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The Künneth Formula in Cyclic Homology

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Introduction

The cyclic homology $HC_*(A)$ of an associative algebra with unit A over a field k of characteristic zero was introduced by A. Connes [C1], and extended to arbitrary commutative rings k in [LQ]. It comes equipped with a natural degree (-2) k-linear map $S\colon HC_*(A)\to HC_{*-2}(A)$. We will occasionally write S as S_A to indicate dependence on the ring A; the map S_A provides $HC_*(A)$ with a natural $k[u]$ co-module structure (as described below) via the isomorphism $HC_*(k)\cong k[u]$, $\deg(u)=2$. Throughout, k will be an arbitrary commutative ring with unit. The principal result of this paper is an Eilenberg-Zilber theorem for cyclic k-modules (Theorem 3.1) whose main applications are Theorem A and Theorem B.

Theorem A. i) *If A and B are two unital k-algebras with A and $HC_*(A)$ k-flat, then there exists a short exact sequence*

$$0\to\Sigma\,\mathrm{Cotor}(HC_*(A),HC_*(B))\xrightarrow{\Psi}HC_*(A\otimes B)\xrightarrow{\Phi}HC_*(A)\square_{k[u]}HC_*(B)\to 0$$

which is natural in A and B (where $\square$ denotes cotensor product).

ii) *If k is a field and $HC_*(B)$ is a quasi-free comodule (see Definition 2.1), $HC_*(B)=k[u]\otimes V_*+W_*$, then*

$$HC_*(A\otimes B)=HC_*(A)\otimes W_*+HH_*(A)\otimes V_*,^1$$

where $HH_*(A)$ is the Hochschild homology of A_*.

As an application one has the following calculation of the cyclic and periodic cyclic homology of $A[t]$ and $A[t,t^{-1}]$:

Theorem B. *If k is a field of characteristic zero and A a unital k-algebra, then:*

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1 This is a sum of k-vector spaces and NOT of $k[u]$-comodules, however, $HC_*(A)\otimes W_*$ is a sub $k[u]$-comodule

i) *The cyclic homology of the polynomial algebra* $A[t]$ *is given by*

$$HC_*(A[t]) = HC_*(A) + \mathrm{Nil}_+ HC_*(A), \quad \text{where } \mathrm{Nil}_+ HC_*(A) = \bigoplus_{\alpha \in \mathbb{N}} (HH_*(A))_\alpha \quad (\alpha = \{1, 2, 3, \ldots\}).$$

ii) *The cyclic homology of the algebra of Laurent polynomials with coefficients in A,* $A[t, t^{-1}]$ *is given by*

$$HC_*(A[t, t^{-1}]) = HC_*(A) + HC_{*-1}(A) + \mathrm{Nil}_+ HC_*(A) + \mathrm{Nil}_- HC_*(A)$$

where $\mathrm{Nil}_- HC_*(A) = \bigoplus_{\alpha \in \mathbb{Z} - (\{0\} \cup \mathbb{N})} (HH_*(A))_\alpha$.

iii) *As a consequence of* i) *and* ii), *one has* $PHC_*(A[t]) = PHC_*(A)$ *and* $PHC_*(A[t, t^{-1}]) = PHC_*(A) + PHC_{*-1}(A)$, *where* $PHC_n(B) = \varprojlim_{S_B} HC_{n+2r}(B)$.

Both the above theorem and corollary generalize to the case when A and B are DG algebras with differential of degree $+1$. The nil groups $\mathrm{Nil}_\pm HC_*(A)$ have interesting geometric applications even in the ungraded case, although we do not explore these applications here. It is worth comparing the decomposition given above for $HC_*(A[t])$ and $HC_*(A[t, t^{-1}])$ with the corresponding result in algebraic K-theory:

$$K_*(A[t]) = K_*(A) + \mathrm{Nil}_+ K_*(A)$$

and

$$K_*(A[t, t^{-1}]) \cong K_*(A) + K_{*-1}(A) + \mathrm{Nil}_+ K_*(A) + \mathrm{Nil}_- K_*(A).$$

Similar decomposition theorems holds true for L-theory and the K-theory of spaces.

Theorem B and Theorem A in the particular case of a group ring were first proven in $[B]_2$. Theorem A was conjectured by the first author and M. Karoubi in May of 1984, both of whom subsequently provided proofs through different arguments (see [K]). The results of this paper (for k a field of characteristic zero) were announced by the first author at Oberwolfach in August of 1984. A somewhat weaker result for the cyclic homology of the product of two cyclic simplicial abelian groups over arbitrary k was proved by the second author in [O]. Künneth formulas for the cyclic homology of algebras have also been proven by Feigin-Tsygan [FT], C. Kassel [Kas], J. Jones and C. Hood. This paper is a combination of [B3] and [O], and being shorter than both of them better suited for publication. As in [B3], one can then identify Ψ with the Loday-Quillen product in cyclic homology ([LQ], Sect. 3) and Φ with the dual of the Connes product in cyclic cohomology, but we will not ellaborate on this here.

The paper is organized as follows. In Sect. I, we recall the definition of algebraic S^1-chain complexes as introduced in [B1], and define the algebraic S^1-action on the tensor product of two such complexes. In Sect. II, we prove the Künneth formula for the tensor product of the two algebraic S^1-chain complexes and in Sect. III, we use "acyclic models" to prove Theorem 3.1. In Sect. IV we derive Theorems A and B.

Section I

Let k be a commutative ring with unit. An algebraic S^1-chain complex (a chain complex equipped with an algebraic circle action) $\tilde{C} \equiv (C_*, d_*, \beta_*)$ consists of a chain complex of k-modules (C_*, d_*), d_*: $C_n \to C_{n-1}$ satisfying $d_{n+1} d_n = 0$, with the algebraic circle action β_* given by k-linear maps β_n: $C_n \to C_{n+1}$ which satisfy $\beta_{n+1}\beta_n = 0$, $d_{n+1}\beta_n + \beta_{n-1} d_n = 0$.

A morphism of algebraic S^1-chain complexes f_*: $(C_*, d_*, \beta_*) \to (C'_*, d'_*, \beta'_*)$ consists of k-linear maps f_n: $C_n \to C'_n$ which commute with the d's and β's.

To an algebraic S^1-chain complex (C_*, d_*, β_*) one can associate the chain complex $({}_\beta C_*, {}_\beta d_*)$ with ${}_\beta C_n = C_n + C_{n-2} + \dots$, ${}_\beta d_n(x_n, x_{n-2}, \dots) = (dx_n + \beta x_{n-2}, dx_{n-2} + \beta x_{n-4}, \dots)$ and the following short exact sequence of chain complexes

$$0 \longrightarrow (C_*, d_*) \xrightarrow{I} ({}_\beta C_*, {}_\beta d_*) \xrightarrow{\pi} \Sigma^2({}_\beta C_*, {}_\beta d_*) \longrightarrow 0. \qquad (*)$$

Here I is the inclusion $I(x_n) = (x_n, 0, \dots, 0)$, Σ denotes the suspension $\Sigma(C_*, d_*) = (B_*, d'_*)$ with $B_{n+1} = C_n$, $B_0 = 0$, $d'_{n+1} = d_n$, and π is the projection $\pi(x_n, x_{n-2}, \dots) = (x_{n-2}, x_{n-4}, \dots)$.

The homology groups $H_*(C_*, d_*)$, resp. $H_*({}_\beta C_*, {}_\beta d_*)$ are by definition the *Hochschild* resp. *cyclic or equivariant* homology of $\tilde{C} \equiv (C_*, d_*, \beta_*)$. The long exact homology sequence associated with the short exact sequence $(*)$ becomes, with the above notation:

$$\longrightarrow HH_*(\tilde{C}) \xrightarrow{I} HC_*(\tilde{C}) \xrightarrow{S} HC_{*-2}(\tilde{C}_*) \longrightarrow HH_{*-1}(\tilde{C}) \longrightarrow \qquad (**)$$

and will be called the Gysin-Connes exact sequence. Obviously a morphism of algebraic S^1-chain complexes f: $\tilde{C} \to \tilde{C}'$ provides a commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & HH_*(\tilde{C}) & \longrightarrow & HC_*(\tilde{C}) & \longrightarrow & HC_{*-2}(\tilde{C}) & \longrightarrow & HH_{*-1}(\tilde{C}) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & HH_*(\tilde{C}') & \longrightarrow & HC_*(\tilde{C}') & \longrightarrow & HC_{*-2}(\tilde{C}') & \longrightarrow & HH_{*-1}(\tilde{C}') & \longrightarrow
\end{array} \qquad (***)$$

Given two algebraic S^1-chain complexes $\tilde{C}'$ and $\tilde{C}''$ one defines the tensor product $\tilde{C}' \otimes \tilde{C}''$ as being the chain complex $(C'_* \otimes C''_*, D_*)$ with

$$(C' \otimes C'')_n = \bigoplus_{k=0}^{n} C'_k \otimes C''_{n-k}, \qquad D_n(x_k \otimes y_{n-k}) = d' x_k \otimes y_{n-k} + (-1)^k x_k \otimes d'' y_{n-k},$$

and with the algebraic circle action $\bar{\beta}_*$ given by $\bar{\beta}_n(x_k \otimes y_{n-k}) = \beta' x_k \otimes y_{n-k} + (-1)^k x_k \otimes \beta'' y_{n-k}$.

We denote by chains$_k$ (resp. S^1-chains$_k$) the category of chain complexes (resp. algebraic S^1-chain complexes) of k-modules by F, T: S^1-chains$_k \rightsquigarrow$ chains$_k$, the functors which associate with (C_*, d_*, β_*) the chain complexes (C_*, d_*) resp. $({}_\beta C_*, {}_\beta d_*)$ and by $F \xrightarrow{I} T \to \Sigma^* T$ the natural transformations which to each S^1-chain complex $\tilde{C}_*$ associates the short exact sequence $(*)$.

Section II

Let k be a commutative ring with unit and let $k[u]$ be the graded commutative algebra generated by u of degree 2. $k[u]$ can also be viewed as a co-commutative coalgebra with commultiplication $\Delta: k[u] \to k[u] \otimes k[u]$ given by

$$\Delta(u^p) = \sum_{i=0}^{p} u^i \otimes u^{p-i}$$

and co-unit given by

$$\varepsilon(u^i) = \begin{cases} 0 & \text{if } i > 0 \\ 1 & \text{if } i = 0. \end{cases}$$

A $k[u]$-comodule is a graded module M_* equipped with the k-linear map Δ_M: $M_* \to k[u] \otimes M_*$ which satisfies the expected axioms. These axioms imply that $\Delta_M(m) = m + u \otimes S(m) + u^2 \otimes S^2(m) + \dots$, where S is a degree -2 k-linear map of M_*. Conversely any $S: M_* \to M_{*-2}$ provides a $k[u]$-comodule structure on M_*, hence the $k[u]$-comodule structures on a graded k-module M_* are in $1-1$ correspondence with the k-linear maps of degree -2.

Example. 1) Suppose V_* is a k-graded module which is k-flat. Then $V_* \otimes k[u]$ is equipped with a canonical $k[u]$-comodule structure given by $S(x \otimes u^n) = x \otimes u^{n-1}$ and $S(x) = 0$. This is called the free $k[u]$-comodule with base V_*.

2) Suppose V_* is a k-graded module and $S = 0$. The $k[u]$-comodule structure given by this S is called the trivial structure.

Definition 2.1. A $k[u]$-comodule M_* is called quasifree if M_* is the direct sum $M'_* + M''_*$ of two $k[u]$-comodules ($S_{M_*} = S_{M'_*} + S_{M''_*}$) with M'_* free ($S_{M'_*}$ surjective) and M''_* trivial ($S_{M''_*} = 0$).

Given two $k[u]$-comodules M_* and N_* one defines the graded vector space $M_* \square_{k[u]} N_*$ and $\Sigma^2 \mathrm{Cotor}_{k[u]}(M_*, N_*)$ as the kernel resp. cokernel of the linear map D: $M_* \otimes N_* \to \Sigma^2(M_* \otimes N_*)$ given by $D(m \otimes n) = S_M(m) \otimes n - m \otimes S_N(n)$; S_M and S_N are the degree (-2) - linear maps which define the $k[u]$-comodule structures of M_* and N_* and $\Sigma^n K_*$ denotes the n-fold suspension of K_*. Since D is a morphism of $k[u]$-comodules with the comodule structure on $M_* \otimes N_*$ given by $S_M \otimes \mathrm{id} + \mathrm{id} \otimes S_N$, $M_* \square_{k[u]} N_*$ and $\Sigma^2 \mathrm{Cotor}_{k[u]}(M_*, N_*)$ are $k[u]$-comodules.

If $\tilde{C}_* = (C_*, d_*, \beta_*)$ is an algebraic S^1-chain complex, then $HC_*(\tilde{C})$ has a $k[u]$-comodule structure induced by $S: HC_*(\tilde{C}) \to HC_{*-2}(\tilde{C})$.

Proposition 2.2. 1) *If $\tilde{C}'_* = (C'_*, d'_*, \beta'_*)$ and $\tilde{C}''_* = (C''_*, d''_*, \beta''_*)$ are two algebraic S^1-chain complexes with C'_* and $HC_*(\tilde{C}_*)$ k-flat, then there exists a (natural) short exact sequence*

$$0 \to \Sigma\, \mathrm{Cotor}_{k[u]}(HC_*(\tilde{C}'), HC_*(\tilde{C}^*)) \xrightarrow{\Psi} HC_*(\tilde{C}' \otimes \tilde{C}'')$$
$$\xrightarrow{\Phi} HC_*(\tilde{C}') \square_{k[u]} HC_*(\tilde{C}'') \to 0.$$

2) *If, moreover, k is a field and $HC_*(\tilde{C}'') = V_* \otimes k[u] + W_*$ is quasi-free where $V_* \otimes k[u]$ is the free part and W_* the trivial part, then*

$$HC_*(\tilde{C}' \otimes \tilde{C}'') = HC_*(\tilde{C}') \otimes V_* + H_*(C'', d''_*) \otimes W_*.$$

Proof of Proposition 2.2. Note that if (C_*, d_*, β_*) is an algebraic S^1-chain complex, then the chain complex $({}_\beta C_*, {}_\beta d_*)$ is a chain complex of free $k[u]$-comodules with ${}_\beta d_*$ being a morphism of $k[u]$-comodules. If (C'_*, d'_*, β'_*) and $(C''_*, d''_*, \beta''_*)$ are two algebraic S^1-chain complexes, we have the following short exact sequence of chain complexes

$$0 \longrightarrow {}_{\tilde\beta}(C'_* \otimes C''_*) \xrightarrow{I} {}_{\beta'}C'_* \otimes {}_{\beta''}C''_* \xrightarrow{D} \Sigma^2({}_{\beta'}C_* \otimes {}_{\beta''}C''_*) \longrightarrow 0. \qquad (*)$$

The differential δ in ${}_\beta C'_* \otimes {}_\beta C''_*$ is given by the tensor product differential, D is defined by $D(\bar x \otimes \bar y) = S'\bar x \otimes \bar y - \bar x \otimes S''\bar y$, $\bar x \in {}_\beta C'_*$, $\bar y \in {}_\beta C''_*$ with S' resp. S'' defined the $k[u]$-comodule structure of ${}_\beta C'_*$ resp. ${}_\beta C''_*$ and I as follows. We formally write

$$\bar x = (x_n, x_{n-2}, x_{n-4}, \ldots) \in {}_{\beta'}C'_n \text{ as } \bar x = \sum_{k=0} x_{n-2k} u^k,$$

$$\bar y = (y_r, y_{r-2}, y_{r-4}, \ldots) \in {}_{\beta''}C_r \text{ as } \bar y = \sum_{k=0} y_{r-2k} v^k$$

and

$$\bar z = (z_s, z_{s-2}, z_{s-4}, \ldots) \in {}_{\tilde\beta}(C'_* \otimes C''_*) \text{ as } \bar z = \sum_{k=0} z_{s-2k} U^k;$$

then I is given by $I(x_m \otimes y_n U^r) = \sum_{l=0}^{r} (x_m u^l) \otimes (y_n v^{r-l})$. The reader can easily check the exactness of this sequence. Moreover, if one equippes ${}_{\beta'}C'_* \otimes {}_{\beta''}C''$ with the degree -2 morphism of chain complexes $\mathscr{S} = S \otimes \mathrm{id} + \mathrm{id} \otimes S$, then ${}_{\beta'}C'_* \otimes {}_{\beta''}C''_*$ is a chain complex of $k[u]$-comodules and both I and D are morphisms of chain complexes of $k[u]$-comodules. Since

$$H_*({}_{\beta'}C'_* \otimes {}_{\beta''}C''_*) = HC_*(\tilde C') \otimes HC_*(\tilde C'')$$

and

$$H_*(D) = S_{HC_*(\tilde C')} \otimes \mathrm{id} - \mathrm{id} \otimes S_{HC_*(\tilde C'')}$$

the long exact sequence for homology induced by $(*)$ is

$$\to \Sigma^3(HC_*(\tilde C'_*) \otimes HC_*(\tilde C''_*)) \to HC_*(\tilde C'_* \otimes \tilde C''_*) \to HC_*(\tilde C'_*) \otimes HC_*(\tilde C''_*) \xrightarrow{S \otimes \mathrm{id} - \mathrm{id} \otimes S}$$
$$\to \Sigma^2(HC_*(\tilde C'_*) \otimes HC_*(\tilde C''_*)) \to \ldots,$$

which clearly provides the following short exact sequence

$$0 \to \Sigma \,\mathrm{Coker}(HC_*(\tilde C'_*), HC_*(\tilde C''_*)) \to HC_*(\tilde C'_* \otimes \tilde C''_*)$$
$$\to HC_*(\tilde C'_*) \square_{k[u]} HC_*(\tilde C''_*) \to 0$$

or equivalently $HC_*(\tilde C'_* \otimes \tilde C''_*) = \mathrm{Ker}\, D + \mathrm{Coker}\, \Sigma D$.

Suppose now that k is a field and $HC_*(\tilde C'') = k[u] \otimes V_* + W_*$ is quasifree. Then D: $HC_*(\tilde C') \otimes HC_*(\tilde C'') \to HC_*(\tilde C') \otimes HC_*(\tilde C'')$ is $D_1 + D_2$ with D_1: $\tilde H C_*(C') \otimes k[u] \otimes V_* \to HC_*(\tilde C') \otimes k[u] \otimes V_*$ defined by $D_1(x \otimes u^n \otimes v) = Sx \otimes u^n \otimes v - x \otimes u^{n-1} \otimes v$ and D_2: $HC_*(\tilde C) \otimes W_* \to HC_*(\tilde C') \otimes W_*$ defined by $D_2(x \otimes w) = Sx \otimes w$. Clearly $\mathrm{Coker}\, D_1 = 0$, $\mathrm{Ker}\, D_1 = HC_*(\tilde C') \otimes V_*$. The Gysin Connes exact sequence of $\tilde C'$ tensored by W_* gives the exact sequence

$$\longrightarrow \Sigma^{-2} HC_*(\tilde{C}')\otimes W_* \xrightarrow{\Sigma^{-1}D_2} H\tilde{C}_*(C')\otimes W_* \xrightarrow{\Sigma B\otimes \mathrm{id}} HH_*(\tilde{C}')\otimes W_* \xrightarrow{I\otimes \mathrm{id}}$$

$$\longrightarrow \Sigma^{-1} HC_*(\tilde{C}')\otimes W_* \xrightarrow{D_2} \Sigma HC_*(\tilde{C}')\otimes W_* \longrightarrow \dots .$$

This implies $HH_*(\tilde{C}')\otimes W_* = \operatorname{Coker}\Sigma^{-1}D_2 + \operatorname{Ker} D_2$, which implies that

$$\operatorname{Ker} D + \operatorname{Coker}\Sigma D = HC_*(\tilde{C}')\otimes V_* + HH_*(\tilde{C}')\otimes W_*.$$

Section III

We recall that a cyclic set (R-module) see [C] or [BF], (X_*, t_*) consists of a simplicial set (R-module) $X_* = (X_n, d_n^i, s_n^i;\ 0\leqq i\leqq n)$ and a cyclic structure $t_* = (t_n\colon X_n\to X_n)$ which satisfies $t_1^{n+1}=\mathrm{id}$, $t_{n-1}d_n^{i-1}=d_n^i t_n$, $t_n s_n^{i-1}=s_n^i t_n$ for $1\leqq i\leqq n$. Let $\mathbb{A}_R$ resp. $\tilde{\mathbb{A}}_R$ denote the category of simplicial R-modules resp. cyclic R-modules (when there is no danger of confusion we will write $\mathbb{A}$, $\tilde{\mathbb{A}}$, chains, S^1-chains instead of $\mathbb{A}_R$, $\tilde{\mathbb{A}}_R$, chains$_R$, S^1-chains$_R$).

As with $\mathbb{A}$, $\tilde{\mathbb{A}}$ is equipped with an internal tensor product

$$(G_n, d_n^i, s_n^i, t_n)\otimes(G'_n, d'^i_n, s'^i_n, t'_n) = (G_n\otimes G'_n, d_n^i\otimes d'^i_n, s_n^i\otimes s'^i_n, t_n\otimes t'_n).$$

With any cyclic R-module (G_n, d_n^i, s_n^i, t_n) one associates the S^1-algebraic chain complex

$$\left(G_n, d_n = \sum_{i=0}^{n}(-1)^i d_n^i, \beta_n = (-1)^n(1-(-1)^{n+1}t_{n+1})s_n^n(1+(-1)^n t_n + \dots(-1)^{n^2}t_n^n\right)$$

denoted by $\tilde{C}(C_*, t_*)$. The purpose of this section is to prove that Hochschild resp. cyclic homology of $\tilde{C}(G_*, t_*)\otimes\tilde{C}(G'_*, t'_*)$ and $\tilde{C}(G_*\otimes G'_*, t_*\otimes t'_*)$ are naturally isomorphic. Precisely if $\mathscr{A}$, $\mathscr{B}$: $\tilde{\mathbb{A}}\times\tilde{\mathbb{A}}\rightsquigarrow S^1$-chains are the functors defined by

$$\mathscr{B}((G_*, t_*), (G'_*, t'_*)) = \tilde{C}(G_*\otimes G'_*, t_*\otimes t'_*),$$
$$\mathscr{A}((G_*, t_*), (G'_*, t'_*)) = \tilde{C}(G_*, t_*)\otimes\tilde{C}(G'_*, t'_*)$$

then we have

Theorem 3.1. There exists the commutative diagram of functors and natural transformations

$$\begin{array}{ccccc}
F\mathscr{A} & \xrightarrow{I_{\mathscr{A}}} & T\mathscr{A} & \xrightarrow{\pi_{\mathscr{A}}} & \Sigma^2 T\mathscr{A} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \Sigma^2 T} \\
F\mathscr{B} & \xrightarrow{I_{\mathscr{B}}} & T\mathscr{B} & \xrightarrow{\pi_{\mathscr{B}}} & \Sigma^2 T\mathscr{B}
\end{array}$$

with f and T inducing isomorphism for homology.

The proof will require the Theorem of acyclic models [M, p. 128] which we will review below.

Let $\mathbf{C}$ be a category and $\mathscr{M}\in\operatorname{ob}\mathbf{C}$ a set of objects called *models*. Given a covariant functor $L\colon \mathbf{C}\to Ab$, $Ab=$ the category of abelian groups one can define a new covariant functor $\underline{L}$: $\mathbf{C}\to Ab$ and a natural transformation η: $\underline{L}\rightsquigarrow L$ by $\underline{L}(K)=$ the free abelian group generated by $X(K)=\bigcup_{M\in\mathcal{M}}(\mathrm{Hom}(M,K)\times L(M))$ for $K\in\mathrm{ob}\,\mathbf{C}$, $\underline{L}(f)\ (\alpha,u)=(f\circ\alpha\cdot u)$ for $f\in\mathrm{Hom}(K,K')$, $\alpha\in\mathrm{Hom}(M,K)$ and $u\in L(M)$, with η^K: $\underline{L}(K)\to L(K)$ given by $\eta^K(\alpha,u)=\alpha(u)$. The functor L is called representable with respect to $\mathcal{M}$ iff η admits a right inverse, i.e., a natural transformation

$$\phi^L\colon L\rightsquigarrow \underline{L}\quad\text{with}\quad \eta\circ\phi^L=\mathrm{id}.$$

Theorem of Acyclic Models [M, p. 128]. 1. *Let A,B: $\mathbf{C}\rightsquigarrow$ chains, be two covariant functors, $f=\{f_i\colon (A)_i\to(B)_i,\ 0\leqq i\leqq n\}$ a natural transformation of chain complex functors through dimension n and $\mathcal{M}$ a set of models in* $\mathbf{C}$. *If A_i is representable for all i, $B(M)$ is acyclic in dimension$>n$ for all $M\in\mathcal{M}$ and $f_n(\mathrm{Im}\,d^A_{n+1})\subset\mathrm{Im}(d^B_{n+1})$ then there exists a natural transformation f: $A\rightsquigarrow B$ extending $\{f_i\}_{i\leqq n}$. Moreover, the extension f is unique up to all higher homotopies.*

2. *If f,g are two natural transformations from A to B and $s=\{s_i\colon (A)_i\to(B)_{i+1},\ i\leqq n-1\}$ a homotopy through dimension n, i.e., $d_{i+1}s_i+s_{i-1}d_i=f_i-g_i$, A_q is representable for $q\geqq n$ and $H_q(B(M))=0$ for $q\geqq n$, $M\in\mathcal{M}$ then there exists a natural chain homotopy s such that $s|A_i=s_i$, $0\leqq i\leqq n-1$.*

This theorem will be applied to $\mathbf{C}=\tilde{A}\times\tilde{A}$ and to the functors $F\mathcal{A}$ and $F\mathcal{B}$.

Models. In [M] p. 130, $M^p_*=(M^p_n,d^i_n,s^i_*)$ is defined to be the free simplicial R-module generated by the standard p-simplex $\Delta[p]$ ($\Delta[p]_n=\mathrm{Hom}_\Delta(\underline{n},\ \underline{p})$) and $\mathcal{M}=\{(M^p_*,M^r_*)|p,r\geqq 0\}$ ob$\mathbb{A}\times\mathbb{A}$ is the set of models used to prove the standard Eilenberg-Zilber theorem. In analogy let M^p_Λ be the free cyclic R-module generated by the cyclic set $\Lambda[p]$. By $\Lambda[p]$ we denote the "free" cyclic set generated by $\Delta[p]$ (see [BF] Definition 1.3). It follows from [BF] Proposition 1.4 that the geometric realization of the underlying simplicial set $\Lambda[p]$ is homotopy equivalent to S^1 by an S^1-equivariant map. let $\mathcal{M}_\Lambda=\{(M^p_\Lambda,M^q_\Lambda)|p,q\geqq 0\}\in\mathrm{ob}\,\tilde{\mathbb{A}}\times\tilde{\mathbb{A}}$.

Representability. The representability for $(F\mathcal{A})_n=\tilde{A}_n$ resp. $(F\mathcal{B})_n=\tilde{B}_n$ can be proven as in [M] Lemma 2.9.1 by using the "free-ness" of our models. Precisely if $x_n\in K_n$, $K\in\mathrm{ob}\,\tilde{\mathbb{A}}$ it induces a simplicial map $\Delta[n]\xrightarrow{\hat{x}_n}K$ and then a cyclic map $\Lambda[n]\xrightarrow{\hat{x}_n}K$. This induces the homomorphism of cyclic R-modules $M^n_\Lambda\xrightarrow{\hat{x}_n}K$. So if $(K,L)\in\mathrm{ob}\,\tilde{\mathbb{A}}\times\tilde{\mathbb{A}}$, $\phi^{\tilde{A}_n}$: $\tilde{A}_n(K,L)=K_n\otimes L_n\to\underline{\tilde{A}_n}(K,L)$ and

$$\phi^{\tilde{B}_n}\colon \tilde{B}_n(K,L)=\sum_{r=0}^{n}K_r\otimes L_{n-r}\to\underline{\tilde{B}_n}(K,L)$$

are defined by the formulas

$$\phi^{\tilde{A}_n}(x_n\otimes y_n)=(\bar{x}_n\times\bar{y}_n,\lambda_n\otimes\lambda_n)\in\mathrm{Hom}_{\tilde{\mathbb{A}}\times\tilde{\mathbb{A}}}(M^n_\Lambda\times M^n_\Lambda,K\times L)\times A_n(M^n_\Lambda,M^n_\Lambda),$$

$$\phi^{\tilde{B}_n}(x_p\otimes x_{n-p})=(\bar{x}_p\times\bar{x}_{n-p},\lambda_p\otimes\lambda_n)\in\mathrm{Hom}_{\tilde{\mathbb{A}}\times\tilde{\mathbb{A}}}(M^p_\Lambda\times M^{n-p}_\Lambda;K\times L)\times B_n(M^p_\Lambda,M^{n-p}_\Lambda)$$

with λ_* the prefered generator of M^n_Λ. It is straightforward to verify $\phi^{\tilde{A}_n}$ and $\phi^{\tilde{B}_n}$ are natural transformations inverse to $\eta^{\tilde{A}_n}$ and $\eta^{\tilde{B}_n}$.

Acyclicity. By definition

$$H_*(T\mathscr{A}(M^n_\Lambda, M^p_\Lambda)) = HC_*(\tilde{C}_*(M^n_\Lambda)\otimes\tilde{C}(M^p_\Lambda))$$

and

$$H_*(T\mathscr{B}(M^n_\Lambda, M^p_\Lambda)) = HC_*(\tilde{C}_*(M^n_\Lambda\otimes M^p_\Lambda)).$$

Proposition 1.4 [BF] in conjunction with Proposition 2.2 (resp. the remark that $M^n_\Lambda\otimes M^p_\Lambda$ is the free cyclic R-module generated by the cyclic set $\Lambda[n] \times\Lambda[p]$) imply that

$$H_*(T\mathscr{A}(M^n_\Lambda, M^p_\Lambda)) \quad \text{resp.} \quad H_*(T\mathscr{B}(M^n_\Lambda, M^p_\Lambda))$$

is isomorphic to zero if $*\geqq 2$.

Proof of Theorem 3.1. Let $F\mathscr{A}'$, $T\mathscr{A}'$, $F\mathscr{B}'$ etc. ... be the graded module valued functors obtained by forgetting d's, and $\beta^{\mathscr{A}}: \Sigma T\mathscr{A}'\to T\mathscr{A}'$ resp. $\beta^{\mathscr{B}}: \Sigma T\mathscr{B}'\to T\mathscr{B}'$ be the natural transformation induced by the corresponding S^1-actions. Let $*$ be the natural self transformation of $T\mathscr{A}', F\mathscr{B}', \dots$ defined by $*_n=(-1)^n\mathrm{Id}$. We will construct the natural transformations $f^{(k)}: \Sigma^{2k}F\mathscr{A}'\to F\mathscr{B}'$, $k=0,1,2,\dots$ which satisfy:

(a)$_i$: $(\beta^{\mathscr{B}}f^{(i)}+f^{(i)}\beta^{\mathscr{A}})$ is natural transformation from $\Sigma^{2i+1}F\mathscr{A}$ to $F\mathscr{B}$.

(b)$_i$: $\beta^{\mathscr{B}}f^{(i)}+f^{(i)}\beta^{\mathscr{A}} = -df^{(i+1)}-f^{(i+1)}d.$

Note that (b)$_i\Rightarrow$(a)$_{i+1}$; indeed composing with $\beta^{\mathscr{A}}$ on the right resp. $\beta^{\mathscr{B}}$ on the left the equality (b)$_i$ implies $\beta df^{(i+1)}+\beta f^{(i+1)}d=df^{(i+1)}\beta+f^{(i+1)}d\beta$ which is clearly equivalent to $(a)_{i+1}$.

If $f^{(i)}$ are constructed, since $T\mathscr{A}'=F\mathscr{A}'+\Sigma^2F\mathscr{A}'+\Sigma^4F\mathscr{A}'+\dots$ resp. $T\mathscr{B}' =F\mathscr{B}'+\Sigma^2F\mathscr{B}'+\Sigma^4F\mathscr{B}'+\dots$ we can define T by the matrix

$$T=\begin{vmatrix} *f^{(0)} & *f^{(1)} & *f^{(2)} & \dots \\ 0 & *f^{(0)} & *f^{(1)} & \dots \\ 0 & 0 & *f^{(0)} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}.$$

Since $d_{T\mathscr{A}}$ resp. $d_{T\mathscr{B}}$ are given by the matrix

$$\begin{vmatrix} d & \beta & 0 & 0 & \dots \\ 0 & d & \beta & 0 & \dots \\ 0 & 0 & d & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

we verify (using (b)$_i$) that $Td_{T\mathscr{A}}=d_{T\mathscr{B}}T$ and we also observe that $TI_{\mathscr{A}}=I_{\mathscr{B}}f$ and $(\Sigma^2T)\pi_{\mathscr{A}}=\pi_{\mathscr{B}}T$, which finish the proof. Suppose $f^{(0)}$ and $f^{(1)}$ have been constructed. Inductively one obtains $-f^{(i+1)}$ as a homotopy from $\beta f^{(i)}+f^{(i)}\beta$ to zero which can be only obtained from Theorem of acyclic modules 2) as extension of $f_0^{(i+1)}=0$, $f_1^{(i+1)}=0$, $f_2^{(i+1)}=0$.

Construction of $f^{(0)}$. One takes $f^{(0)}=*f$ whose f is given by the shuffle map.

Construction of $f^{(1)}$. We want to obtain $-f^{(1)}$ as a homotopy from $\beta f^{(0)} + f^{(0)}\beta$ to zero, in other words we want $v_n = (-1)^n f_n^{(1)}$, $v_n\colon (\Sigma^2 F\mathscr{A})_n \to (F\mathscr{B})_n$ to satisfy $\beta f^0 + f^0\beta = dv + vd$. Again by applying the acyclic model theorem, it suffices to have v_n given for $n=0,1,2$ hence $v_2\colon A_0 \to B_2$. We take $v_2\colon G_0 \otimes H_0 = A_0 \to (G\otimes H)_2 = B_2 = G_2 \otimes H_2$, $G., H. \in \mathrm{ob}\,\tilde{\mathbb{A}} \times \tilde{\mathbb{A}}$ defined by

$$v_2(g_0 \otimes h_0) = \tau_2^G s_1^0 s_0^0(g_0) \otimes s^0 \tau_1^H s_0^0(h_0) + s_1^0 s_0^0(g_0) \otimes s_1^0 s_0^0(h_0)$$

where τ^G and τ^H are the cyclic structures of $G.$ resp. $H.$ (the definition of v_2 on morphisms is obvious).

The resulting transformations induce an isomorphism in homology because $f^{(0)}$ is the shuffle map up to sign.

Section IV

Proof of Theorem A. Given an R-algebra A the Hochschild resp. cyclic homology of A are calculated by the algebraic S^1-chain complex

$$(T_n(A), d_n^i, s_n^i, t_n) \quad \text{with} \quad T_n(A) = \underbrace{A \otimes \ldots \otimes A}_{n+1}$$

$$d_n^i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } i \leqq n-1 \\ a_n a_0 \otimes \ldots \otimes a_i \otimes \ldots \otimes a_{n-1} & \text{if } i = n \end{cases}$$

$$s_n^i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n$$

$$t_n^A(a_0 \otimes \ldots \otimes a_n) = a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.$$

Theorem 3.1 implies that Hochschild resp. cyclic homology of $\tilde{C}(T_*(A\otimes B), t_*^A)$, and of $\tilde{C}(T_*(A), t_*^A) \otimes \tilde{C}(T_*(B), t_*^B)$ are naturally isomorphic. Theorem A follows then from Proposition 2.2.

Proof of Corollary B. This follows from the calculation of the Hochschild resp. cyclic homology of $k[t]$ resp. $k[t, t^{-1}]$ given in [LQ] Sect. 2. In both cases the cyclic homology is a quasifree $k[u]$-comodule with

$$W_* = \begin{cases} k & \text{if } * = 0 \\ 0 & \text{if } * \neq 0 \end{cases} \quad \text{resp.} \quad W_* = \begin{cases} k & \text{if } * = 0, 1 \\ 0 & \text{if } * \neq 0, 1 \end{cases},$$

and

$$V_* = \begin{cases} \bigoplus_{\alpha \in \mathbb{N}} k_\alpha & \text{if } * = 0 \\ 0 & \text{if } * \neq 0 \end{cases} \quad \text{resp.} \quad V_* = \begin{cases} \bigoplus_{\alpha \in \mathbb{Z} \setminus 0} k_\alpha & \text{if } * = 0 \\ 0 & \text{if } * \neq 0 \end{cases};$$

here k_α denotes a copy of k. Q.E.D.

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