3. Analysis of the obstruction

Proposition 1.2.22 above identifies a condition sufficient to guarantee injectivity of $\alpha_n(G)$ in even dimensions, and a similar analysis works in odd dimensions after crossing G with $\mathbb{Z}$. The purpose of this section is to indicate the relationship between the injectivity of $\beta_m(G)$ and the first Dehn function of G when G is finitely-presented. In dimension 2 (cf. Theorem 2.1.3 below), $\alpha(G, L_G)$ is an isomorphism with arbitrary coefficients when the first Dehn function of G is of polynomial type. In higher dimensions, the injectivity of $\beta_m(G)$ follows if one can show that a certain natural class of projection maps are admissible epimorphisms

(Theorem 1.3.5 below). This section is in preliminary form; a sequel to this paper will contain a much more detailed analysis of Conjecture A, along with complete proofs of the results stated in this section.

We begin by recalling the definition of Dehn functions. Let $\mathscr{P} = \langle \mathscr{S} | \mathscr{W} \rangle$ be a finite presentation of a discrete group G. Then there is a short-exact sequence

$$F' \rightarrowtail F \twoheadrightarrow G \tag{1.3.1}$$

where F is the free group on the (finite) set of generators $\mathscr{S}$ and F' the subgroup of F normally generated by the (finite) set of relators $\mathscr{W}$. We take L_F to be the standard word-length metric on F which takes the value 1 on each generator, with L_G the standard word-length function on G induced by L_F and the projection $F \twoheadrightarrow G$. For $w \in \mathscr{W}$, denote its image in F' by $\bar{w}$. Any element $y \in F'$ may be written as

$$y = \bar{w}_1^{x_1} \bar{w}_2^{x_2} \dots \bar{w}_n^{x_n} \tag{1.3.2}$$

where for each i, $w_i^{\pm 1} \in \mathscr{W}$, $x_i \in F$ and $\bar{w}^x = x\bar{w}x^{-1}$. The *area* of y, written $Area_{\mathscr{P}}(y)$, is the minimum number n such that y can be written as in (1.3.2). A map $f : \mathbb{N} \to \mathbb{N}$ is called an *isoperimetric function* for the presentation if

$$Area_{\mathscr{P}}(y) \leqslant f(n)$$

for all relations y with $L_F(y) \leqslant n$. Among all isoperimetric functions associated to $\mathscr{P}$ there is a minimal one, $f_{\mathscr{P}}$, referred to as the *Dehn function* of the presentation $\mathscr{P}$.

Dehn functions are due to Gersten [4,5]. We say that the Dehn function is of *polynomial type* if it is bounded above by a polynomial function.

Some notation. We will write $H_1(F'; \mathbb{Q})$ as $R = R(1)$, and in general for $n \geqslant 1$ denote $\otimes^n H_1(F'; \mathbb{Q})$ by $R(n)$. Recall that as a subgroup of F, F' is normally generated by elements of the form $(\bar{w})^g$ where $w \in \mathscr{W}$ and $g \in F$. The image of $(\bar{w})^g$ in $H_1(F'; \mathbb{Q})$ only depends on $\bar{w}$ and the image of $g \in G$. Thus R is spanned as a vector space over $\mathbb{Q}$ by $\{[(\bar{w})^g]\}_{w \in \mathscr{W}, g \in G}$, where $[(\bar{w})^g]$ denotes the image of $(\bar{w})^g \in F'$ under the canonical map $F' \to R$. From this we see there is a natural surjection

$$\mathbb{Q}[G][\mathscr{W}] \twoheadrightarrow R, \quad (g, w) \mapsto [(\bar{w})^g]. \tag{1.3.3}$$

Let $P(n) = \otimes^n(\mathbb{Q}[G][\mathscr{W}])$. The map in (1.3.3) induces a surjection of n-fold tensor products

$$p_n : P(n) \twoheadrightarrow R(n), \quad ((g_1, w_1), \dots, (g_n, w_n)) \overset{p_n}{\mapsto} ([(\bar{w}_1)^{g_1}], \dots, [(\bar{w}_n)^{g_n}]). \tag{1.3.4}$$

The word-length function on F' induces a weight function on $\mathscr{W}$; together with the word-length function on G we get a p.f. G-module structure on $P(n)$. This induces a (quotient) p.s. G-module structure on $R(n)$, where in both cases the G-action is the diagonal one. Let $T_n = \prod_1^n (G \times \mathscr{W})$, so that $P(n) = \mathbb{Q}[T_n]$. For each $x \in T_n$, let $[x]$ denote its image in $H_0(G; P(n)) = P(n)_G$, and $[p_n(x)]$ its image in $R(n)_G$. For $[p_n(x)] \neq 0$, let $\mathbb{Z}_{[p_n(x)]}$ be the copy of $\mathbb{Z}$ generated by this element. Again, the p.s. G-module structure above induces a quotient p.s. module structure on $R(n)_G$, and so by restriction a p.s. module structure on $\mathbb{Z}_{[p_n(x)]}$ for each $[p_n(x)] \neq 0$. Finally for each such $[p_n(x)]$ we may restrict the p.s. module structure on $R(n)_G$ to $\mathbb{Z}_{[p_n(x)]}$. It is not hard to show that this is the same as the p.f. module structure induced by the (quotient) length function on $\mathbb{Z}_{[p_n(x)]}$. Note that for each x there is a canonical word-length metric on $\mathbb{Z}_x =$ the subgroup of $P(n)$ generated by x and a projection $\mathbb{Z}_x \to \mathbb{Z}_{[p_n(x)]}$ which is an isomorphism of abelian groups.

Theorem 1.3.5. *If the projection map $p_x : \mathbb{Z}_x \to \mathbb{Z}_{[p_n(x)]}$ is an admissible epimorphism for each x with $[p_n(x)] \neq 0$, then the map $\beta_n(G)$ is injective.*

We give a sketch of the proof; a more detailed proof (including a proof of the two technical lemmas below) will appear in the sequel. The hypothesis of the theorem implies that the projection map $\mathbb{Z}_x \to \mathbb{Z}_{[p_n(x)]}$ is a p-equivalence (i.e., a p-bounded isomorphism of p.s. modules). This in turn implies that the p-bounded homomorphism $\mathbb{Z}_x \to \mathbb{Q}$ induced by the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ (equipped with standard p.f. module structure) factors by the projection p_x. Denote the p-bounded homomorphism $\mathbb{Z}_{[p_n(x)]} \to \mathbb{Q}$ by $\phi_{[p_n(x)]}$. Now let V be a subspace of $R(n)_G$ spanned by a *finite* number of elements $\{[p_n(x_1)], [p_n(x_2)], \ldots, [p_n(x_n)]\}$. Then V inherits a p.s. module structure from its embedding into $R(n)_G$.

Lemma 1.3.6. *Let $0 \neq a \in V$. Then there exists $1 \leqslant i \leqslant n$ and a p-bounded extension $\phi : V \to \mathbb{Q}$ of $\phi_{[p_n(x_i)]} : \mathbb{Z}_{[p_n(x_i)]} \to \mathbb{Q}$ with $\phi(a) \neq 0$.*

Lemma 1.3.7. *If V is a finite-dimensional subspace of $R(n)_G$ and $\phi : V \to \mathbb{Q}$ a p-bounded homomorphism, then ϕ extends to a p-bounded homomorphism $\phi : R(n)_G \to \mathbb{Q}$.*

The proof of Lemma 1.3.6 uses in an essential way the fact that V is finite-dimensional, while Lemma 1.3.7 is an analogue of the Hahn–Banach theorem. Together they imply Theorem 1.3.5. For suppose $0 \neq y \in H_0(G; B_n) = R(n)_G$. Because homology has finite supports, the image of y lies in a finite-dimensional subspace of the type considered in Lemma 1.3.6, which guarantees the existence of a p-bounded homomorphism $\phi : V \to \mathbb{Q}$ with $\phi(y) \neq 0$. By Lemma 1.3.7, this homomorphism may be extended over all of $R(n)_G$, yielding an element of $PH^0(G; A_n^*)$ which pairs non-trivially with the image of y. This implies $\beta_n(G)(y) \neq 0$. Varying y then implies the injectivity of $\beta_n(G)$.

One may consider a hypothesis similar to that in Theorem 1.3.5, but without passing to G-coinvariants. For $x \in T_n$, $p_n(x)$ denotes the image in $R(n)$. If $p_n(x) \neq 0$, we denote by $\mathbb{Z}_{p_n(x)}$ the copy of $\mathbb{Z}$ in $R(n)$ generated by $p_n(x)$, and by $\mathbb{Z}_x$ the corresponding copy of $\mathbb{Z}$ in $P(n)$ generated by x.

Theorem 1.3.8. *Suppose that G is a finitely-presented group with Dehn function of polynomial type. Then for each $n \geqslant 1$ and $x \in T_n$ with $p_n(x) \neq 0$, the projection map p_n induces an equivalence of p.s. modules $\mathbb{Z}_x \to \mathbb{Z}_{p_n}(x)$.*

In fact, the condition on the Dehn function of G directly implies the result for $n = 1$, and the result for $n > 1$ follows directly from the case $n = 1$.

It follows from Theorem B of the introduction (proved below in Section 2) that $\beta_1(G)$ is an injection when the first Dehn function is of polynomial type. In fact, using the five-term exact sequence the hypothesis of Theorem 1.3.5 is easily verified in this case. However, unlike the situation in Theorem 1.3.8, the case $n > 1$ in Theorem 1.3.5 does not follow in any obvious way from the case $n = 1$.

1.4. The map $\alpha_1(G, L_G)$

As we have observed in the introduction, $\alpha_i(G, L_G)$ depends only on G and the choice of length function. Thus if G is generated by a set S and L_G is the word-length function determined by a function $f : S \to \mathbb{N}^+$ (i.e., by the word-length metric L_F determined by f on the free group F generated by S, together with the natural surjection $F \twoheadrightarrow G$) then L_G and so also $\alpha_i(G, L_G)$ is independent of the choice of relator set W in a presentation $P = \langle S|W\rangle$ of G which uses S as the set of generators. We verify Theorem B of the introduction by proving the following three Lemmas. We assume that G_{ab}, the abelianization of G, is finitely generated.

Lemma 1.4.1. *Let $\bar{G} = G_{ab}/G_{ab}^{\text{torsion}}$. Then there is a system of generators $\bar{S}$ for $\bar{G}$ and a weight function $\bar{f} : \bar{S} \to \mathbb{N}^+$ for which $\alpha_1(\bar{G}, L_{\bar{G}})$ is injective, where $L_{\bar{G}}$ is the word-length function determined by $\bar{f}$.*

Proof. By the assumption on G_{ab}, $\bar{G} \cong \mathbb{Z}^r$ for some finite integer $r \geqslant 0$. Let $\bar{S} = \{\bar{x}_1, \ldots, \bar{x}_r\}$ be a basis for $\bar{G}$, and set $f_{\bar{S}}(\bar{x}_i) = 1$ for all i. Let $L_{\bar{G}}$ be the word-length function determined by $f_{\bar{S}}$ (in other words, the standard word-length function associated with this set of generators). Then $\alpha_1(\bar{G}, L_{\bar{G}})$ is an isomorphism as observed above. □

Lemma 1.4.2. *Given $\bar{S}$ and $f_{\bar{S}}$ as in the previous Lemma, there exists a generating set S for G and proper function $f : S \to \mathbb{N}^+$ so that $(G, L_G) \twoheadrightarrow (\bar{G}, L_{\bar{G}})$ is surjection of groups with word-length function.*

Proof. We first choose a set of elements $S' \subset G$ which maps isomorphically to $\bar{S} \subset \bar{G}$ under the surjection $G \twoheadrightarrow \bar{G}$, and set $f_{S'}(x_i') = f_{\bar{S}}(\bar{x}_i)$ where $x_i' \in S'$ maps to $\bar{x}_i$. Let S'' be an arbitrary set of generators for $ker(G \twoheadrightarrow \bar{G})$ and let $f_{S''} : S'' \to \mathbb{N}^+$ a proper function on S''. Then $S = S' \coprod S''$ is a generating set for G equipped with proper function $f = f_{S'} \coprod f_{S''}$. Setting L_G to be the word-length function determined by f completes the proof. □

Lemma 1.4.3. *Suppose $\phi : (G, L_G) \to (H, L_H)$ is a p-bounded homomorphism of groups with word-length function. If $\phi_1 : H_1(G; \mathbb{Q}) \to H_1(H; \mathbb{Q})$ and $\alpha_1(H, L_H)$ are injective, then $\alpha_1(G, L_G)$ is injective.*

Proof. This follows from the naturality of $\alpha_1(G, L_G)$ with respect to p-bounded homomorphisms of groups equipped with word-length function. □

Taking $(H, L_H) = (\bar{G}, L_{\bar{G}})$, these three Lemmas together imply Theorem B.

2. Higher Dehn functions

2.1. Dehn functions and simplicial resolutions

We begin by considering a variant of the Dehn function associated to a presentation. As already noted, an element $w \in F'$ can be written as

$$w = w_1^{x_1} w_2^{x_2} \dots w_n^{x_n} \tag{2.1.1}$$

where for each i, $w_i^{\pm 1} \in \mathscr{W}$ and $x_i \in F$, and $Area_{\mathscr{P}}(w)$ is the minimum number n such that w can be written as in (2.1.1). Analogously, define $Area'_{\mathscr{P}}(w)$ as the smallest integer m' such that w can be written as in (2.1.1) with $m' = \sum_{i=1}^{k} L(w_i) + 2L(x_i)$. Let $f'_{\mathscr{P}}$ be the minimal isoperimetric function defined using $Area'_{\mathscr{P}}$ instead of $Area_{\mathscr{P}}$. The inequalities $Area_{\mathscr{P}} \leqslant Area'_{\mathscr{P}}$ and $f_{\mathscr{P}} \leqslant f'_{\mathscr{P}}$ are obvious. The following result is due to Gersten [8].

Lemma 2.1.2. *Let $M = max\{L(w) \mid w \in \mathscr{W}\}$. Then*

$$f'_{\mathscr{P}}(n) \leqslant (2M) f_{\mathscr{P}}(n)^2 + (2n + M) f_{\mathscr{P}}(n).$$

In preparation for what follows, we will need to recall some terminology and constructions used in [18]. For standard properties of simplicial sets, we refer the reader to [12].

For a simplicial group $\Gamma.$ set

$$\Gamma_n^{-1} = \Gamma_n,$$
$$\Gamma_n^k = \bigcap_{i=0}^{k} ker(\partial_i : \Gamma_n \twoheadrightarrow \Gamma_{n-1}) \quad \text{for } k \geqslant 0, n \geqslant 1,$$
$$\Gamma_0^0 = \partial_1(\Gamma_1^0).$$

For $0 \leqslant k < n$ and $n \geqslant 1$ there is a split short-exact sequence

$$\Gamma_n^k \rightarrowtail \Gamma_n^{k-1} \overset{\partial_k}{\twoheadrightarrow} \Gamma_{n-1}^{k-1}$$

with the splitting induced by the restriction of s_k to Γ_{n-1}^{k-1}. When $k = n$ there is an exact sequence

$$\Gamma_n^n \rightarrowtail \Gamma_n^{n-1} \overset{\partial_n}{\to} \Gamma_{n-1}^{n-1}$$

and the Kan extension property implies

$$\pi_n(\Gamma.) = \Gamma_n^n / (\partial_{n+1}(\Gamma_{n+1}^n))$$

where $\Gamma.$ is viewed here as a simplicial set with basepoint $1 \in \Gamma_0$. One also has

$$\pi_0(\Gamma.) = \Gamma_0 / \Gamma_0^0.$$

We say that $\Gamma.$ is a *resolution* if $\pi_n(\Gamma.) = 0$ for all $n > 0$. This is equivalent to the condition that $\Gamma_n^{n-1} \overset{\partial_n}{\to} \Gamma_{n-1}^{n-1}$ is a surjection for all $n \geqslant 1$ (note that it need not be a split-surjection).

A simplicial group $\Gamma.$ equipped with word-length function $L.$ is a simplicial group $\Gamma. = \{[n] \mapsto \Gamma_n\}_{n\geqslant 0}$ where L_n is a word-length function on Γ_n for each n, and all face and degeneracy maps are p-*bounded*. The simplicial group together with its word-length function will be written as a pair $(\Gamma., L.)$. A word-length function on an augmented simplicial group $\Gamma.^+$ is a word-length function $L.$ on the associated simplicial group $\Gamma.$ ($\Gamma_n = \Gamma_n^+$ for $n \geqslant 0$) together with a word-length function L_{-1} on Γ_{-1} induced by L_0 and the augmentation map $\varepsilon : \Gamma_0 \twoheadrightarrow \Gamma_{-1}$. The resulting augmented simplicial group together with word-length function is written as $(\Gamma.^+, L.^+)$. The associated simplicial group with word-length function $(\Gamma., L.)$ is gotten by restricting to simplicial dimensions $n \geqslant 0$. A map ϕ of simplicial or augmented simplicial groups is p-*bounded* if it is p-bounded in each simplicial degree.

Occasionally we need to keep track of generating sets, in which case they are included in the notation. As always, we assume generating sets are countable. We call $(F, \mathbb{X}, L)$ a *triple* when F is the free group with basis $\mathbb{X}$ equipped with a function $f : \mathbb{X} \to \mathbb{N}^+$, and L_F is the word-length metric induced by f. This definition extends to the augmented simplicial setting. A triple $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ indicates (i) $(\Gamma.^+, L.^+)$ is an augmented simplicial group with word-length, (ii) $(\Gamma_n, \mathbb{X}_n, L_n)$ is a triple for each $n \geqslant 0$ and (iii) Γ_{-1} is generated by $\mathbb{X}_{-1} = \mathbb{X}_0$. We do not put any additional restriction on the face and degeneracy maps when including a generating set $\mathbb{X}.^+$ (although in practice it can always be arranged for $\mathbb{X}.^+$ to be closed under degeneracies).

For an augmented simplicial group $\Gamma.^+$, we will denote the kernel $ker(\Gamma.^+ \twoheadrightarrow \Gamma_{-1})$ of the simplicial augmentation map as $\Gamma(\varepsilon).^+$. This is an augmented simplicial subgroup of $\Gamma.^+$ with $\Gamma(\varepsilon)_{-1} = \{1\}$.

We say that a free resolution $(\Gamma.^+, L.^+)$ or $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ is *type* $P(m)$ if $\Gamma(\varepsilon).^+$, viewed as a *simplicial set*, admits a simplicial contraction through dimension $(m-1)$ which is p-bounded in each degree (with respect to $L.^+$). The resolution is *type* P if $\Gamma(\varepsilon).^+$ admits a simplicial contraction (of simplicial sets) which is p-bounded in all degrees. Type P is slightly stronger than being type $P(m)$ for all m.

Theorem 2.1.3. *If G is a finitely-presented group with polynomial Dehn function f_G, then $\alpha_2(G, L_G^{\mathrm{st}})$ is an isomorphism.*

Proof. As above, we denote the finite set of generators of G by $\mathscr{S}$ and the finite set of relators by $\mathscr{W}$. Let Γ_0 be the free group on $\mathbb{X}_0 = \mathscr{S}$, and Γ_1 the free group on $\mathbb{X}_1 = \mathscr{S} \coprod \mathscr{W}$. Let $\varepsilon : \Gamma_0 \twoheadrightarrow G$ be the obvious projection. The natural inclusion $\mathbb{X}_0 \hookrightarrow \mathbb{X}_1$ determines a monomorphism $s_0 : \Gamma_0 \rightarrowtail \Gamma_1$. Define ∂_i on Γ_1 $(i = 0, 1)$ as the unique homomorphism determined on generators by

$$\begin{aligned} &\partial_i(s) = s \quad \text{if } s \in \mathscr{S},\ i = 0, 1 \\ &\partial_0(w), = id, \quad \partial_1(w) = \bar{w} \end{aligned} \tag{2.1.4}$$

where $\bar{w}$ denotes w viewed as an element of Γ_0. For $m \geqslant 2$ let Γ_m be the free group on

$$\mathbb{X}_m = \coprod \{s(\mathbb{X}_1)\}$$

where the coproduct is over all iterated degeneracies from dimension 1 to dimension m. Finally let $\Gamma_{-1} = G$. The partial simplicial structure on $\{\Gamma_n\}_{-1 \leqslant n \leqslant 1}$ defined above admits a unique extension to an augmented simplicial structure on $\Gamma.^{+} = \{\Gamma_n\}_{n \geqslant -1}$. The word-length function $L.^{+} = \{L_n\}_{n \geqslant -1}$ on $\Gamma.^{+}$ is the standard one in dimensions -1 and 0. In dimension 1 it is the metric determined by the function $x \mapsto L_0(x)$, $w \mapsto L_0(\bar{w})$ where $x \in \mathscr{S}$ and $w \in \mathscr{W}$. In dimensions $\geqslant 2$ it is the unique metric defined on generators by $L_m(s(x)) = L_1(x)$ where $x \in \mathbb{X}_1$ and s is an iterated degeneracy from Γ_1 to Γ_m. Then $(\Gamma.^{+}, L.^{+})$ is an augmented free p-bounded simplicial group which is $(G, L_{-1} = L_G)$ in dimension -1, and equipped with a word-length metric in non-negative degrees.

Claim 2.1.5. *If the Dehn function of G is polynomial, then there is a* p*-bounded section of sets* $s_1' : \Gamma_0^0 \rightarrowtail \Gamma_1^0$ *which is a left-inverse to* ∂_1 *restricted to* Γ_1^0.

Proof. As $\Gamma_0^0 = ker(\varepsilon)$ is the subgroup of Γ_0 normally generated by the relators $\mathscr{W}$, an element $\bar{w} \in \Gamma_0^0$ may be written

$$\bar{w} = \bar{w}_1^{x_1} \bar{w}_2^{x_2} \ldots \bar{w}_n^{x_n} \tag{2.1.6}$$

as in (2.1.1), where $x_i \in \Gamma_0$ and $w_i^{\pm} \in \mathscr{W} \subset \mathbb{X}_1$. We use the convention of (2.1.4) to distinguish between $w \in \mathscr{W}$ and $\bar{w} = \partial_1(w) \in \Gamma_0$. A p-bounded section s_1' exists if and only if there are constants $C, n > 0$ such that for all $w \in \Gamma_0^0$, there exists a choice of w_i and x_i in (2.1.6) for which

$$L_1(w_1^{s_0(x_1)} w_2^{s_0(x_2)} \ldots w_n^{s_0(x_n)}) \leqslant C(1 + L_0(w))^n.$$

Since

$$L_1(w_1^{s_0(x_1)} w_2^{s_0(x_2)} \ldots w_n^{s_0(x_n)}) \leqslant \sum_{j=1}^{n} L_1(w_i) + 2L_0(x_i). \tag{2.1.7}$$

Gersten's Lemma 2.1.4 implies the left-hand side of (2.1.7) is quadratically bounded by the Dehn function f_G of G. Then f_G polynomial implies the claim. $\square$

Continuing with the proof of Theorem 2.1.3, we see that $(\Gamma.^{+}, L.^{+})$ is type $P(1)$ as defined above. By Theorem A.1 of the appendix, there is an inclusion of augmented simplicial groups with word-length (and generating sets)

$$(\Gamma.^{+}, \mathbb{X}.^{+}, L.^{+}) \hookrightarrow (\widetilde{\Gamma}.^{+}, \widetilde{\mathbb{X}}.^{+}, \widetilde{L}.^{+})$$

where $(\widetilde{\Gamma}.^{+}, \widetilde{\mathbb{X}}.^{+}, \widetilde{L}.^{+})$ is a type P resolution and $\widetilde{\Gamma}_i = \Gamma_i$ for $i = 0, 1$. Now set $D^n(\Gamma., \mathbb{Q}) = PH^1(\Gamma_{n-1}; \mathbb{Q})$ for $n \geqslant 1$, and 0 for $n = 0$. Similarly, let $E^n(\Gamma., \mathbb{Q}) = H^1(\Gamma_{n-1}; \mathbb{Q})$ for $n \geqslant 1$, and 0 for $n = 0$. There are coboundary maps $\delta^n = \sum_{i=1}^{n} (-1)^i (\partial_i)^* : E^n(\Gamma., \mathbb{Q}) = H^1(\Gamma_{n-1}; \mathbb{Q}) \to E^{n+1}(\Gamma., \mathbb{Q}) = H^1(\Gamma_n; \mathbb{Q})$ making $(E^*(\Gamma., \mathbb{Q}), \delta^*)$ a cocomplex. Because each face map ∂_i is p-bounded, we also get a well-defined coboundary map $\delta^n : D^n(\Gamma., \mathbb{Q}) = PH^1(\Gamma_{n-1}; \mathbb{Q}) \to D^{n+1}(\Gamma., \mathbb{Q}) = PH^1(\Gamma_n; \mathbb{Q})$ given by the same expression. In addition, since (Γ_n, L_n) is a free group with word-length metric, there is for each $n \geqslant 0$ an inclusion$D^n(\Gamma., \mathbb{Q}) \hookrightarrow E^n(\Gamma., \mathbb{Q})$ which is clearly compatible with the coboundary maps just defined, yielding an inclusion of cocomplexes

$$(D^*(\Gamma., \mathbb{Q}), \delta^*) \hookrightarrow (E^*(\Gamma., \mathbb{Q}), \delta^*). \tag{2.1.8}$$

By Theorem A.12 of the appendix, there are isomorphisms

$$\begin{aligned} PH^*(G;\mathbb{Q}) &\cong H^*(D^*(\Gamma.,\mathbb{Q}),\delta^*) \\ H^*(G;\mathbb{Q}) &\cong H^*(E^*(\Gamma.,\mathbb{Q}),\delta^*) \end{aligned} \tag{2.1.9}$$

under which the inclusion of (2.1.8) induces the transformation

$$PH^*(G;\mathbb{Q}) \to H^*(G;\mathbb{Q}). \tag{2.1.10}$$

Because the generating set for Γ_i is finite for $i=0,1$, the map in (2.1.8) is an isomorphism for $*=1,2$. Together with the injectivity of the map for $*=3$, Theorem A.12 implies the map in (2.1.10) is an isomorphism for $*=1,2$. □

If A is a p.s. G-module, it is a p.s. Γ_i-module via the augmentation $\Gamma_i \twoheadrightarrow G$. One may then replace the coefficient module $\mathbb{Q}$ by A in the above discussion. The result is again an inclusion of cocomplexes

$$(D^*(\Gamma.,A),\delta^*) \hookrightarrow (E^*(\Gamma.,A),\delta^*) \tag{2.1.11}$$

and isomorphisms

$$\begin{aligned} PH^*(G;A) &\cong H^*(D^*(\Gamma.,A),\delta^*) \\ H^*(G;A) &\cong H^*(E^*(\Gamma.,A),\delta^*) \end{aligned} \tag{2.1.12}$$

under which the inclusion of (2.1.11) induces the transformation

$$PH^*(G;A) \to H^*(G;A). \tag{2.1.13}$$

The finiteness of Γ_i for $i=0,1$ implies

Corollary 2.1.14. *If G is a finitely-presented group with Dehn function of polynomial type and A is a p.s. G-module, then* $PH^*(G;A) \cong H^*(G;A)$ *for* $*=0,1,2$.

2.2. Higher Dehn functions and cohomology

Suppose π is an HF^∞ group, i.e. one with a classifying space $B\pi$ the homotopy type of a CW complex with finitely many cells in each dimension. One can show this is equivalent to the condition that there is a simplicial set $X.$ with $X_0=*$ and $|X|\simeq B\pi$ where $X.$ has finitely many simplices in each dimension. The augmented Kan loop group $GX.^+$ of $X.$ is π in dimension -1, and in dimension $n\geqslant 0$ is the free group on generators $\mathbb{X}_n = X_{n+1} - s_0(X_n)$, which is a finite set. We write $GX.^+$ as $\Gamma.^+$. For $n\geqslant 0$, L_n is the standard word-length function associated with the set of generators $\mathbb{X}_n$. The generating set $\mathbb{X}_0$ determines a generating set for π, and we take L_{-1} to be the standard word-length function on π associated with those generators. Then $L.^+=\{L_n\}_{n\geqslant -1}$ is an augmented simplicial word-length function, and $(\Gamma.^+, L.^+)$ is an augmented free simplicial resolution with word-length function, with Γ_n generated by a finite set $\mathbb{X}_n$ and L_n the standard word-length function determined by $\mathbb{X}_n$. We assume for this section that $(\Gamma.^+, L.^+)$ is as just described. As discussed in the previous section, the fact $\Gamma.^+$ is a resolution implies the homomorphism

$$\partial'_{n+1} : \Gamma^n_{n+1} \to \Gamma^n_n \tag{2.2.1}$$

induced by the restriction of ∂_{n+1} to Γ^n_{n+1}, is surjective for all $n \geqslant 0$. We now define the nth Dehn function f^Γ_n associated to $\Gamma.^+$ as the smallest $\mathbb{N}$-valued function for which there exists a section of sets $s'_{n+1} : \Gamma^n_n \to \Gamma^n_{n+1}$ with $\partial'_{n+1} \circ s'_{n+1} = id$ and

$$L_{n+1}(s'_{n+1}(x)) \leqslant f^\Gamma_n(L_n(x)) \quad \forall x \in \Gamma^n_n. \tag{2.2.2}$$

The Dehn function of $(\Gamma.^+, L.^+)$ is $f.^\Gamma = \{f^\Gamma_n\}_{n \geqslant 0}$. An element $x \in \Gamma^n_n$ induces a map $\phi_x : S^n \to |\Gamma.|$ and an element $y \in \Gamma^n_{n+1}$ with $\partial_{n+1}(y) = x$ induces a null-homotopy of ϕ_x. So this definition is the simplicial analogue of the classical geometric situation where one bounds the volume of a null homotopy of a map $S^n \to M$ by a function evaluated on the volume of (the image of) S^n. We call two functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ p-*equivalent* if there are polynomials p_1, p_2 such that $f_1 \leqslant p_2 \circ f_2$ and $f_2 \leqslant p_1 \circ f_1$. A long and tedious argument using simplicial identities proves the following

Theorem 2.2.3. *If $f.^\Gamma$ is the Dehn function for $(\Gamma.^+, L.^+)$, then $\Gamma(\varepsilon).^+ = ker(\Gamma.^+ \twoheadrightarrow \pi)$ admits an extra degeneracy $s.' = \{s'_{n+1}\}_{n \geqslant -1}$ satisfying the property that*

$$L_{n+1}(s'_{n+1}(x)) \leqslant f'_n(L_n(x)) \quad \forall x \in \Gamma(\varepsilon)_n$$

where f'_n is p-equivalent to f^Γ_n for all n.

Note that the converse is obvious, since $\Gamma(\varepsilon)^n_n = \Gamma^n_n$. If the Dehn function f^Γ_m is polynomial for $m \leqslant n$, then $(\Gamma.^+, L.^+)$ is a type $P(n+1)$ resolution, and if it is polynomial for all m then $(\Gamma.^+, L.^+)$ is type P. Finite generation in each degree implies $PH^1(\Gamma_n; A) = H^1(\Gamma_n; A)$ for $n \geqslant 0$, where A is an arbitrary p.s. π-module. By the same method as above we have

Theorem 2.2.4. *Let A be a p.s. π-module. If f^Γ_m is polynomial for $m \leqslant n$, then $(\Gamma.^+, L.^+)$ is a type $P(n+1)$ resolution of π, and the map*

$$PH^*(\pi; A) \to H^*(\pi; A)$$

is an isomorphism for $* \leqslant n+2$. If f^Γ_m is polynomial for all m, then π satisfies condition (PC1) for all coefficients A.

In the proof of Theorem 2.1.3, we showed how to construct a simplicial group from a presentation of π. If that presentation is finite, Γ_0 and Γ_1 are finitely generated. If π is HF^∞, this free simplicial group can be extended to a resolution $\Gamma.$ of the type used in this section. By Lemma 2.1.4, f^Γ_0 is p-equivalent to the Dehn function associated with the presentation. This justifies the term higher Dehn functions when referring to $\{f^\Gamma_n\}_{n \geqslant 0}$.

Question 2.2.5. *If f^Γ_0 is polynomial, is f^Γ_n polynomial for all $n > 0$?*

A stronger version of the same question is

Question 2.2.6. *Is f^Γ_n polynomially equivalent to f^Γ_0 for all $n > 0$?*

We conclude this section with an alternative definition of higher Dehn functions analogous to that given in [1]. We first assume, as before, that $\mathbb{Q}$ admits a resolution over $\mathbb{Q}[\pi]$ which is free and finitely generated in each dimension. This may then be written as

$$\mathscr{R}(\pi)_* = \mathbb{Q} \overset{\varepsilon}{\leftarrow} \mathbb{Q}[\pi][S_0] \overset{d_1}{\leftarrow} \mathbb{Q}[\pi][S_1] \overset{d_2}{\leftarrow} \cdots \tag{2.2.7}$$

where each S_i is a finite set and each differential d_i is a $\mathbb{Q}[\pi]$-module homomorphism. Taking the weight function on each set S_i to be identically 1 gives each term a p.f. π-module structure. Note also that as each d_i is π-equivariant and each S_i is finite, the differentials will be linearly bounded with respect to this p.f. π-module structure. Then $\{f_n\}_{n\geqslant 0}$ is a sequence of isoperimetric functions for this resolution if there is a chain contraction

$$\{s_n : \mathscr{R}(\pi)_{n-1} \to \mathscr{R}(\pi)_n\}_{n\geqslant 0}$$

over $\mathbb{Q}$ with

$$|s_{n+1}(a)| \leqslant f_n(|a|)$$

for all $a \in \mathscr{R}(\pi)_n = \mathbb{Q}[\pi][S_n]$, for all $n \geqslant 0$ (where for $x \in \mathscr{R}(\pi)_m$, $|x|$ is the semi-norm of x in the p.f. module $\mathscr{R}(\pi)_m$ as defined in (1.1.4)). If each f_n is a minimal isoperimetric function, then it is natural to call the sequence the (higher) Dehn functions associated to the resolution $\mathscr{R}(\pi)_*$. Because each S_i is finite, there are equalities

$$PHom_\pi(\mathbb{Q}[\pi][S_n]; A) = Hom_\pi(\mathbb{Q}[\pi][S_n]; A) \tag{2.2.8}$$

Proposition 2.2.9. *If each of the Dehn functions $\{f_n\}_{n\geqslant 0}$ is of polynomial type, there is an isomorphism*

$$PH^*(\pi; A) \overset{\cong}{\to} H^*(\pi; A).$$

Proof. If the Dehn functions are all of polynomial type, the Comparison Theorem yields an isomorphism

$$PH^*(\pi; A) \cong H^*(\{PHom_\pi(\mathbb{Q}[\pi][S_n]; A), (d_n)^*\}_{n\geqslant 01}).$$

The result then follows from the isomorphism in (2.2.8). □

As an application we have

Corollary 2.2.10. *If A is a p.s. G-module and G is either finitely generated nilpotent or synchronously combable, then there is an isomorphism*

$$PH^*(\pi; A) \overset{\cong}{\to} H^*(\pi; A).$$

Proof. For nilpotent groups, all the Dehn functions are polynomial, so we may apply the above methods. The result may alternatively be proved by induction on the length of the lower central series and the fact that for abelian central extensions, hypothesis 1.1.1H(k) can be shown to hold for all k, allowing for a comparison of Serre spectral sequences. For the second case, we appeal to Gersten's argument in [7]. Gersten's argument not only shows that G is of type FP^{∞} (cf. [3]), but that the Dehn functions associated to the cellular chain complex of the universal cover, which is of the type in (2.2.7), are all polynomial (not just the first one). The result follows by the above proposition. □

2.3. Linearly and uniformly bounded cohomology

Probably the strongest constraint one can impose on the cochain level while still retaining enough functoriality for the homological algebra machinery of Section 1.1 is a linear (or Lipschitz) constraint. Thus, an l-*semi-normed G-module* (l.s. G-module) is defined as in (1.1.1) except that in (1.1.1)(ii) we replace n by 1. A homomorphism $f : M \to M'$ of l.s. G-modules is a $\mathbb{Q}[G]$- module homomorphism for which there exists $C_1, C_2 > 0$ such that for all $x' \in \Lambda_{M'}$ there exists $x \in \Lambda_M$ with

$$\eta_{gx'}(f(a)) \leqslant C_1 C_2 (1 + \eta_{gx}(a))$$

for all $g \in G$ and $a \in M$. As before, the constant C_1 may vary with f but is independent of x', while C_2 may vary with x' but is independent of the other parameters. Note this is slightly more rigid than what one gets when replacing n by 1 in (1.1.2). The set of linearly bounded l.s. G-module homomorphisms from M to M' is denoted by $LHom_G(M, M')$; not requiring f to commute with the action of G produces the larger vector space $LHom(M, M')$ on which G acts with fixed-point set $LHom_G(M, M')$.

Finally, an l.f. resolution of $\mathbb{Q}$ over $\mathbb{Q}[G]$ is defined as in (1.1.9), except that the differentials are homomorphisms of l.f. G-modules, and the chain contraction is required to be linearly bounded. Admissible monomorphisms and epimorphisms are defined in the same manner as before, with linear replacing polynomial. In this context, Propositions 1.1.5, 1.1.6 and Lemma 1.1.8 carry over to the linear setting. Moreover, the bar resolution described prior to the Comparison Theorem is an l.f. resolution as the reader may easily verify. This provides the resolution for defining the linearly bounded cohomology of G with coefficients in an l.s. G module M:

$$LH^*(G; M) \stackrel{\text{def}}{=} LH^*_G(EG_*; M) = H^*(LHom_G(EG_*, M), \delta^*).$$

It is not clear at this point if there is a useful Serre spectral sequence in linearly bounded cohomology (for reasons discussed below). However, the method of proof of the Comparison Theorem does carry over, yielding

Theorem 2.3.1 (*Linear Comparison Theorem*). *Let* (R_*, d_*) *be an l.f. resolution of* $\mathbb{Q}$ *over* $\mathbb{Q}[G]$ *and M an l.s. G-module. Then there is an isomorphism*

$$LH^*_G(EG_*; M) \cong LH^*_G(R_*; M) = H^*(LHom_G(R_*, M), \delta^*).$$

Suppose that the resolution R_* satisfies the finiteness condition mentioned in the previous section; i.e., $R_n = \mathbb{Q}[G][S_n]$ with S_n finite for each $n \geqslant 0$. Then there is an equality

$$LHom_G(\mathbb{Q}[G][S_n]; A) = Hom_G(\mathbb{Q}[G][S_n]; A). \tag{2.3.2}$$

It is reasonable to ask under what conditions such a resolution can exist. The answer is: when G is word-hyperbolic. This is proved by Mineyev in [15] (if we worked over $\mathbb{Z}$ instead of $\mathbb{Q}$ then the work of Mineyev and others shows that such a resolution exists *if and only if* G is word-hyperbolic). Combining Mineyev's results with the above yields

Theorem 2.3.4. *If G is word-hyperbolic, there is an isomorphism*

$$LH^*(G; M) \overset{\cong}{\to} H^*(G; M)$$

for any l.s. G-module M.

In particular, this implies

Corollary 2.3.5. *If G is a finitely-generated free group, equipped with the standard word-length metric, then*

$$LH^*(G; M) = 0$$

for all $* > 1$.

This suggests that the linear analogue of Corollary 1.2.17 may hold.

We conclude this section with a short discussion of bounded cohomology. Because we do not require word-length functions to be proper, we could define the length function L_G on G by $L(x) = 1$ if $x \neq 1$. Because this length function is bounded, the *PHom*() groups used in the computation of $PH^*(G; M)$ are simply those that are uniformly bounded on basis vectors, yielding an isomorphism

$$PH^*(G; M) \cong H_b(G; M)$$

where the right-hand side denotes the bounded cohomology groups of G in the p.s. G-module M.

Any word-length function on G may be realized as the word-length function induced by a free group equipped with word-length metric (F, L_F) via an appropriate surjection $F \twoheadrightarrow G$. In fact, there is a universal example of such. Given (G, L_G) let F be the free group on elements $\{1 \neq g \in G\}$ and let L_F be the word-length metric induced by L_G, viewed as a weight function on the set $G - \{1\}$. Then

$$F' \rightarrowtail F \twoheadrightarrow G$$

is a short-exact sequence of groups with word-length, and so as before there is an associated Serre spectral and five-term exact sequence. As in Corollary 1.2.18 one has

Corollary 2.3.6. *There is a sequence*

$$\begin{aligned} H^0(G; A_m^*) = H_b^0(G; A_m^*) \twoheadrightarrow H_b^2(G; A_{m-1}^*) &\cong H_b^4(G; A_{m-2}^*) \\ &\cong \cdots \cong H_b^{2m}(G; \mathbb{Q}) \end{aligned} \tag{2.3.7}$$

where the maps in the sequence occur as differentials in the $E_2^{*,*}$- term of the appropriate Serre spectral sequence for p-bounded cohomology (*and the groups A_k^* are as defined in Section* 1.2).

In this way one can realize bounded $2m$-dimensional cohomology classes on G as G-invariant elements of A_m^*. This application to bounded cohomology will be further examined in the sequel to this paper.

Appendix A. Type P resolutions

The following results first appeared in [18]. We have included them here as they are an essential ingredient in the proofs appearing in Sections 2.1 and 2.2. We begin with a demonstration of the existence of type P resolutions.

Theorem A.1. *Let $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ be a triple, where $(\Gamma.^+, L.^+)$ is a p-bounded augmented free simplicial group with $\pi = \Gamma_{-1}$. Then there is an inclusion*

$$\imath : \Gamma.^+ \hookrightarrow \widetilde{\Gamma}.^+$$

where $(\widetilde{\Gamma}.^+, \widetilde{\mathbb{X}}.^+, \widetilde{L}.^+)$ is a triple and $(\widetilde{\Gamma}.^+, \widetilde{L}.^+)$ is a type P resolution of π. Moreover, if $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ is type $P(m)$, then the construction can be done so that $(\widetilde{\Gamma}_n^+, \widetilde{\mathbb{X}}_n^+, \widetilde{L}_n^+) = (\Gamma_n^+, \mathbb{X}_n^+, L_n^+)$ for $n \leqslant m$.

Proof. We first give the general construction. Denote $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ by $(\Gamma(0).^+, \mathbb{X}(0).^+, L(0).^+)$. Note that $\Gamma(0)(\varepsilon)_{-1} = \{1\}$, so taking $s_0' : \Gamma(0)(\varepsilon)_{-1} \to \Gamma(0)(\varepsilon)_0$ as the inclusion of the trivial group shows that $(\Gamma(0).^+, \mathbb{X}(0).^+, L(0).^+)$ is type $P(0)$.

By induction, we may assume that a p-bounded free simplicial group triple $(\Gamma(m-1).^+, \mathbb{X}(m-1).^+, L(m-1).^+)$ has been constructed such that $\Gamma(m-1)(\varepsilon).^+$ is $(m-2)$-connected, and admits a contracting degeneracy $\{s'_{p+1}\}_{0 \leqslant p \leqslant m-2}$ through dimension $(m-2)$ which is p-bounded.

Let $\mathbb{X}(m)'_m$ equal the *set* $\Gamma(m-1)(\varepsilon)_{(m-1)} - \{1\}$. For $1 \neq g \in \Gamma(m-1)(\varepsilon)_{(m-1)}$, we denote by $[g]$ the corresponding generator in $\mathbb{X}(m)'_m$. Let

$$\begin{aligned} &\mathbb{X}(m)_j = \mathbb{X}(m-1)_j \quad \text{for } j \leqslant (m-1),\\ &\mathbb{X}(m)_m = \mathbb{X}(m-1)_m \coprod \mathbb{X}(m)'_m,\\ &\mathbb{X}(m)_n = \mathbb{X}(m-1)_n \coprod \{s(\mathbb{X}(m)'_m)\} \quad \text{for } n > m \end{aligned} \tag{A.2}$$

where the last coproduct is over all iterated degeneracies s from dimension m to dimension n. Face maps are determined by the following values on generators

$$\begin{aligned} &\partial_j([g]) = s'_{m-1}\partial_j(g) \quad \text{for } 0 \leqslant j < m,\\ &\partial_m([g]) = g. \end{aligned} \tag{A.3}$$

Proceeding as before, we define $\Gamma(m)_{-1}=\pi$ and $\Gamma(m)_n$ to be the free group on $\mathbb{X}(m)_n$ for $n\geqslant 0$. $L(m).^+$ is uniquely defined and determined by the following four properties:

(i) It equals $L(m-1).^+$ on $\mathbb{X}(m-1).^+$ (A.4)

(ii) If f is the proper function on $\mathbb{X}(m)'_m$ determined by $L(m-1)_{m-1}$ restricted to $\Gamma(m-1)(\varepsilon)_{(m-1)}$, then $L(m)_m$ is the metric induced by f when restricted to the free group F_m generated by $\mathbb{X}(m)'_m$. (A.4)

(iii) If $x\in\Gamma(m)_m$ is written as $x=w_1w_2\dots w_p$ with w_{2i-1} in $\Gamma(m-1)_m$ and $w_{2i}\in F_m$, then

$$L(m)_m(x)=\sum_{i=1}^{p}L(m)_m(w_i) \tag{A.4}$$

(iv) If $s:\Gamma(m)_m\to\Gamma(m)_n$ is an iterated degeneracy, then $L(m)_n(s(x))=L(m)_m(x)$. If $w=w_1w_2\dots w_q\in\Gamma(m)_n$ is a product of degenerate elements $w_{2i}=s(x_i)$, $x_i\in F_m$ and elements w_{2i-1} in $\Gamma(m-1)_n$, then $L(m)_n(w)=\sum_{i=1}^{q}L(m)_n(w_i)$. (A.4)

That $L(m-1).^+$ is a metric implies $L(m).^+$ is again a metric in each non-negative degree, and the contracting degeneracy $s.'$ for $\Gamma(m).^+$ is now extended through dimension $(m-1)$ as the set map

$$\begin{aligned}&s'_m(1)=1,\\&s'_m(g)=[g]\quad\text{for } g\neq 1.\end{aligned} \tag{A.5}$$

(A.3) and (A.5) guarantee that $s.'$ satisfies the required simplicial identities through dimension $(m-1)$. (A.3)–(A.5) and induction imply that all of the degeneracy maps (including $s.'$) through dimension $(m-1)$ and all of the face maps through dimension m are p-bounded. Let

$$(\widetilde{\Gamma}.,\widetilde{\mathbb{X}}.,\widetilde{L}.)=\varinjlim_m\{(\Gamma(m).^+,\mathbb{X}(m).^+,L(m).^+)\}. \tag{A.6}$$

Then $(\widetilde{\Gamma}.,\widetilde{\mathbb{X}}.,\widetilde{L}.)$ is a type P resolution. The inclusion of generating sets $\mathbb{X}.\hookrightarrow\widetilde{\mathbb{X}}.$ induces the simplicial group monomorphism $\imath:\Gamma.^+\to\widetilde{\Gamma}.^+$, which is the identity on $\pi=\Gamma_{-1}=\widetilde{\Gamma}_{-1}$.

Finally if $(\Gamma.^+,\mathbb{X}.^+,L.^+)$ is type $P(m)$, then in the above sequence we may start with $(\Gamma(m).^+,\mathbb{X}(m).^+,L(m).^+)=(\Gamma.^+,\mathbb{X}.^+,L.^+)$ and continue with the construction by adding generators in simplicial dimensions $n>m$. This verifies the second part of the theorem. □

Example A.7. Let π be a countable group equipped with an $\mathbb{N}$-valued word-length function L. Let $\Gamma.^+=GB.\pi^+$ be the augmented Kan loop group of the non-homogeneous bar construction on π (this is the augmented simplicial group associated to the usual Kan loop group $GB.\pi$; cf. [12]). Then the word length function L determines a proper function on the set of n-simplices of $B.\pi$ in the standard way:

$$L([g_1,\dots,g_n])=\sum_{i=1}^{n}L(g_i)$$

and thus by restriction a proper $\mathbb{N}^+$-valued function on the generating set $\mathbb{X}_{n-1} = B_n\pi - s_0(B_{n-1}\pi)$ of $(GB\pi)_{n-1}$ for all $n \geqslant 1$. In non-negative dimensions we then take L_n to be the metric determined by this proper function. This produces a resolution $(\Gamma.^+, \mathbb{X}.^+, L.^+)$ to which we may apply the above extension theorem. Note also that the word-length functions L_n arising from this construction are $\mathbb{N}$-valued, making the word-length function $\widetilde{L}.$ constructed above $\mathbb{N}$-valued as well.

We summarize this as

Corollary A.8. *Every countable group π admits a type P resolution where the word-length function in non-negative degrees is an $\mathbb{N}$-valued metric. Moreover, if $C(\pi)$ is the category whose objects are p-bounded augmented free simplicial groups equal to π in dimension -1 and equipped with word-length metrics in non-negative degrees, and whose morphisms are p-bounded simplicial group homomorphisms inducing the identity on π_0, then the full subcategory whose objects are type P resolutions is cofinal in $C(\pi)$.*

For the remainder of the section we assume $\Gamma.^+$ is a type P resolution of G. We construct a contraction of $\Gamma.^+$ viewed as an augmented simplicial set. To begin with, $\Gamma(\varepsilon).$ admits a simplicial contraction $s.' = \{s'_{n+1} : \Gamma(\varepsilon)_n \to \Gamma(\varepsilon)_{n+1}\}_{n\geqslant 0}$ which is p-bounded. Now choose a section $s(0) : \Gamma_{-1} \rightarrowtail \Gamma_0$, $\varepsilon_0 \circ s(0) =$ identity, with $s(1) = 1$ and which is minimal with respect to word-length. Define $s(n) = s_0^{(n)} \circ s(0) : \Gamma_{-1} \rightarrowtail \Gamma_n$. Note that

$$\begin{aligned} &\varepsilon_n \circ s(n) = \text{identity} \quad \forall n \geqslant 0, \\ &\partial_i \circ s(n) = s(n-1) \quad \forall n \geqslant 1, 0 \leqslant i \leqslant n, \\ &s_i \circ s(n-1) = s(n) \quad \forall n \geqslant 1, 0 \leqslant i \leqslant n-1. \end{aligned} \tag{A.9}$$

For $n = -1$, set $\widetilde{s}_{n+1} = \widetilde{s}_0 = s(0)$. Note that for arbitrary $g \in \Gamma_n$, $g(s(n)(\varepsilon_n(g)))^{-1} \in \Gamma(\varepsilon)_n$. Then when $n \geqslant 0$

$$\widetilde{s}_{n+1}(g) = s'_{n+1}(g(s(n)(\varepsilon_n(g)))^{-1})s(n+1)(\varepsilon_n(g)). \tag{A.10}$$

This defines a map of sets $\widetilde{s}_{n+1} : \Gamma_n \to \Gamma_{n+1}$. The simplicial identities imply $\widetilde{s}_{*+1} = \{\widetilde{s}_{n+1} : \Gamma_n \to \Gamma_{n+1}\}_{n\geqslant -1}$ is a simplicial contraction of simplicial sets, which by construction is p-bounded for each $n \geqslant -1$.

Recall from Section 2.1 that $D^n(\Gamma., \mathbb{Q}) = PH^1(\Gamma_{n-1}; \mathbb{Q})$ for $n \geqslant 1$, and $D^n(\Gamma., \mathbb{Q}) = 0$ for $n \leqslant 0$. As all face maps of $\Gamma.$ are p-bounded, there is a well-defined homomorphism $\delta^n = \sum_{i=0}^{n-1}(-1)^i \partial_i^* : D^n(\Gamma., \mathbb{Q}) \to D^{n+1}(\Gamma., \mathbb{Q})$, making $(D^*(\Gamma., \mathbb{Q}), \delta^*)$ a cocomplex. Similarly, one defines the cocomplex $(E^*(\Gamma., \mathbb{Q}), \delta^*)$ in the same fashion with H^1 in place of PH^1. As we observed in Section 2.1, there is an inclusion of cocomplexes

$$(D^*(\Gamma., \mathbb{Q}), \delta^*) \hookrightarrow (E^*(\Gamma., \mathbb{Q}), \delta^*). \tag{A.11}$$

Theorem A.12. *For $n \geqslant 1$ there is an isomorphism of cohomology groups*

$$PH^n(G; \mathbb{Q}) \cong H^n(D^*(\Gamma., \mathbb{Q}), \delta^*).$$

Moreover, the inclusion of cocomplexes $(D^*(\Gamma., \mathbb{Q}), \delta^*) \hookrightarrow (E^*(\Gamma., \mathbb{Q}), \delta^*)$ induces, upon passing to cohomology, the transformation $PH^*(G; \mathbb{Q}) \to H^*(G; \mathbb{Q})$.

Proof. Fix an $m \geqslant 0$ and consider the augmented simplicial abelian group

$$C(m).^{+} = \{[n] \mapsto C_m((B\Gamma_n);\mathbb{Q})\}_{n \geqslant -1}.$$

The p-bounded contraction $\widetilde{s}_{*+1}$ on $\Gamma.^{+}$ defined above induces a p-bounded $\mathbb{Q}$-vector space contraction on $C(m).^{+}$ for each $m \geqslant 0$ given by

$$B_m\Gamma_n \ni [g_1,\ldots,g_m] \mapsto [\widetilde{s}_{n+1}(g_1),\ldots,\widetilde{s}_{n+1}(g_m)] \in B_m\Gamma_{n+1} \quad n \geqslant -1.$$

Applying $PHom(\ ,\mathbb{Q})$ to the associated complex $C(m)_*$, we get a cocontraction above dimension 0, yielding

$$H^n(PHom(C(m)_*,\mathbb{Q})) = 0 \quad \text{for } n > 0$$
$$H^0(PHom(C(m)_*,\mathbb{Q})) = PHom(C_m(BG;\mathbb{Q}),\mathbb{Q}).$$

Applying $PHom(\ ,\mathbb{Q})$ to the bi-complex $C_{*,*} = \{C_*(m)\}_{m \geqslant 0}$ produces a bi-cocomplex. From the computation of $H^*(PHom(C(m)_*,\mathbb{Q}))$, we see that filtering by columns produces an E_1-term which collapses to the cocomplex $(PHom(C_*(BG;\mathbb{Q}),\mathbb{Q}),\delta^*)$ whose cohomology is $PH^*(G;\mathbb{Q})$. Filtering by rows on the other hand yields an E_1-term with $E_1^{p,q} = PH^p(\Gamma_q;\mathbb{Q})$. Now Γ_q is free and L_q is a metric for $q \geqslant 0$, so by Corollary 1.2.17 $E_1^{p,q} = 0$ for $p > 1$. The $E_2^{0,*}$-line, which is the cohomology of $(E_1^{0,*}, d_1^{0,*})$, is $\mathbb{Q}$ for $* = 0$ and 0 for $* > 0$. There is an isomorphism of cocomplexes $(D^*(\Gamma.,\mathbb{Q}),\delta^*) \cong (E_1^{1,*-1}, d_1^{1,*-1})$, hence

$$PH^n(G;\mathbb{Q}) = E_2^{1,n-1} = H^n(D^*(\Gamma.,\mathbb{Q}),\delta^*)$$

for all $n \geqslant 1$. Applying $Hom(\ ,\mathbb{Q})$ in place of $PHom(\ ,\mathbb{Q})$ and repeating the same line of reasoning produces an isomorphism

$$H^n(G;\mathbb{Q}) = E_2^{1,n-1} = H^n(E^*(\Gamma.,\mathbb{Q}),\delta^*).$$

Finally, the natural transformation $PH^*(G;\mathbb{Q}) \to H^*(G;\mathbb{Q})$ in the above context is induced by a map of bicocomplexes coming from the natural transformation $PHom(\ ,\mathbb{Q}) \to Hom(\ ,\mathbb{Q})$. On the level of spectral sequences, this induces a map on the $E_1^{1,*}$ line corresponding to the inclusion of (A.11) above, completing the proof. □

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