

Subexponential group cohomology and the K -theory of Lafforgue's algebra $\mathcal{A}_{\max}(\pi)$

R. Ji · C. Ogle

Received: March 2006
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Keywords p -bounded cohomology · Lafforgue algebras · Topological K -theory

0 Introduction

Let $K_*(X)$ denote the generalized homology of X with coefficients in the spectrum $\underline{K}(\mathbb{C})$ [1], π a finitely generated discrete group and $B\pi$ its classifying space. For any Banach algebra $A(\pi)$ with $\mathbb{C}[\pi] \subseteq A(\pi)$, there is an assembly map $K_*(B\pi) \rightarrow K_*^t(A(\pi))$ for the topological K -theory of $A(\pi)$. In this paper, we are interested in the “maximal unconditional completion” of $\mathbb{C}[\pi]$ in the reduced group C^* -algebra $C_r^*(\pi)$. This algebra, denoted $\mathcal{A}_{\max}(\pi)$, was introduced by Lafforgue [10] (the definition is recalled below). The main result of the paper is as follows.

Theorem 1 *Let π be a finitely-generated group and $x = (x_n, x_{n-2}, \dots)$ denote an element of $K_n(B\pi) \otimes \mathbb{C} \cong (H_*(B\pi) \otimes K_*(\mathbb{C})) \otimes \mathbb{C}$. Suppose there exists a cohomology class $[c] \in H^n(B\pi; \mathbb{C})$ represented by an n -cocycle of subexponential growth with $\langle c, x_n \rangle \neq 0$. Then x is sent by the assembly map to a nonzero element of $K_n^t(\mathcal{A}_{\max}(\pi)) \otimes \mathbb{C}$. In particular, if all rational homology classes of π are detected by cocycles of this type, then the assembly map for the topological K -theory of $\mathcal{A}_{\max}(\pi)$ is rationally injective.*

An n -cochain c is λ -exponential ($\lambda > 1$) if there is a constant C (depending on λ) for which $\|c(g_1, \dots, g_n)\| \leq C(\lambda^{\sum_{i=1}^n L(g_i)})$ for all $[g_1, \dots, g_n]$ in $C_n(B\pi)$, where $L(-)$ is the standard word-length function associated to a set of generators of π . It is subexponential if it is λ -exponential for all $\lambda > 1$. For finitely-generated groups two distinct presentations of π yield linearly equivalent word-length functions, so the notion of λ -exponential does not

C. Ogle (✉)
Department of Mathematics, Ohio State University, OH, USA
e-mail: ogle@math.ohio-state.edu

R. Ji
Department of Mathematics
Indiana University · Purdue University Indianapolis,
425 N. University Boulevard Indianapolis, IN 46202, USA
e-mail: ronji@math.iupui.edu

depend on the choice of presentation of π . The proof of Theorem 1 is based on the recent work of Puschnigg [13, 14], in which the theory of local cyclic homology is developed. Given the results of these two papers, our task essentially reduces to showing that subexponential group cocycles extend to continuous local cyclic homology cocycles on $\mathcal{A}_{\max}(\pi)$. This is accomplished using relatively standard techniques from cyclic homology and appealing to two key results from Puschnigg [13].

In an earlier draft of this paper, the authors proved the weaker result that cohomology classes of polynomial growth paired with $K_*^t(\mathcal{A}_{\max}(\pi))$ in the manner described by the above theorem, following the ideas of Connes and Moscovici [6]. It is clear that our original approach could have been extended to include a more restrictive subexponential growth condition modeled on the subexponential technical algebra described in [8, 9]. We are indebted to the referee for suggesting this stronger result as well as its connection to [13]. There is substantial evidence that the class of groups satisfying the property that all of its rational homology is detected by cocycles of polynomial growth is quite large, possibly containing all finitely-generated discrete groups whose first Dehn function is of polynomial type [12, Conj. A]. If this pattern persists for Dehn functions of subexponential growth, one might expect a similar density of subexponential cohomology classes for Olshanskii groups.

1 Proof of the main result

We recall some basic facts about Hochschild and cyclic homology [4, 5, 11], consistent with the notation and conventions of [13]. We assume throughout that the base field is \mathbb{C} , with all algebras and tensor products being over \mathbb{C} . For an associative unital algebra A , let $\Omega^n A = A \otimes A^{\otimes n}$, and set $\Omega^*(A) = \bigoplus_n \Omega^n(A)$. Define $b_n: \Omega^n(A) \rightarrow \Omega^{n-1}(A)$ by the equation

$$b_n(a_0, a_1, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n (a_n a_0, a_1, \dots, a_{n-1}).$$

This is a differential, and the resulting complex $(\Omega^*(A), b)$ is the Hochschild complex of A , also denoted by $C_*(A)$. Its homology is denoted $HH_*(A)$. The Hochschild complex admits a degree one chain map B_* given by

$$B_n(a_0, a_1, \dots, a_n) = \sum_{i=0}^n (-1)^{ni} (1, a_i, a_{i+1}, \dots, a_n, a_0, a_1, \dots, a_{i-1}) - \sum_{i=0}^n (-1)^{ni} (a_i, 1, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1})$$

and this defines a 2-periodic bicomplex $\mathcal{B}(A)_{**}$ with $\{\mathcal{B}(A)_{2p,q} = \otimes^{q+1} A, \mathcal{B}(A)_{2p-1,q} = 0\}$. The differentials are $b_q: \mathcal{B}(A)_{2p,q} \rightarrow \mathcal{B}(A)_{2p,q-1}$ and $B_q: \mathcal{B}(A)_{2p,q} \rightarrow \mathcal{B}(A)_{2p-2,q+1}$. The total complex $T_*(A) = \left(\bigoplus_{2p+q=*} \mathcal{B}(A)_{2p,q}, b + B \right)$ of the resulting bicomplex $\mathcal{B}(A)_{**} = \{\mathcal{B}(A)_{**}; b, B\}$ is clearly $\mathbb{Z}/2$ -graded (and contractible). There is a decreasing filtration by subcomplexes $F^k T_*(A) = (b(\Omega^k(A)) \bigoplus_n \Omega^{n \geq k}(A), b + B)$, and the periodic cyclic complex $CC_*^{\text{per}}(A)$ is the completion of $T_*(A)$ with respect to this filtraton: $CC_*^{\text{per}}(A) = \varprojlim_k T_*(A)/F^k T_*(A)$, with homology $HC_*^{\text{per}}(A)$. For locally convex topological algebras,

projectively completing the tensor products in the complex $\Omega^*(A)$ yields topological Hochschild and periodic cyclic homology.

Note on notation In [13, 14] the author defines $\Omega^n(A)$ as $\Omega^n A = \tilde{A} \otimes A^{\otimes n}$, where $\tilde{A} = A_+ = A \oplus \mathbb{C}$ denotes A with a unit adjoined. This is referred to as the Hochschild complex of A , and its homology as the Hochschild homology of A . However, this definition actually produces the normalized Hochschild complex $C_*(A_+)_n$ of A_+ , and its homology is $HH_*(A_+) = HH_*(A) \oplus HH_*(\mathbb{C})$. Dividing out by the additional copy of \mathbb{C} occurring in \tilde{A} in dimension 0 of $C_*(A_+)_n$ yields the reduced complex $C_*(A_+)_{\text{red}}$, and the inclusion $C_*(A) \hookrightarrow C_*(A_+)_{\text{red}}$ induces an isomorphism in homology, but is not an isomorphism of complexes [11, 1.4.2]. In order for the chain-level isomorphisms in [13, Lemmas 3.2 and 3.3] to hold, one must use $C_*(\mathbb{C}[\pi])$, not the larger $C_*(\mathbb{C}[\pi]_+)_{\text{red}}$. This decomposition leads to the homogeneous decomposition stated in [13, Lemma 3.9], a result used in this paper. For these reasons, we will assume throughout that our algebra A is unital, and that $\Omega^*(A)$ denotes the usual Hochschild complex of A .

We recall the definition of local cyclic homology [13, 14]. Let A be a Banach algebra, U its unit ball. Given a compact subset K of U , A_K is the completion of the subalgebra of A generated by K with respect to the largest submultiplicative seminorm η such that $\eta(K) \leq 1$. Next, for an auxiliary Banach algebra A' , let $\eta(-)_{N,m}$ be the largest seminorm on $\Omega^*(A')$ satisfying

$$\eta_{N,m}(a_0, a_1, a_2, \dots, a_n)_{N,m} \leq \frac{1}{c(n)!} (2 + 2c(n))^m N^{-c(n)} \|a_0\|_{A'} \cdots \|a_n\|_{A'} \tag{1.1}$$

for each n , where $N \geq 1$, $m \in \mathbb{N}$ and $c(2n) = c(2n + 1) = n$. For each N the boundary maps b and B are bounded with respect to the family of seminorms $\{\eta(-)_{N,m}\}_{m \in \mathbb{N}}$. Thus the completion of $T_*(A')$ with respect to this family of seminorms again yields a complex, denoted $T_*(A')_{(N)}$. In particular this applies when $A' = A_K$. Note that an inclusion of compact subspaces $K \hookrightarrow K'$ induces a continuous homomorphism of Banach algebras $A_K \rightarrow A_{K'}$, and that if $N < N'$ the identity map on $T^*(A')$ induces a natural morphism of complexes $T_*(A')_{(N)} \rightarrow T_*(A')_{(N')}$.

Definition 1.2 [13, Definition 3.4] In terms of the above notation, the local cyclic homology of a Banach algebra A is

$$HC_*^{\text{loc}}(A) = \varinjlim_{\substack{K \subset U \\ N \rightarrow \infty}} H_*(T_*(A_K)_{(N)}).$$

A Banach space X is said to have the *Grothendieck approximation property* if the set of finite rank operators is dense in the space of bounded operators $\mathcal{L}(E)$ with respect to the compact-open topology. A Banach algebra has this property when its underlying Banach space does. In [14], Puschnigg proves that for Banach algebras satisfying this property, the above inductive system used in the definition of $HC_*^{\text{loc}}(A)$ can be replaced by a much smaller countable directed system. Precisely, let $V_0 \subset V_1 \subset \dots \subset V_j \subset \dots \subset A$ be any increasing sequence of finite-dimensional subspaces of A for which $\bigcup V_j$ is dense in A . Let \bar{B}_r be the closed ball of radius r in A . Set $K_j = V_j \cap \bar{B}_{\frac{1}{j+1}}$.

Proposition 1.3 [13, Proposition 3.5; 14, Theorem 3.2] *If A is a Banach algebra with the Grothendieck approximation property, then*

$$HC_*^{\text{loc}}(A) = \varinjlim_{\substack{j \rightarrow \infty \\ N \rightarrow \infty}} H_*(T_*(A_{K_j})_{(N)}).$$

We now specialize to the case A is a Banach algebra completion of $\mathbb{C}[\pi]$. There is a well-known decomposition of the Hochschild complex [3, 7] $C_*(\mathbb{C}[\pi]) = \bigoplus C_*(\mathbb{C}[\pi])_{\langle g \rangle}$ the sum being indexed over the set $\{\langle g \rangle\}$ of conjugacy classes of π . The boundary maps b_n preserve summands, so that for each $\langle g \rangle$ $(C_*(\mathbb{C}[\pi])_{\langle g \rangle}, b_*)$ is a subcomplex and direct summand of $(C_*(\mathbb{C}[\pi]), b_*)$. Because the differential B also preserves this decomposition, it extends to the bicomplex $\mathcal{B}(A)_{**}$. These sum decompositions then yield sum-decompositions of the corresponding algebraic homology groups

$$HH_*(\mathbb{C}[\pi]) = \bigoplus HH_*(\mathbb{C}[\pi])_{\langle g \rangle} \tag{1.4}$$

and for each $\langle g \rangle$ a summand $HC_*^{\text{per}}(\mathbb{C}[\pi])_{\langle g \rangle}$ of $HC_*^{\text{per}}(\mathbb{C}[\pi])$, where $HH_*(\mathbb{C}[\pi])_{\langle g \rangle}$ resp. $HC_*^{\text{per}}(\mathbb{C}[\pi])_{\langle g \rangle}$ is the homology of $(C_*(\mathbb{C}[\pi])_{\langle g \rangle}, b_*)$ resp. $(CC_*^{\text{per}}(\mathbb{C}[\pi])_{\langle g \rangle}, b_*)$. In general, extension of these decompositions to the topological complexes associated to completions of the group algebra is problematic. However, for weighted ℓ^1 completions, the summands persist (cf. [8]). For local cyclic homology, one has a similar result. To describe it we need some more terminology.

Let $\ell^1(\pi)$ be the ℓ^1 algebra of the discrete group π . As in [2], let $\ell_\lambda^1(\pi)$ denote the completion of $\mathbb{C}[\pi]$ with respect to the largest seminorm v_λ satisfying $v_\lambda(g) \leq \lambda^{L(g)}$, where L is the word-length function on π associated to a finite symmetric set of generators, and $\lambda \geq 1$. Obviously $\ell_1^1(\pi) = \ell^1(\pi)$, and for each $\lambda > 1$, $\ell_\lambda^1(\pi)$ is a subalgebra of $\ell^1(\pi)$ and a Banach algebra, with $\{\ell_\lambda^1(\pi)\}_{\lambda \geq 1}$ forming an inductive system. The Banach algebra $\ell^1(\pi)$ has the Grothendieck approximation property described above. Given a finite generating set S of π with associated word-length metric L_S , let V_j be the linear span of $\{g \mid L_S(g) \leq j\}$, with K_j defined in terms of V_j as before.

Proposition 1.5 [13, Lemma 3.7] *For each generating set S and associated word-length function L on π associated with S , there is a natural isomorphism*

$$“\lim_{j \rightarrow \infty}” \ell^1(\pi)_{K_j} \cong “\lim_{\substack{\lambda \rightarrow 1 \\ \lambda > 1}}” \ell_\lambda^1(\pi).$$

The quotation marks here indicate that these limits are actually objects in the category of Ind-Banach algebras (i.e., the category whose objects are formal inductive limits over the category of Banach algebras – [14]). As a consequence, one has the following analog in local cyclic homology of the “Principe d’Oka” due to Bost [2].

Theorem 1.6 [13, Corollary 3.8, Lemma 3.9] *Let π be finitely-generated with generating set S and associated word-length function L . Then there is an isomorphism*

$$HC_*^{\text{loc}}(\ell^1(\pi)) \cong \lim_{\substack{\lambda \rightarrow 1, \lambda > 1 \\ N \rightarrow \infty}} H_*(CC_*(\ell_\lambda^1(\pi))_{(N)}).$$

Moreover $HC_*^{\text{loc}}(\ell^1(\pi))$ admits a decomposition as a topological direct sum indexed by the set of conjugacy classes of π .

Next we recall the following definition and results, due to Lafforgue [10]: $\mathcal{A}(\pi)$ is a sufficiently large good completion of the group algebra $\mathbb{C}[\pi]$ inside of $C_r^*(\pi)$ if (i) it is admissible, (ii) it is a Fréchet algebra with seminorms $\{\eta_i(-)\}$ satisfying $\eta_i(\sum_g \lambda_g g) = \eta_i(\sum_g |\lambda_g| g)$; $|\lambda_g| \leq |\lambda'_g| \forall g$ implies $\eta_i(\sum_g \lambda_g g) \leq \eta_i(\sum_g \lambda'_g g)$ and (iii) it contains the Banach subalgebras $\ell_\lambda^1(\pi)$ for all $\lambda > 1$. There is a maximal such subalgebra of $C_r^*(\pi)$, denoted $\mathcal{A}_{\text{max}}(\pi)$, which is the completion of $\mathbb{C}[\pi]$ in the seminorm $\|f\|_{\text{max}} := \| |f| \|_r$, $\|-\|_r$ being the reduced C^* norm. For our purposes, we will only state the following result for $\mathcal{A}_{\text{max}}(\pi)$; a more general result is given in [13]. In what follows, $\langle e \rangle$ denotes the conjugacy class of the identity

element; the summand indexed by this element is referred to in [13] as the homogeneous summand (cf. also [3]).

Theorem 1.7 [13, Proposition 4.5], [6] *There is a decomposition*

$$HC_*^{\text{loc}}(\mathcal{A}_{\max}(\pi)) \cong HC_*^{\text{loc}}(\mathcal{A}_{\max}(\pi))_{(e)} \oplus (HC_*^{\text{loc}}(\mathcal{A}_{\max}(\pi))_{(e)})^C$$

analogous to Burghlelea’s decomposition for $HC_*^{\text{per}}(\mathbb{C}[\pi])$. Moreover the inclusions $\ell_\lambda^1(\pi) \hookrightarrow \mathcal{A}_{\max}(\pi)$ induce an isomorphism on the homogeneous summands of local cyclic homology groups

$$HC_*^{\text{loc}}(\ell^1(\pi))_{(e)} \cong HC_*^{\text{loc}}(\mathcal{A}_{\max}(\pi))_{(e)}.$$

We define a cohomology theory on $B\pi$, which we will pair with the right-hand side of (1.6). Let S and L be as above. Let $B_*(\pi)$ denote the nonhomogeneous bar complex on π . Then $H^*(B\pi; \mathbb{C})$ is computed as the cohomology of the cocomplex $C^*(B\pi; \mathbb{C}) = \text{Hom}(B_*(\pi); \mathbb{C})$, a typical n -cochain represented by a map $\phi: (\pi)^n \rightarrow \mathbb{C}$.

Definition 1.8 A cochain $\phi: (\pi)^n \rightarrow \mathbb{C}$ is called λ -exponential if there exists a constant $C_\lambda > 0$ with

$$|\phi(g_1, g_2, \dots, g_n)| \leq C_\lambda \left(\lambda^{\sum_{i=1}^n L(g_i)} \right)$$

for all $(g_1, g_2, \dots, g_n) \in B_n(\pi)$.

For fixed λ , the set of λ -exponential cochains form a subcocomplex $C_\lambda^*(B\pi; \mathbb{C})$ of $C^*(B\pi; \mathbb{C})$, and the λ -exponential cohomology of $B\pi$ is $H_\lambda^*(B\pi; \mathbb{C}) := H^*(C_\lambda^*(B\pi; \mathbb{C}))$. A cohomology class $[c] \in H^n(B\pi; \mathbb{C})$ is λ -exponential if it lies in the image of the natural map $H_\lambda^*(B\pi; \mathbb{C}) \rightarrow H^*(B\pi; \mathbb{C})$ induced by the inclusion $C_\lambda^*(B\pi; \mathbb{C}) \hookrightarrow C^*(B\pi; \mathbb{C})$. The subexponential cocomplex is $C_{\text{se}}^*(B\pi; \mathbb{C}) := \bigcap_{\lambda > 1} C_\lambda^*(B\pi; \mathbb{C})$, and the subexponential cohomology of π is $H_{\text{se}}^*(B\pi; \mathbb{C}) := H^*(C_{\text{se}}^*(B\pi; \mathbb{C}))$. Finally, a cohomology class $[c]$ as above is subexponential if it is in the image of the canonical map $H_{\text{se}}^*(B\pi; \mathbb{C}) \rightarrow H^*(B\pi; \mathbb{C})$.

In order to describe the pairing, we construct a variant of Puschnigg’s local theory. First, note that $HC_*^{\text{loc}}(A)$ is a $\mathbb{Z}/2$ -graded theory whose algebraic counterpart is not cyclic homology but periodic cyclic homology. Returning to the bicomplex $\mathcal{B}(A)_{**}$, we let $\mathcal{BC}(A)_{pq} = \{\mathcal{B}(A)_{pq}\}$ for $p \geq 0$, and 0 for $p < 0$. Algebraically this is the quotient of $\mathcal{B}(A)_{**}$ by the subbicomplex $\{\mathcal{B}(A)_{pq}\}_{p < 0}$. Set $TC_*(A) = \left(\bigoplus_{2p+q=*} \mathcal{BC}(A)_{2p,q}, b + B \right)$. This is a connective complex [usually denoted $CC_*(A)$] whose homology computes the algebraic cyclic homology of A . Let $C_*^{\text{cy}}(A) = \{C_n^{\text{cy}}(A) = C_n(A)/(1 - \tau_{n+1}), b\}_{n \geq 0}$ denote the usual cyclic complex of A , formed as the quotient of the Hochschild complex by $(1 - \tau_*)$, where $\tau_{n+1}(a_0, \dots, a_n) = (-1)^{n+1}(a_n, a_0, \dots, a_{n-1})$. There is a projection map $TC_*(A)$, which sends $\mathcal{BC}_{0q}(A) = C_q(A)$ surjectively to $C_q^{\text{cy}}(A)$, $(1 - \tau_{q+1})(C_q(A)) \mapsto 0$, and sends $\mathcal{BC}_{pq}(A)$ to 0 for $p > 0$. As we are over a field of char. 0, this is a quasi-isomorphism. Now the completions used in the definition of local cyclic homology apply to both $TC_*(A)$ and $C_*^{\text{cy}}(A)$. Thus

Definition 1.9 For a Banach algebra A , let

$$HCC_*^{\text{iloc}}(A) = \varinjlim_{\substack{K \subset U \\ N \rightarrow \infty}} H_*(TC_*(A_K)_{(N)}),$$

$$HCC_*^{\text{loc}}(A) = \varinjlim_{\substack{K \subset U \\ N \rightarrow \infty}} H_*(C_*^{\text{cy}}(A_K)_{(N)}),$$

where the notation is as in Definition 1.2 above. The projection maps of complexes $T_*(A') \rightarrow TC_*(A') \rightarrow C_*^{cy}(A')$ are clearly bounded with respect to the seminorms used above for any of the auxiliary Banach algebras A_K , yielding homomorphisms on homology groups

$$HC_*^{loc}(A) \rightarrow HCC_*^{iloc}(A) \rightarrow HCC_*^{loc}(A) \tag{1.10}$$

induced by the obvious projection maps of complexes. Here HCC_*^{loc} stands for “connective local cyclic homology”, as distinguished from HC_*^{loc} , while “iloc” refers to the “intermediate local” connective theory, which simply serves as a bridge between the left and right-hand sides of (1.10). Finally, we note that for algebras with the Grothendieck approximation property, one could similarly define homology groups using the smaller countable direct limits described above in Proposition 1.3. As we are interested in the case $A = \ell^1(\pi)$, we will use the directed systems of Theorem 1.6.

Definition 1.11 Let π be finitely-generated with generating set S and associated word-length function L . Then

$$HCC_*^{aploc}(\ell^1(\pi)) := \lim_{\substack{\lambda \rightarrow 1, \lambda > 1 \\ N \rightarrow \infty}} H_*(C_*^{cy}(\ell_\lambda^1(\pi))_{(N)}).$$

By the same reasoning as in [13, Lemma 3.9], we see that $HCC_*^{aploc}(\ell^1(\pi))$ admits a decomposition as a topological direct sum indexed by the set of conjugacy classes of π . Moreover, the canonical map $HC_*(\mathbb{C}[\pi]) \rightarrow HCC_*^{aploc}(\ell^1(\pi))$ obviously preserves the decomposition. In particular, $HC_*(\mathbb{C}[\pi])_{(e)}$ maps to $HCC_*^{aploc}(\ell^1(\pi))_{(e)}$. Note that, unlike the original local cyclic theory of [13], there is no reason to believe that $HCC_*^{aploc}(\ell^1(\pi))$ and $HCC_*^{loc}(\ell^1(\pi))$ (defined as $HCC_*^{loc}(-)$ of an Ind-Banach algebra in the sense of [14]) agree. Nevertheless, there is an evident map $HC_*^{loc}(\ell^1(\pi)) \rightarrow HCC_*^{aploc}(\ell^1(\pi))$, which we may fit into a diagram

$$\begin{CD} K_*^{top}(\mathbb{C}[\pi]) @>>> HC_*(\mathbb{C}[\pi]) \\ @VVV @VVV \\ K_*^{top}(\ell^1(\pi)) @>>> HC_*^{loc}(\ell^1(\pi)) @>>> HCC_*^{aploc}(\ell^1(\pi)). \\ @VVV @VVV @VVV \\ K_*^{top}(\mathcal{A}_{max}(\pi)) @>>> HC_*^{loc}(\mathcal{A}_{max}(\pi)) @>>> HCC_*^{aploc}(\ell^1(\pi))_{(e)} \end{CD} \tag{1.12}$$

The vertical maps on the left are induced by the evident inclusion of topological algebras. The group $K_*^{top}(\mathbb{C}[\pi])$ denotes the topological K -theory of $\mathbb{C}[\pi]$ equipped with the fine topology, and the top horizontal map is the Chern character of [15]. The second and third horizontal maps are the Chern character of [13, 14] to local cyclic homology. The two maps to the group in the lower right corner are induced by Theorem 1.6, Theorem 1.7 and the evident decomposition of $HCC_*^{aploc}(\ell^1(\pi))$ into summands indexed by conjugacy classes. The commutativity of the diagram follows from the compatibility of Chern characters. Theorem A now follows from

Lemma 1.13 Let $[\phi] \in H_{se}^n(B\pi; \mathbb{C})$. Let $[c]$ denote the image of $[\phi]$ in $H^n(B\pi; \mathbb{C})$, and let τ_c denote the standard n -dim. cyclic group cocycle on $\mathbb{C}[\pi]$ formed by choosing a normalized representative c of $[c]$ and extending it over $C_n^{cy}(\mathbb{C}[\pi])$. Then ϕ induces a map $(\phi)_*: HCC_n^{aploc}(\ell^1(\pi))_{(e)} \rightarrow \mathbb{C}$ such that the composition

$$HC_n(\mathbb{C}[\pi]) \rightarrow HCC_n^{aploc}(\ell^1(\pi)) \rightarrow HCC_n^{aploc}(\ell^1(\pi))_{(e)} \xrightarrow{(\phi)_*} \mathbb{C}$$

equals the map induced by $[\tau_c]$.

Proof For fixed N, m let $D_{N,m,n} = c(n)!(2 + 2c(n))^{-m} N^{c(n)}$ (cf. (1.1)). Now also fix $\lambda > 1$, and consider an element $x = \sum_i \gamma_i (g_{0i}, g_{1i}, \dots, g_{ni})$ in the completed complex $C_n^{cy}(\ell^1_\lambda(\pi))_{(N)}$. For τ_c as above, we have

$$\begin{aligned} |\tau_c(x)| &\leq \sum_{g_{0i} g_{1i} \dots g_{ni} = 1}^i |\gamma_i| |\tau_c(g_{0i}, g_{1i}, \dots, g_{ni})| \\ &= \sum_i |\gamma_i| |c([g_{1i}, \dots, g_{ni}])| \\ &\leq \sum_i |\gamma_i| C \lambda^{L(g_{1i})} \dots \lambda^{L(g_{ni})} \\ &\leq (CD_{N,m,n}) \eta_{N,m} \left(\sum_i \gamma_i (g_{0i}, g_{1i}, \dots, g_{ni}) \right) = (CD_{N,m,n}) \eta_{N,m}(x) < \infty. \end{aligned} \tag{1.14}$$

We are using the seminorm defined in (1.1) for the auxiliary algebra $A' = \ell^1_\lambda(\pi)$ with norm $\| \sum \gamma_i g_i \|_{\ell^1_\lambda(\pi)} := \sum_i |\gamma_i| \lambda^{L(g_i)}$. Thus, for each $\lambda > 1$ and $N > 0$, τ_c extends to a continuous cyclic n -cocycle on $C_*^{cy}(\ell^1_\lambda(\pi))_{(N)}$, defining a homomorphism $H_n(C_*^{cy}(\ell^1_\lambda(\pi))_{(N)}) \rightarrow \mathbb{C}$. These extensions are obviously compatible with the inclusions associated to the directed system occurring in the definition of $HCC_*^{aploc}(\ell^1(\pi))$ in (1.1) above, and factor through the projection $HCC_*^{aploc}(\ell^1(\pi)) \rightarrow HCC_*^{aploc}(\ell^1(\pi))_{(e)}$. Denoting the induced homomorphism on the direct limit by $(\phi)_*$ yields the result. \square

Combining this with the commuting diagram now completes the proof of main result stated in the introduction.

Acknowledgement C. Ogle would like to thank D. Burghela for many helpful conversations in the preparation of this paper.

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