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ON THE K-THEORY AND CYCLIC HOMOLOGY OF A SQUARE-ZERO IDEAL I

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1. Introduction

The purpose of this paper is to establish an integral connection between the relative K-theory and cyclic homology of a square-zero ideal $I \subset R$, R a ring with unit. We restrict attention to the case when R is a split extension of R/I by I. The outline of the paper is as follows: in Section 2, we compute a direct summand of $\mathrm{HC}_*(R, I)$, where $\mathrm{HC}_*(R, I)$ is graded in analogy to relative K-theory and fits into a long-exact sequence $\cdots \to \mathrm{HC}_n(R) \to \mathrm{HC}_n(R/I) \xrightarrow{\partial} \mathrm{HC}_{n-1}(R, I) \to \mathrm{HC}_n(R) \to \cdots$. In Section 3, we construct double-brackets symbols (after Loday), and prove their relevant properties. In Section 4, we prove that the summand of $\mathrm{HC}_{n-1}(R, I)$ generated by these symbols splits off of $K_n(R, I)$ after inverting n. By severely restricting the quotient rings R/I under consideration, we get some information on the remaining piece of $K_n(R, I)$. We also give an application to the stable K-groups $K_*^m(R, A)$ of Hatcher, Igusa and Waldhausen. It is a theorem of Goodwillie's that rationally there is an isomorphism $K_*(R_\bullet, I_\bullet) \otimes Q \overset{\cong}{\to} \mathrm{HC}_{*-1}(R_*, I_*) \otimes Q$ when $I_\bullet \subset R_\bullet$ is a nilpotent ideal (generalizing earlier results of Burghelea and Staffeldt). Therefore the splitting mentioned is only of interest when taken integrally. The author would like to thank Chuck Weibel for his careful reading of an earlier version of this paper and for the many helpful suggestions which followed.

We now state the main results.

Theorem 1.1. *For a split extension $R \cong R/I \oplus I$ of a ring with unit R/I by a square-zero ideal I ($a_1 a_2 = 0$ for each $a_1, a_2 \in I$) one has (for all $n \geq 1$)*

$$\mathrm{HC}_{n-1}(R, I) \cong \left(\bigoplus_{p+q=n} H_p\left(\mathbb{Z}/q; \bigotimes_R^q I\right)\right) \oplus ?$$

where the group on the left is the $(n-1)$st relative cyclic homology group of the pair (R, I) over the integers, and where $\mathbb{Z}/q$ (on the right) acts by the non-trivial representation $t(a_1 \otimes \cdots \otimes a_q) = (-1)^{q+1} a_q \otimes a_1 \otimes \cdots \otimes a_{q-1}$ with tensor products being over R.

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Theorem 1.2. *For R as in Theorem 1.1, there exist for each $n \geq 1$ maps $\phi_n : \bigotimes_R^n I/(1-t) \cong H_0(\mathbb{Z}/n; \bigotimes_R^n I) \to K_n(R, I)$ and $c_n : K_n(R, I) \to \bigotimes_R^n I/(1-t)$ such that $c_n \circ \phi_n$ is multiplication by n (for each n), where $K_n(R, I)$ is the nth relative algebraic K-group of the pair (R, I). Thus there exists a splitting*

$$K_n(R, I) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \cong \left(\bigotimes_R^n I/(1-t)\right) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \oplus T_n(R, I) \quad (n \geq 1).$$

If $R/I \subset Q$, then $T_n(R, I)$ is torsion, and if $R/I \cong Q$, then $T_n(R, I) = 0$, for all $n \geq 1$. Finally, if $1/n \notin R/I$, then the map c_n is in general not divisible by n (so that inversion of n is necessary for the above splitting to hold).

It is worth noting that the map c_n in Theorem 1.2 factors by the Hurewicz homomorphism and so Theorem 1.2 is also true with $K_n(R, I)$ replaced by $H_n(\text{fibre}(B\text{GL}(R)^+ \to B\text{GL}(R/I^+))) \cong H_n(M^*(I))_{\text{GL}(R/I)}$, where $M^*(I) = \text{Id} + M(I) = \text{Ker}(\text{GL}(R) \to \text{GL}(R/I))$. It would seem that modular invariant theory could be applied to $H_*(M^*(I))_{\text{GL}(R/I)}$ to recover this last result.

Suslin has shown in [4] that for commutative fields F there exist maps $K_n^{\text{M}}(F) \xrightarrow{f_n} K_n(F) \xrightarrow{g_n} K_n^{\text{M}}(F)$ ($K_n^{\text{M}}(F)$ = the Milnor K-theory of F) such that $g_n \circ f_n$ is multiplication by $(n-1)!$ for each $n \geq 1$. The above splitting theorem is similar in spirit to Suslin's, with $\bigotimes_R^n I/(1-t)$ playing the role of the Milnor K-group $K_n^{\text{M}}(F)$ – splitting theorems of the type in Theorem 1.2 were conjectured to exist by Loday [8]. The necessity of inverting n can be explained by the existence of torsion elements in $K_n(R, I)$ arising from degenerate Hochschild n-chains which cannot be detected by the chern classes c_n. The fact that the splitting induces, for commutative rings R, an R/I-module structure on a summand of $K_n(R, I)$ is closely related to a result of Weibel [20] who has shown that the relative K-theory $K_*(A, A_*)$ of a graded algebra $A = \{A_i\}_{i \geq 0}$ ($A_* = \{A_i\}_{i > 0}$) over R admits the structure of a continuous module over the ring of Witt vectors $W(R)$.

2. A summand of $HC_*(R, I)$

Most of the notation will be as in Loday and Quillen's paper [9] to which we refer throughout this section – other references are [3] and [7]. An ingredient in the proof of Theorem 1.1 described in the previous section is the notion of Hochschild and cyclic homology over a non-commutative ground ring. Although this is a natural extension of the commutative case, it has not appeared in the literature and so we give the relevant constructions here. It is straightforward to show that the basic constructions and results (in particular Proposition 4.2) of [9] carry over to the case of a non-commutative ground ring R (except for products).

Thus, in [9] one considers the Hochschild complex $(\bigotimes_k^{*+1} R, b)$ and the acyclic Hochschild complex $(\bigotimes_k^{*+1} R, b')$. These generate the Connes–Tsygan double complexes $\underline{C}_{**}(R; k)$ and $C_{**}(R; k)$ which compute the cyclic and periodic homology of

R over the commutative ground ring k (see also [7]). In these situations, R is an algebra over k, with equivalent left and right module structures over k. This induces a k-module structure on the Hochschild and Connes–Tsygan double complexes, with the maps $b, b', (1-t)$ and N in the Connes–Tsygan complexes being k-linear.

Now let k be an arbitrary ring. By an algebra R over k we will mean an associative algebra in the usual sense, with both left and right pairing maps (which do *not* necessarily induce equivalent module structures on R). In this case we will define $\bigotimes_k^n R$ to be $\bigotimes_{\mathbb{Z}}^n R/\sim$, where the relations $\sim$ are generated by the n types of relations

$$r_1 s \otimes r_2 \otimes \cdots \otimes r_n \sim r_1 \otimes s r_2 \otimes \cdots \otimes r_n,$$

$$r_1 \otimes r_2 s \otimes r_3 \otimes \cdots \otimes r_n \sim r_1 \otimes r_2 \otimes s r_3 \otimes \cdots \otimes r_n,$$

$$\vdots$$

$$r_1 \otimes r_2 \otimes \cdots \otimes r_n s \sim s r_1 \otimes r_2 \otimes \cdots \otimes r_n$$

where $r_i \in R$, $1 \le i \le n$ and $s \in k$. The last relation indicates that $\bigotimes_k^n R$ is a 'cyclic' tensor product (with the copies of R arranged in a circle). Note that here we are really considering the Hochschild complex as a covariant functor in *two* variables: $\bigotimes_{(\,)}^{*}(\,): \mathbf{rings} \times \mathbf{rings} \to \mathbf{ab.groups}_*$. As in the case of k commutative one forms the complexes $(\bigotimes_k^{*+1} R, b)$, $(\bigotimes_k^{*+1} R, b')$ (and verifies that they are in fact, complexes). For unital R, $(\bigotimes_k^{*+1} R, b')$ is acyclic by the same argument as given in [9, Section 1]. The homology of $(\bigotimes_k^{*+1} R, b)$ will be denoted by $\mathrm{HH}_*(R; k)$ (and similarly for the relative groups) – if k is omitted it is assumed to be $\mathbb{Z}$. It is easy to verify that these complexes generate double complexes $C_{**}(R; k)$ and $\underline{C}_{**}(R; k)$; the homology of their associated total complexes is denoted by $\mathrm{HPer}_*(R; k)$ and $\mathrm{HC}_*(R; k)$ respectively (and similarly for the relative groups).

If $\mathbb{Z}/n = \langle t \mid t^n \rangle$ acts on a module M, the norm operator $N = 1 + t + t^2 + \cdots + t^{n-1}$ defines a map $N_*: H_0(\mathbb{Z}/n; M) \to H^0(\mathbb{Z}/n; M)$ induced by $x \to \sum_{i=0}^{n-1} x \cdot t^i$, $x \in M$. For notational purposes, it will be useful to use the Tate homology and cohomology groups $\hat{H}^0(\mathbb{Z}/n; M)$ and $\hat{H}_0(\mathbb{Z}/n; M)$ which one can define by the exact sequence

$$0 \to \hat{H}_0(\mathbb{Z}/n; M) \to H_0(\mathbb{Z}/n; M) \xrightarrow{N_*} H^0(\mathbb{Z}/n; M) \to \hat{H}^0(\mathbb{Z}/n; M) \to 0.$$

For an arbitrary ring R, the complex $(\mathrm{HH}_*(R), B)$ (as in [9] or [3]) can be viewed as a cochain complex whose cohomology we will call the DeRham cohomology of the ring $R: H_{\mathrm{DR}}^*(R) = H^*(\mathrm{HH}_*(R), B)$. The following computation is essentially Example 4.3 of [9]; we sketch the proof for the sake of completeness.

Lemma 2.1. *Let k be a ring with unit, I a k bi-module considered as a ring, with $I^2 = 0$. Let $R = k \oplus I$. Then over the ground ring k we have for each $n \ge 0$*

$$\mathrm{HH}_n(R, I; k) = H_0\Big(\mathbb{Z}/n+1; \bigotimes_k^{n+1} I\Big) \oplus H^0\Big(\mathbb{Z}/n; \bigotimes_k^{n} I\Big), \tag{2.1}$$

$$H_{\mathrm{DR}}^n(R, I; k) = \hat{H}_0\Big(\mathbb{Z}/n+1; \bigotimes_k^{n+1} I\Big) \oplus \hat{H}^0\Big(\mathbb{Z}/n; \bigotimes_k^{n} I\Big), \tag{2.2}$$

$$\mathrm{HC}_n(R,I;k)=\mathrm{HH}_n(R,I;k)/\mathrm{im}(B)\oplus H_{\mathrm{DR}}^{n-2}(R,I;k)\oplus H_{\mathrm{DR}}^{n-4}(R,I;k)\cdots, \tag{2.3}$$

$$\mathrm{HPer}_j(R,I;k)=\bigoplus_{i\geq 0} H_{\mathrm{DR}}^{2i+j}(R,I;k)\quad (j=0,1). \tag{2.4}$$

Proof. Throughout this proof, all tensor products are assumed to be over k. The normalized Hochschild complex $(R\oplus \bar{R}^*, d_*)$ in dimension n is $(k\oplus I)\otimes(\bigotimes^n I)\cong k\otimes(\bigotimes^n I)\oplus(\bigotimes^{n+1} I)$, with $d_n:\bigotimes^{n+1} I\to 0$ (since $I^2=0$), $d_n: k\otimes(\bigotimes^n I)\to\bigotimes^n I$ given by

$$\begin{aligned} d_n(1\otimes r_1\otimes\cdots\otimes r_n) &= r_1\otimes\cdots\otimes r_n+(-1)^n r_n\otimes r_1\otimes\cdots\otimes r_{n-1}\\ &=(1-t)(r_1\otimes\cdots\otimes r_n). \end{aligned}$$

So $\mathrm{HH}_n(I)$ is the sum of $\bigotimes^{n+1} I/(1-t)$ and $\operatorname{Ker} d_n: k\otimes(\bigotimes^n I)\to\bigotimes^n I$, where $t:\bigotimes^{n+1} I\to\bigotimes^{n+1} I$ is given by $t(r_0\otimes\cdots\otimes r_n)=(-1)^n(r_n\otimes r_0\otimes\cdots\otimes r_{n-1})$. This proves (2.1). The operator $B_n:\mathrm{HH}_n(R,I)\to\mathrm{HH}_{n+1}(R,I)$ is then given by

$$\begin{aligned} (0,N): H^0(\mathbb{Z}/n;\textstyle\bigotimes^n I)\oplus H_0(\mathbb{Z}/n+1;\bigotimes^{n+1} I)&\\ \to H_0(\mathbb{Z}/n+2;\textstyle\bigotimes^{n+2} I)\oplus H^0(\mathbb{Z}/n+1;\bigotimes^{n+1} I),& \end{aligned}$$

and so the equality $H_{\mathrm{DR}}^n(R,I)\cong\hat{H}_0(\mathbb{Z}/n+1;\bigotimes^{n+1} I)\oplus\hat{H}^0(\mathbb{Z}/n;\bigotimes^n I)$ follows from the definition of $\hat{H}^0$ and $\hat{H}_0$. To compute $\mathrm{HC}_*(R,I)$, we note that the same argument as given in [9, Proposition 4.2] shows that for any ring k, $\mathrm{HC}_*(k\oplus I,I)$ can be computed as the homology of the double complex $\underline{C}_{**}(I;k)$ of the ring without unit I. Since $I^2=0$, the vertical differentials in the double complex $\underline{C}_{**}(I;k)$ are all zero. The spectral sequence E_{**}^* associated to the double complex $C_{**}(I;k)$ (which filters first by rows, then columns and converges to $\mathrm{HC}_*(I)$) collapses at E_{**}^2, yielding $E_{**}^2=E_{**}^\infty$ with a trivial filtration on the E^∞ term, and with $E_{p,q}^\infty\cong H_p(\mathbb{Z}/q+1;\bigotimes^{q+1} I)$.

This yields $\mathrm{HC}_n(I)\cong\bigoplus_{p+q=n+1} H_p(\mathbb{Z}/q;\bigotimes^q I)$ which rewritten in terms of the groups $\hat{H}^0,\hat{H}_0$ and HH_n yields $\mathrm{HC}_n(R,I;k)=\mathrm{HC}_n(I;k)\cong\mathrm{HH}_n(I;k)/\mathrm{im}(B)\oplus H_{\mathrm{DR}}^{n-2}(I;k)\oplus H_{\mathrm{DR}}^{n-4}(I;k)\oplus\cdots$. Finally, to compute HPer_*, we note that the inverse systems $\{\mathrm{HC}_0(I;k)\leftarrow\mathrm{HC}_2(I;k)\leftarrow\cdots\leftarrow\mathrm{HC}_{2j}(I;k)\leftarrow\cdots\}$ and $\{\mathrm{HC}_1(I;k)\leftarrow\mathrm{HC}_3(I;k)\leftarrow\mathrm{HC}_5(I;k)\to\cdots\mathrm{HC}_{2j+1}(I;k)\to\cdots\}$ are Mittag–Leffler, and hence the $\varprojlim^1$ terms vanish; it is easy to check that on $\mathrm{HC}_n(I;k)\cong\mathrm{HH}_n(I;k)/\mathrm{im}(B)\oplus H_{\mathrm{DR}}^{n-2}(I;k)\oplus\cdots$, the operator S shifts everything left by 1, sending $\mathrm{HH}_n(I;k)$ to zero, $H_{\mathrm{DR}}^{n-2}(I;k)$ into $\mathrm{HH}_{n-2}(I;k)/\mathrm{im}(B)$ and $H_{\mathrm{DR}}^{n-2j}(I;k)$ to $H_{\mathrm{DR}}^{n-2j}(I;k)$ for $j>1$. Thus $\mathrm{HPer}_i(I)=\varprojlim_S \mathrm{HC}_{i+2n}(I)$ is given as in (2.4). $\square$

We note that in the above example, the spectral sequence

$$E_{**}^2\Rightarrow H_*\left(\mathrm{Tot}_*\left(\underline{B}_{**}(k\oplus I)\bigoplus_{\mathrm{red}}\right),d_*\right)$$

of [9, Theorem 1.9] collapses at E_{**}^3, which explains why the decomposition for

$\mathrm{HC}_*(I)$ is exactly analogous to that for $\mathrm{HC}_*(A)$ (over k a field of characteristic zero) when A is smooth over k [9, Theorem 2.9]. It is also worth noting that the computation of $\mathrm{HC}_*(I)$ can be used to show that the results of [7] are in some sense the best possible. The main result of [7] is that for a (simplicial) nilpotent ideal $I \subset R$, $\mathrm{HPer}_*(R) \to \mathrm{HPer}_*(R/I)$ is an isomorphism over a field of characteristic zero. Letting $R = k \oplus I$, $I^2 = 0$ (with trivial simplicial structure), the above computation of $\mathrm{HPer}_*(R, I)$ yields p-torsion for any prime p (by choosing I large enough). So $\mathrm{HPer}_*(R, I) \neq 0$ over k in general unless $Q \subset k$. Over arbitrary k, Goodwillie shows that the map $(h!S^h): \mathrm{HC}_{*+2h}(R, I) \to \mathrm{HC}_*(R, I)$ is zero for $* < h$. By the above, one sees in general that $(h-i)!S^h: \mathrm{HC}_{*+2h}(R, I) \to \mathrm{HC}_*(R, I)$, $* < h$ is not zero if $i > 0$, $1/(h-i)! \notin R$ for a square-zero ideal I.

For a square-zero ideal $I \subset R$, Goodwillie's result mentioned previously reduces the computation of the rational groups $K_*(R, I) \otimes Q$ to computing $\mathrm{HC}_{*-1}(R, I) \otimes Q$. We now compute part of this group for a ring R, over an arbitrary ground ring k when $I^2 = 0$ and $R \cong R/I \oplus I$. The point of considering the Hochschild complex functor as a bi-functor on the category of rings (as mentioned above) is that one has the following change-of-base-rings lemma:

Proposition 2.2. *Let k, l be rings with unit, $f: k \to l$ a ring homomorphism. If A is an algebra over l, then it is naturally an algebra over k (via the map f), and with this algebra structure over k there are natural maps $f_\#: \mathrm{HH}_*(A; k) \to \mathrm{HH}_*(A; l)$, and $\tilde{f}_\#: \mathrm{HC}_*(A; k) \to \mathrm{HC}_*(A; l)$ induced by a natural transformation η_+ from tensor products of l-algebras over k to tensor products of l-algebras over l (so that the same applies to the groups HPer_*, H^*_{DR}).*

Proof. Since we are restricting to l-algebras whose k-algebra structure is induced by f, the natural transformation η_+ simply arises by replacing '$\otimes_k$' by '$\otimes_l$' everywhere. Note that η_+ depends on f, and that we must restrict to algebras whose structure over k is induced by f. However, since η_+ is a natural transformation, all possible diagrams involving η_+ in 1 dimension which can commute do in fact commute. $\square$

Lemma 2.3. *Let R and R/I be algebras with unit over k, where $I \subset R$ is a square-zero ideal such that $R \cong R/I \oplus I$. Then there is a natural splitting*

$$\mathrm{HC}_*(R, I; k) \cong \mathrm{HC}_*(R, I; R/I) \oplus ?$$

(and similarly for HH_* and HPer_*).

Proof. By Lemma 2.1 and Proposition 2.2, the map $k \xrightarrow{\tau} R/I$ induces maps $\bigoplus_{p+q=n+1} H_p(\mathbb{Z}/q; \bigotimes_k^q I) \cong \mathrm{HC}_n(k \oplus I, I; k) \xrightarrow{i_n} \mathrm{HC}_n(R, I; k) \to \mathrm{HC}_n(R, I; R/I) \cong \bigoplus_{p+q=n+1} H_p(\mathbb{Z}/q; \bigotimes_{R/I}^q I)$. Each map is induced by a natural map of Connes-Tsygan double complexes; this implies that the composition is induced for each p, q by the map τ:

$$H_p\Big(\mathbb{Z}/q; \bigotimes_k^q I\Big) \xrightarrow{i_n} H_p\Big(\mathbb{Z}/q; \bigotimes_{R/I}^q I\Big).$$

We need to show that $\bar{i}_n : \mathrm{HC}_n(k \oplus I, I; k) \to \mathrm{HC}_n(R, I; k)$ factors as $\bar{i}_n = \alpha_n \circ i_n$ for some $\alpha_n : \mathrm{HC}_n(R, I; R/I) \to \mathrm{HC}_n(R, I; k)$. This follows by an easy diagram-chasing argument applied to the map of reduced Connes–Tsygan double complexes which induces i_n by noting that

(i) $I^2 = 0$ implies $\mathrm{Ker}(\bigotimes_k^q I \to \bigotimes_{R/I}^q I) \subset \mathrm{im}(b)$ for each q, where $b : \bigotimes_k^{q+1} I \to \bigotimes_k^q I$ is the Hochschild boundary map, and

(ii) the k-algebra map τ induces a surjection (for each q) $\bigotimes_k^q I \twoheadrightarrow \bigotimes_{R/I}^q I$.

The same argument applies also to the functors HH_* and HPer_*. Finally, we note that the R-module structure on $\mathrm{HH}_*(R, I; k)$ for commutative R induces an R-module (and R/I-module) structure on the summand $H_0(\mathbb{Z}/n+1; \bigotimes_{R/I}^{n+1} I)$ of $\mathrm{HC}_n(R, I; k)$ via the obvious projection (by Lemma 2.1). □

Remark 2.4. For (R, I) as in the above lemma, the splitting described implies that the pair (R, I) will usually not satisfy excision (even rationally) with respect to the functor HC_*. A necessary condition would be that $\bigotimes_k^n I/(1-t) \twoheadrightarrow \bigotimes_{R/I}^n I/(1-t)$ is an isomorphism, and this will usually only happen if the R/I-module structure on I is a trivial extension of the k-module structure on I.

It is easy to compute lower bounds for the rank of $\bigotimes_{R/I}^n I/(1-t)$ as a $\mathbb{Z}$-module (or k-module for commutative k).

Proposition 2.5. *Suppose R/I maps onto a commutative division field D and m = minimal number of generators of $\tilde{I} = I \otimes_{R/I} D$ as a vector space over D. Define $\mathrm{rk}_D(M) = \mathrm{rk}(M \otimes_{R/I} D)$ = the rank of $M \otimes_{R/I} D$ as a vector space over D. Then*

$$\mathrm{rk}_D\Big(\bigotimes_{R/I}^n I/(1-t)\Big) \geq \begin{cases} \left[\dfrac{m^n}{n}\right] & \text{if } n \text{ is odd,} \\[2ex] \left[\dfrac{m^n - m}{n}\right] & \text{if } n \text{ is even.} \end{cases}$$

Proof. $\mathrm{rk}_D(\bigotimes_{R/I}^n I) \geq m^n$. $t(r_1 \otimes \cdots \otimes r_n) = (-1)^{n+1} r_n \otimes r_1 \otimes \cdots \otimes r_{n-1}$. If n odd, t does not change signs, so $\mathrm{rk}_D(\bigotimes_{R/I}^n I/(1-t)) \geq \mathrm{rk}_D(\bigotimes_{R/I}^n I)/n$. For n even, $(-1)^{n+1} = -1$ hence $\mathrm{im}(\Delta I) \subset \bigotimes_{R/I}^n I/(1-t)$ is 2-torsion, where Δ is the diagonal map. Subtracting this gives $[(m^n - m)/n)]$ as a lower bound. □

3. Double-brackets symbols

We will give an alternative construction of the double-bracket construction of Loday [8]. This map has the advantage that its image in $H_*(B\mathrm{GL}(R))$ under the

Hurewicz map $K_*(R) \to H_*(BGL(R^+)) \stackrel{\cong}{=} H_*(BGL(R))$ is simple to describe (Lemma 3.1(iv) below). We do not actually show here that these two definitions of double-brackets are equivalent (although in fact they are; in particular, for $n=2$ they agree with the symbols $\langle a_1, a_2\rangle$ of Dennis and Stein).

Recall that for a ring R with unit and $I \subset R$ a two-sided ideal, one has $K_n(R, I) = \pi_n(K(R, I))$, where $K(R, I) = \mathrm{fibre}(BGL(R)^+ \to BGL(R/I^+))$ and $n \geq 1$.

Lemma 3.1. *Let R be any ring with unit, and $I \subset R$ a square-zero ideal, with $K_n(R, I) \to K_n(R)$ injective for all n. For n elements $a_1, \ldots, a_n \in I$ ($n \geq 1$), there is a map $\phi(a_1, \ldots, a_n): S^n \to K(R, I)$, defined up to homotopy which satisfies the following properties:*

(i) $[\phi(a_1, \ldots, a_n)] = (-1)^{n+1}[\phi(a_n, a_1, \ldots, a_{n-1})]$;

(ii) $[\phi(a_1, \ldots, a_n)] = 0$ *if* $a_i = 0$ *for any* i;

(iii) $[\phi(a_1, \ldots, a_i + a_i', \ldots, a_n)] = [\phi(a_1, \ldots, a_i, \ldots, a_n)] + [\phi(a_1, \ldots, a_i', \ldots, a_n)]$;

(iv) *The image of* $[\phi(a_1, \ldots, a_n)] \in \pi_n(K(R, I))$ *under the composition* $\pi_*(K(R, I)) \to K_*(R) \xrightarrow{H} H_*(GL(R))$ *is represented by the homology class of the n-fold shuffle product* $S([e_{12}(a_1)], \ldots, [e_{n1}(a_n)])$ *if* $n > 1$, *and by the image of the unit* $(1 + a_1) \in R^* = GL_1(R) \to GL(R) \twoheadrightarrow H_1(GL(R))$ *if* $n = 1$;

(v) *For* $a_1, \ldots, a_n \in I$ *and* $r \in R$, $[\phi(a_1 r, a_2, \ldots, a_n)] - [\phi(a_1, r a_2, a_3, \ldots, a_n)] = 0$.

Proof. Since $I^2 = 0$, the map $a \to [1 + a] \in K_1(R)$ is well defined on sums; $(a + b) \to [(1 + a)(1 + b)] = [(1 + (a + b)]$. If $n > 1$, then n elements $a_1, \ldots, a_n \in I$ induce a linear representation (as noted in [8]) $\Phi(a_1, \ldots, a_n): \mathbb{Z}^n \to GL_n(R)$ given by $\Phi(a_1, \ldots, a_n)(t_i) = e_{i,i+1}(a_i)$ for $1 \leq i \leq n-1$, $\Phi(a_1, \ldots, a_n)(t_n) = e_{n,1}(a_n)$. Passing to classifying spaces yields a map $B\Phi(a_1, \ldots, a_n): T^n \to BGL(R)$, where $T^n = B(\mathbb{Z}^n)$ is the n-torus. Now the n distinct inclusions $i_k: \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ ($i_k(s_i) = t_i$ if $i < k$, $i_k(s_i) = t_{i+1}$ if $i \geq k$) induce the n distinct inclusions $Bi_k: T^{n-1} \to T^n$, and under the standard cubical cell decomposition for T^n the $(n-1)$-skeleton $(T^n)^{n-1}$ is given by $(T^n)^{n-1} = \bigcup_{k=1}^n Bi_k(T^{n-1})$. For each k, the composition

$$\mathbb{Z}^{n-1} \xrightarrow{i_k} \mathbb{Z}^n \xrightarrow{\Phi(a_1, \ldots, a_n)} GL(R)$$

factors through a triangular subgroup $H_k \subset GL(R)$ generated by $\{e_{12}(a_1), \ldots, e_{n1}(a_n)\} - \{e_{k,\alpha_k}(a_k)\}$, where $\alpha_k = k+1$ if $1 \leq k \leq n-1$, $\alpha_k = 1$ if $k = n$. (A subgroup $T(R) \subset GL(R)$ is called *triangular* if there exists an ordering α of $\mathbb{Z}^+ = \{1, 2, \ldots\}$, agreeing with the standard ordering on a cofinal subset of $\mathbb{Z}^+$, such that $(g)_{ij} = 0$ if $\alpha(i) \not< \alpha(j)$, for all $g \in T(R)$.) So one can form the space $\bigcup_\alpha BT^\alpha(R)$, where the union ranges over all triangular subgroups of $GL(R)$. The space $\bigcup_\alpha BT^\alpha(R)$ is acyclic, with $\pi_1(\bigcup_\alpha BT^\alpha(R)) \cong \mathrm{St}(R)$ (as shown in [15]), and so the inclusion $\bigcup_\alpha BT^\alpha(R) \to BGL(R)$ is natural with respect to the plus construction yielding $* \simeq (\bigcup_\alpha BT^\alpha(R))^+ \to BGL(R)^+$. By the above, we have a commutative diagram

$$\begin{tikzcd}
(T^n)^{n-1} = \bigcup_{k=1}^{n} Bi_k(T^{n-1}) \arrow[r, "B\Phi(a_1,\dots,a_n)|_{(T^n)^{n-1}}"] \arrow[d] & \bigcup_{k=1}^{n} BH_k \subset \bigcup_{\alpha} BT^{\alpha}(R) \arrow[d] \\
T^n \arrow[r, "B\Phi(a_1,\dots,a_n)"] & BGL(R)
\end{tikzcd}$$

which after performing the plus construction on the right, becomes

$$\begin{tikzcd}
(T^n)^{n-1} \arrow[r, "B\Phi^+|_{(T^n)^{n-1}}"] \arrow[d] & \left(\bigcup_{\alpha} BT^{\alpha}(R)\right)^+ \arrow[d] \\
T^n \arrow[r, "B\Phi^+"] & BGL(R)^+
\end{tikzcd}$$

As $(\bigcup_{\alpha} BT^{\alpha}(R))^+$ is contractible, $B\Phi(a_1,\dots,a_n)^+$ yields a map, well defined up to homotopy,

$$S^n = T^n/(T^n)^{n-1} \xrightarrow{\overline{B\Phi}(a_1,\dots,a_n)^+} BGL(R)^+,$$

which is null-homotopic when mapped to $BGL(R/I)^+$, and hence lifts to $K(R,I)$ uniquely up to homotopy, since $K_*(R,I) \to K_*(R)$ is injective. This map, as well as its lifting, will be denoted by $\phi(a_1,\dots,a_n)$. We now verify its properties; again, since $K_n(R,I) \to K_n(R)$ is injective, it suffices to consider the map into $BGL(R)^+$.

We will assume $n>1$, as the case $n=1$ is trivial.

(i) The representations $\Phi(a_1,\dots,a_n)$ and $\Phi(a_n,a_1,\dots,a_{n-1})$ are related in the following way: The permutation matrix in $GL_n(R) \subset GL(R)$ corresponding to the permutation $\sigma_n = (1,2,\dots,n)$ induces an inner automorphism P_n of $GL_n(R)$ (and hence of $GL(R)$) via conjugation. Furthermore, σ_n induces an automorphism α_n of $\mathbb{Z}^n$ by $\alpha_n(t_i) = t_{i+1}$, $1 \le i \le n-1$, $\alpha_n(t_n) = t_1$. One can now check that $P_n \circ \Phi(a_1,\dots,a_n) = \Phi(a_n,a_1,\dots,a_{n-1}) \circ \alpha_n$. Passing to classifying spaces and performing the plus construction induces a commuting cube

$$\begin{tikzcd}
 & S^n \arrow[rr, "\phi(a_n,a_1,\dots,a_{n-1})"] & & BGL(R)^+ \\
T^n \arrow[ur, "c"] \arrow[rr, "B\Phi(a_n,a_1,\dots,a_{n-1})"] & & BGL(R) \arrow[ur] & \\
 & S^n \arrow[uu, "\overline{B\alpha_n}"] \arrow[rr, "\phi(a_1,\dots,a_n)"] & & BGL(R)^+ \arrow[uu, "P_n^+"'] \\
T^n \arrow[uu, "B\alpha_n"] \arrow[ur, "c"] \arrow[rr, "B\Phi(a_1,\dots,a_n)"] & & BGL(R) \arrow[uu, "P_n"] \arrow[ur] &
\end{tikzcd}$$

where c is the collapsing map $T^n \to T^n/(T^n)^{n-1}$ and $\overline{B\alpha_n}: S^n \to S^n$ is the map induced by $B\alpha_n : T^n \to T^n$ after collapsing $(T^n)^{n-1}$. Note that $\overline{B\alpha_n}$ is well defined, as $\overline{B\alpha_n}$ sends $(T^n)^{n-1}$ to $(T^n)^{n-1}$ by a cyclic rotation. Now the map $P_n^+ : B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R)^+$ is homotopic to the identity, since $(P_n^+)_*$ induces the identity in homology and $B\mathrm{GL}(R)^+$ is a simple space (alternatively, because for any $A \in \mathrm{GL}(R)$, $A \cdot (P_n(A))^{-1}$ is a product of elementary matrices). However, $[\overline{B\alpha_n}] = (-1)^{n+1} \cdot \mathrm{id} \in [S^n, S^n]$ for the reason that the self-map $B\alpha_n : T^n \to T^n$ is a diffeomorphism which changes the orientation (and hence the fundamental class $\mu_n \in H_n(T_n; \mathbb{Z})$) by the sign $(-1)^{n+1}$. Since the collapsing map $T^n \to T^n/(T^n)^{n-1}$ is a degree-1 map (which can be chosen so as to preserve orientation) $(\overline{B\alpha_n})_* = (-1)^{n+1} \cdot \mathrm{id} : H_n(S^n) \to H_n(S^n)$, and so $[\overline{B\alpha_n}] = (-1)^{n+1} \cdot \mathrm{id} \in [S^n, S^n]$. Thus $[\phi(a_1, \ldots, a_n)] = [\phi(a_n, a_1, \ldots, a_{n-1})] \cdot [\overline{B\alpha_n}] = (-1)^{n+1}[\phi(a_n, a_1, \ldots, a_{n-1})]$.

(ii) If $a_k = 0$ for some k, then the representation $\phi(a_1, \ldots, a_n) : \mathbb{Z}^n \to \mathrm{GL}(R)$ factors through the triangular subgroup H_k (defined above). Hence $B\phi(a_1, \ldots, a_n) : T^n \to B\mathrm{GL}(R)$ factors through $BH_k \subset \bigcup_\alpha BT^\alpha(R)$ and so $B\phi(a_1, \ldots, a_n)^+$ factors through $(\bigcup_\alpha BT^\alpha(R))^+ \simeq *$, implying $[\phi(a_1, \ldots, a_n)] = 0$.

(iii) By (i) it suffices to show that $\phi(a_1, \ldots, a_n)$ is linear in the first coordinate. For $a_1, a_1' \in I$, let $\beta : \mathbb{Z}^2 \to \mathrm{GL}(R)$ be the representation $\beta(t_1) = e_{12}(a_1)$, $\beta(t_1') = e_{12}(a_1')$. Let $\gamma : \mathbb{Z}^{n-1} \to \mathrm{GL}(R)$ be the representation $\gamma(t_2) = e_{23}(a_2), \ldots, \gamma(t_n) = e_{n1}(a_n)$. Let $i_1, i_2 : \mathbb{Z} \to \mathbb{Z}^2$ be the two inclusions, and $\Delta : \mathbb{Z} \to \mathbb{Z}^2$ the diagonal embedding: $\Delta(t) = i_1(t) + i_2(t)$. The representations $\Phi(a_1, \ldots, a_n)$, $\Phi(a_1', \ldots, a_n)$ and $\Phi(a_1 + a_1', \ldots, a_n)$ are given by $(i_1 \times \gamma), (i_2 \times \gamma)$ and $(\Delta \times \gamma)$. Let $B(\beta \times \gamma) : T^2 \times T^{n-1} \to B\mathrm{GL}(R)$ be the map on classifying spaces. Let $\Delta_2 \subset T^2$ be the 2-simplex representing the 2-chain $[t_1, t_2] \in C_2(B\mathbb{Z}^2; \mathbb{Z})$ in the bar resolution of $\mathbb{Z}^2$. Let $f = B(\beta \times \gamma)|_{\Delta_2 \times T^{n-1}}$, and $f^+ : \Delta_2 \times T^{n-1} \xrightarrow{f} B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$. On $\partial\Delta_2 \times T^{n-1}$, f^+ collapses $\Delta_2^{(0)} \times T^{n-1} \cup \partial\Delta_2 \times (T^{n-1})^{n-2}$ to a point, up to homotopy (as it factors through $\bigcup_\alpha BT^\alpha(R)^+ \simeq *$), and so $f^+|_{\partial\Delta_2 \times T^{n-1}}$ induces a map $(\bigvee^3 S^1) \wedge S^{n-1} \to B\mathrm{GL}(R)^+$ which represents the loop sum $s = \phi(a_1, \ldots, a_n) - \phi(a_1 + a_1', \ldots, a_n) + \phi(a_1', \ldots, a_n)$. As f^+ extends over $\Delta_2 \times T^{n-1}$, and collapses $\Delta_2^{(0)} \times T^{n-1} \cup \Delta_2 \times (T^{n-1})^{(n-2)}$ to $*$ by the same upper triangular matrix argument as above, the map $(\bigvee^3 S^1) \wedge S^{n-1} \xrightarrow{s} B\mathrm{GL}(R)^+$ extends to a map $(\Delta_2/\Delta_2{}^{(0)}) \wedge S^{n-1} \xrightarrow{\bar{s}} B\mathrm{GL}(R)^+$ yielding the required null-homotopy of s. This completes the proof of (iii).

(iv) The fundamental class μ_n of T^n is represented by the class of the n-fold shuffle product $[\sum_{\sigma \in \Sigma_n} (-1)^\sigma [t_{\sigma(1)}, \ldots, t_{\sigma(n)}]]$ in $C_n(B\mathbb{Z}^n; \mathbb{Z})$. $T^n \to S^n$ is a degree-1 map, and so the statement follows from the fact that $B\Phi(a_1, \ldots, a_n)_*$ maps the shuffle product $\sum_{\sigma \in \Sigma_n} (-1)^\sigma [t_{\sigma(1)}, \ldots, t_{\sigma(n)}]$ to the shuffle product that appears in the statement of (iv) above.

(v) This follows from a construction in [2, Chapters 1 and 3], where the map from cyclic homology into K-theory is constructed first on the level of base-pointed simplices. Specifically, for each $a_1, \ldots, a_n \in R$ satisfying $a_1 a_2 \cdot \cdots \cdot a_n = a_2 a_3 \cdot \cdots \cdot a_n a_1 = \cdots = a_n a_1 a_2 \cdot \cdots \cdot a_{n-1} = 0$ $(n \geq 2)$ and n distinct indices $i_1, \ldots, i_n \in \mathbb{Z}^+$, a map $\bar{f} = \bar{f}(a_1, \ldots, a_n; i_1, \ldots, i_n) : S(X(n-1)) = S(\Delta_{n-1}/\Delta_{n-1}^0) \to B\mathrm{GL}(R)^+$ is constructed satisfying the properties (where $X(n-1) = \Delta_{n-1}/\Delta_{n-1}^0$)

$$\text{(1)} \qquad \bar f|_{S(\partial_j X(n-1))} = \partial_j \bar f = \begin{cases} \bar f(a_2,\dots,a_n a_1; i_2,\dots,i_n) & \text{if } j=0, \\ \bar f(a_1,\dots,a_j a_{j+1},\dots,a_n; i_1,\dots,\hat{i}_{j+1},\dots,i_n) & \text{if } 1\le j<n; \end{cases}$$

(2) $\bar f \simeq *$ if $a_i=0$ for some i;

(3) If $n=3$, $\bar f(a_1,a_i,a_3; i_1,i_2,i_3)$ is a canonical null-homotopy representing the Jacobi identity for Dennis–Stein symbols;

(4) If $a_1a_2=a_2a_3=\cdots=a_na_1=0$, then $\bar f(a_1,\dots,a_n; i_1,\dots,i_n)$ factors as

$$S(X(n-1)) \twoheadrightarrow S(\Delta_{n-1}/\Delta_{n-1}^{n-2}) \simeq S^n \xrightarrow{\phi(a_1,\dots,a_n)} B\mathrm{GL}(R)^+$$

(and hence is independent of the choice of $i_1,\dots,i_n$).

Thus, if $r\in R$ and $a_1,\dots,a_n\in I$, we have a map $\bar f(a_1,r,a_2,\dots,a_n; 1,2,\dots,n+1)$: $SX(n)\to B\mathrm{GL}(R)^+$ which by property (2) collapses all of the faces $S(\partial_j X(n))$, except $S(\partial_1 X(n))$ and $S(\partial_2 X(n))$, to the basepoint in $B\mathrm{GL}(R)^+$. The result is a space homotopy equivalent to $(S^n\vee S^n)\bigcup_\alpha e^{n+1}$, $\alpha: S^n\to S^n\vee S^n$ the 'pinch' map, and a map $\bar f'(a_1,r,a_2,\dots,a_n; 1,\dots,n+1): (S^n\vee S^n)\bigcup_\alpha e^{n+1}\to B\mathrm{GL}(R)^+$ which on the boundary is independent of the indices (up to homotopy) and so yields a null-homotopy of $\phi(a_1r,a_2,\dots,a_n)-\phi(a_1,ra_2,\dots,a_n)$ by property (4). This completes the proof of Lemma 3.1. $\square$

Corollary 3.2. *For R,I as above the map $\bigotimes^n I/(1-t)\xrightarrow{\phi_n} K_n(R,I)$ constructed in* Lemma 3.1 factors as $\bigotimes^n I/(1-t)\twoheadrightarrow \bigotimes_R^n I/(1-t)\xrightarrow{\phi_n} K_n(R,I)$, *where the R-module structure on I is induced by the inclusion $I\subset R$.*

Proof. The projection map $\bigotimes^n I/(1-t)\twoheadrightarrow\bigotimes_R^n I/(1-t)$ is induced by adding the relation $a_1r\otimes a_2\otimes\cdots\otimes a_n \sim a_1\otimes ra_2\otimes\cdots\otimes a_n$ together with its cyclic permutations. But properties (i) and (v) show that this relation is compatible with the map constructed. Hence ϕ_n factors as desired. $\square$

Remark 3.3. Weibel has pointed out that for commutative R the kernel of ϕ_n is not necessarily an R-submodule of $\bigotimes_R^n I/(1-t)$. An example of this occuring for $n=2$ is given in [18].

4. A splitting theorem for symbols

We now construct the splitting described in Section 1, and give some applications.

Lemma 4.1. *If R is a ring with unit and $I\subset R$ a square-zero ideal with $R\cong R/I\oplus I$, then there is a map $c_n: K_n(R,I)\to\bigoplus_{R/I}^n I/(1-t)$ $(n\ge 1)$ such that the composition $\bigotimes_{R/I}^n I/(1-t)\xrightarrow{\phi_n} K_n(R,I)\xrightarrow{c_n}\bigotimes_{R/I}^n I(1-t)$ is multiplication by n.*

Proof. Consider the composition $\bigotimes_R^n I/(1-t)\xrightarrow{\phi_n} K_n(R,I)\xrightarrow{i_n} K_n(R)\xrightarrow{D_n}\mathrm{HH}_n(R)$, where D is the Dennis map, defined as, for example, in [5] and so one might at first suspect that, by restricting to the relative group $K_n(R, I)$ for R, I as above, one can lift the Dennis map D_n integrally to a map $\tilde{D}_n : K_n(R, I) \to \mathrm{HC}_{n-1}(R, I)$ satisfying the identity $D_n = B \circ \tilde{D}_n$. This would be more or less equivalent to showing that the chern class c_n is divisible by n, i.e., that it is really the nth component of a chern character which for the algebraic K-groups would satisfy $\mathrm{ch}_n = (-1)^{n-1} n c_n$. This would show that $\bigotimes^n_{R/I} I/(1-t)$ splits off of $K_n(R, I)$ integrally. But a simple example shows that in general this cannot be done.

In the case $n = 2$, one knows that for commutative R, $K_2(R, I)$ is generated by Dennis–Stein symbols (we use the presentation due to Maazen and Steinstra given in [10] – the result is originally due to Dennis and Stein). Thus if I is a square-zero ideal, $R = R/I \oplus I$, then $K_2(R, I) \cong D_2(R, I)$ is generated by $\{\langle a, b\rangle \mid a \in I \text{ or } b \in I\}$ with relations generated by $\langle a, c\rangle + \langle b, c\rangle = \langle a + b - acb, c\rangle$, $\langle ab, c\rangle - \langle a, bc\rangle + \langle ca, b\rangle = 0$, $\langle a, b\rangle = -\langle b, a\rangle$. A straightforward computation shows that $D(\langle a, b\rangle)$ is represented by the cycle $(1-ab)^{-1} \otimes a \otimes b - (1-ab)^{-1} \otimes b \otimes a \in \bigotimes^3 R =$ Hochschild 2-chains. $\langle 1, a\rangle = 0$ by the Jacobi identity, and by the sum relation, one has $0 = \langle 1, a\rangle + \langle 1, a\rangle = \langle 2 - a, a\rangle = \langle 2, a\rangle + \langle -a, a\rangle = \langle 2, a\rangle - \langle a, a\rangle$. The map $c_2 : K_2(R, I) \to \bigotimes^2_{R/I} I/(1-t)$ factors by the Dennis map, and hence sends both $\langle 2, a\rangle$ and $\langle a, a\rangle$ to zero by the above formula for $D(\langle a, b\rangle)$. However, if $\frac{1}{2} \notin R$ both $\langle 2, a\rangle$ and $\langle a, a\rangle$ can represent non-trivial 2-torsion in $K_2(R, I)$. Thus c_2 must kill this 2-torsion, and so in general c_2 would not be divisible by 2. If $\frac{1}{2} \in R$, then I is an algebra over $\mathbb{Z}[\frac{1}{2}]$ and so by Weible's theorem (see [18, 19]), $K_2(R, I)$ is a module over $\mathbb{Z}[\frac{1}{2}]$ (as is $\bigotimes^2_{R/I} I/(1-t)$), so division by 2 is not a problem. This condition also appears in Van der Kallen's original result that $K_2(R[\varepsilon]/(\varepsilon^2), (\varepsilon)) \cong \Omega^1_{R/\mathbb{Z}}$ for commutative R with $\frac{1}{2} \in R$. We can state the above lemma in terms of a splitting theorem.

Theorem 4.3. *If $R \cong R/I \oplus I$ is a ring with $I^2 = 0$, then*

$$K_n(R, I) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \cong \left(\bigotimes_{R/I}^{n} I/(1-t)\right) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \oplus T_n(R, I)$$

and

$$H_n(B\Gamma(R, I))_{\mathrm{GL}(R/I)} \otimes \mathbb{Z}\left[\frac{1}{n}\right] \cong \bigotimes_{R/I}^{n} I/(1-t) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \oplus U_n(R, I)$$

for each $n \geq 1$.

We now give some applications of the above theorem.

Proposition 4.4. *Let $R = \mathbb{Z}/2\ [(\mathbb{Z}/2)^m]$ be the mod 2 group ring of $(\mathbb{Z}/2)^m = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ (m times). Let I_m denote the augmentation ideal of R. Then for n odd, $K_n(R, I) \cong \bigotimes^n_{\mathbb{Z}/2} I_m/(1-t) \oplus$ (2-group). The rank of $\bigotimes^n_{\mathbb{Z}/2} I_m/(1-t)$ as a vector space over $\mathbb{Z}/2$ is bounded below by $[(2^m - 1)^n/n]$; hence if $m > 1$, the rank of $K_{2n-1}(R, I)$ over $\mathbb{Z}/2$ grows exponentially as a function of n.*

$$D_n : K_n(R) \xrightarrow{H} H_n(B\mathrm{GL}(R)^+) \cong H_n(B\mathrm{GL}(R)) \xrightarrow{f_n} \mathrm{HH}_n(\mathbb{Z}[\mathrm{GL}(R)])$$
$$\xrightarrow{i_n} \mathrm{HH}_n(M(R)) \xrightarrow[\cong]{\mathrm{Tr}_{n+1}} \mathrm{HH}_n(R)$$

where H is the Hurewicz homorphism, f_n is induced on n-chains by $[g_1, \dots, g_n] \to (g_1 \cdots g_n)^{-1} \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$, i_n induced on normalized chains by the ring extension of the inclusions $(\mathrm{GL}_m(R) \to M_m(R)\}$ and Tr_{n+1} is induced by the trace map

$$M^0 \otimes \cdots \otimes M^n \to \sum_{i_0, i_1, \dots, i_n} \mathrm{Tr}(M^0_{i_0, i_1} \otimes M^1_{i_1, i_2} \otimes \cdots \otimes M^n_{i_n, i_0}) \in R \otimes \bar{R}^n$$
$$= \text{normalized Hochschild } n\text{-chains.}$$

Tr_{n+1} is the isomorphism arising from the Morita equivalence of $M(R)$ and R. Now by Lemma 3.1(iv), the image of $\phi_n(a_1 \otimes \cdots \otimes a_n)$ under H is represented for $n > 1$ by the n-fold shuffle product $S([e_{12}(a_1)], [e_{23}(a_2), \dots, [e_{n1}(a_n)])$ in $C_n(\mathrm{GL}(R); \mathbb{Z})$. It is easy to see that under the map $C_n(\mathrm{GL}(R); \mathbb{Z}) \to R \otimes \bar{R}^n$ defined above, that out of the $n!$ elements in this shuffle product, the only ones that map non-trivially are those that differ by a cyclic permutation from $[e_{12}(a_1), \dots, e_{n1}(a_n)]$, yielding

$$(D_n i_n \phi_n)(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma \in \mathbb{Z}/n} (-1)^{\sigma_1} \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} = B(a_1 \otimes \cdots \otimes a_n)$$

(where B is the transfer map $\mathrm{HC}_{n-1}(R) \to \mathrm{HH}_n(R)$ defined as, for example, in [9, Section 1]). For $n = 1$, one has

$$(D_1 i_1 \phi_1)(a_1) = (1 + a_1)^{-1} \otimes (1 + a_1) = (1 - a_1) \otimes a_1 = (1 \otimes a_1) - a_1 \otimes a_1.$$

Note that by $\mathrm{HH}_n(R)$ we mean $\mathrm{HH}_n(R; \mathbb{Z})$. $\mathbb{Z} \to R/I$ induces $\mathrm{HH}_n(R; \mathbb{Z}) \to \mathrm{HH}_n(R; R/I)$ by Proposition 2.2, and by Lemma 2.1, $\mathrm{HH}_n(R; R/I) = \mathrm{HH}_n(R/I \oplus I; R/I) \cong H_0(\mathbb{Z}/n+1; \otimes^{n+1}_{R/I} I) \oplus H^0(\mathbb{Z}/n; \otimes^n_{R/I} I)$ for $n \geq 1$. The usual map from invariants to coinvariants maps $H^0(\mathbb{Z}/n; \otimes^n_{R/I} I) \to H_0(\mathbb{Z}/n; \otimes^n_{R/I} I)$, and so under $\mathrm{HH}_n(R) \to \mathrm{HH}_n(R; R/I) \xrightarrow{p_2} H^0(\mathbb{Z}/n; \otimes^n_{R/I} I) \to H_0(\mathbb{Z}/n; \otimes^n_{R/I} I)$, $(D_n i_n \phi_n)(a_1 \otimes \cdots \otimes a_n)$ maps to $n(a_1 \otimes \cdots \otimes a_n)$ for all $n > 1$. Defining c_n then to be the sequence of maps $K_n(R, I) \to K_n(R) \xrightarrow{D} \mathrm{HH}_n(R) \to \mathrm{HH}_n(R; R/I) = H_0(\mathbb{Z}/n+1; \otimes^{n+1}_{R/I} I) \oplus H^0(\mathbb{Z}/n; \otimes^n_{R/I} I) \xrightarrow{p_2} H^0(\mathbb{Z}/n; \otimes^n_{R/I} I) \to H_0(\mathbb{Z}/n; \otimes^n_{R/I} I) = \otimes^n_{R/I} I/(1-t)$ proves the lemma. □

Note that this map can be factored through the relative groups $H_n(\Gamma(R, I))_{\mathrm{GL}(R/I)}$ and $\mathrm{HH}_n(R, I)$, but it is easier to describe by passing to the absolute case and then projecting back.

Remark 4.2. The above maps fit into a commutative diagram

$$\begin{tikzcd}
\bigotimes^n_{R/I} I/(1-t) \arrow[r, "\phi_n"] \arrow[d] & K_n(R, I) \arrow[d, "D_n"] \\
\mathrm{HC}_{n-1}(R, I) \arrow[r, "B"] & \mathrm{HH}_n(R, I)
\end{tikzcd}$$

Proof. The rank of I_m over $\mathbb{Z}/2$ is $2^m - 1$. So applying Proposition 2.5 yields the lower bound. Also, $K_*(R, I)$ is a 2-group by Weibel's theorem [19], since $K_*(R, I) \otimes \mathbb{Z}[\frac{1}{2}] = 0$. $\square$

Lemma 4.5. *Let (R, I) be a split extension, $I^2 = 0$, with $R/I \cong \mathbb{Z}/m$, or $R/I \subset Q$. Then $\phi_n : \bigotimes^n_{R/I} I/(1-t) \to K_n(R, I)$ is a rational isomorphism for all $n \geq 1$. If $R/I \cong Q$, ϕ_n is an integral equivalence.*

Proof. The case R/I finite is trivial, since then $K_*(R, I) \otimes Q = 0$. So assume $R/I \subset Q$. We recall that Staffeldt, in his proof of [12, Theorem 1] produces a sequence of isomorphisms (where $\mathcal{O} = R/I \subset Q$)

$$\begin{aligned}
&\pi_*(\text{fibre}(B\mathrm{GL}_m(R)^+ \to B\mathrm{GL}(\mathcal{O})^+)) \\
&\quad \cong \mathrm{Prim}(H_*(\Gamma(R, I); Q)_{\mathrm{GL}(\mathcal{O})}), \quad \Gamma(R, I) = \mathrm{Ker}(\mathrm{GL}(R) \to \mathrm{GL}(R/I)), \\
&\quad \cong \mathrm{Prim}(H^{\mathrm{L}}_*(M(I \otimes Q); Q)_{\mathrm{gl}(\mathcal{O})}) \quad \text{where } H^{\mathrm{L}}_* = \text{Lie algebra homology}, \\
&\quad \cong \mathrm{Prim}(H^{\mathrm{L}}_*(M(I \otimes Q); Q)_{\mathrm{gl}(Q)}) \\
&\quad \cong \mathrm{HC}_{*-1}(I) \otimes Q \quad \text{by [9, Theorem 6.2].}
\end{aligned}$$

The map $H_*(\Gamma(R, I); Q) \to H^{\mathrm{L}}_*(M(I \otimes Q); Q)$ above is induced by matrix logarithm, and as $I^2 = 0$, $M_m(I) = m \times m$ matrices in I is a square-zero ideal in $M_m(R)$, so that $\log : \Gamma_m(R, I) \to M_m(I)$ is just given by $\log(X) = X - 1$. This gives a well-defined integral map $H_*(\Gamma(R, I))_{\mathrm{GL}(\mathcal{O})} \to H^{\mathrm{L}}_*(M(I))_{\mathrm{gl}(\mathcal{O})} \cong (\bigwedge^*(M(I)), d = 0)_{\mathrm{gl}(\mathcal{O})}$ induced by $[g_1, \ldots, g_n] \to \log(g_1) \wedge \cdots \wedge \log(g_n) = (g_1 - 1) \wedge \cdots \wedge (g_n - 1)$, and this maps to $\mathrm{HC}_{*-1}(I)$ by $M_1 \wedge \cdots \wedge M_n \to \sum_{\sigma \in \Sigma_{n-1}} (-1)^\sigma \mathrm{Tr}_n(M_1 \wedge M_{\sigma(2)} \wedge M_{\sigma(3)} \wedge \cdots \wedge M_{\sigma(n)})$ as in [9]. Now the case $n = 1$ is easy, so we restrict to the case $n > 1$. By Lemma 3.1, the image of $\phi_n(a_1 \otimes \cdots \otimes a_n)$ in $H_n(\mathrm{GL}(I))$ is represented by the cycle

$$\sum_{\sigma \in \Sigma_n} (-1)^\sigma [e_{\sigma(1)\sigma(2)}(a_{\sigma(1)}), \ldots, e_{\sigma(n)\sigma(1)}(a_{\sigma(n)})].$$

Under the log map above, this maps to the cycle

$$\sum_{\sigma \in \Sigma_n} (-1)^\sigma m_{\sigma(1)\sigma(2)}(a_{\sigma(1)}) \wedge \cdots \wedge m_{\sigma(n)\sigma(1)}(a_{\sigma(n)}) \in \textstyle\bigwedge^n M(I),$$

$m_{ij}(a) = e_{ij}(a) - \mathrm{Id}$, and under the trace map, this goes to $(n!) a_1 \otimes \cdots \otimes a_n \in \mathrm{HC}_{n-1}(I)$. Tensoring with Q shows that $\mathrm{im}(\phi_n) \subset K_n(R, I) \otimes Q$ surjects isomorphically onto $\mathrm{HC}_{n-1}(R, I) \otimes Q = \bigotimes^n I/(1-t) \otimes Q$ under Staffeldt's map, assuming $I \otimes Q$ is finitely generated. A simple direct limit argument removes that condition. In particular, for $\mathcal{O} = Q$ and $I \oplus \mathcal{O}$ a Q-algebra, $I^2 = 0$ the results of Weibel and Staffeldt imply in this case that $\phi_n : \bigotimes^n_Q I/(1-t) \to K_n(\mathcal{O} \oplus I, I)$ is an integral equivalence for all $n \geq 1$.

Furthermore, since $c_n \phi_n$ is multiplication by n, by Lemma 4.1 we have that $\mathrm{Ker}(c_n)$ is torsion for all n; in particular the group $T_n(R, I)$ in the decomposition

$$K_n(R,I)\otimes\mathbb{Z}\left[\frac{1}{n}\right]\cong\left(\bigotimes_{R/I}^{n} I/(1-t)\right)\otimes\mathbb{Z}\left[\frac{1}{n}\right]\oplus T_n(R,I)$$

(Theorem 4.3) is torsion for all n. □

The torsion subgroups of $K_n(R,I)$ not coming from $\otimes_{R/I}^n I/(1-t)$ are not in general known for $n>2$ even when $R/I\subset Q$, but with R/I containing non-invertible primes. Note that in the case of the ring of dual integers of $\mathbb{Z}$, $R=\mathbb{Z}(\varepsilon)$, one has $I=(\varepsilon)$ and $R/I\cong\mathbb{Z}$. One then recovers a result due to Soulé, from the above constructions, namely

Corollary 4.6. *$K_n(\mathbb{Z}(\varepsilon),\varepsilon)\cong\mathbb{Z}\oplus$ (torsion) for n odd, and $K_n(\mathbb{Z}(\varepsilon),\varepsilon)\cong$ (torsion) for n even.*

Proof. $\otimes^n(\varepsilon)/(1-t)\cong\mathbb{Z}$ for n odd, $\mathbb{Z}/2$ for n even, and L_n is a rational equivalence (see also [8, Proposition 3.3]. □

More generally, one might suspect from the above constructions that there should be an integral connection between $K_n(R,I)$ and the part of $\mathrm{HC}_{n-1}(R,I)$ not coming from $\otimes_R^n I/(1-t)$, after inverting n. When R is an algebra over Q this follows trivially by the theorems of Weibel, Goodwillie and Staffeldt mentioned above. When R is not an algebra over Q, it is probably necessary to require that at least $R/I\to R/I\otimes Q$ is injective. For example, it follows from [1] and [6] that $K_3(\mathbb{Z}/p(\varepsilon),(\varepsilon))\cong\mathbb{Z}/p\oplus\mathbb{Z}/p$, when p is a prime, $p\neq 3$ and $R(\varepsilon)=R[\varepsilon]/(\varepsilon^2)$. A computation shows that $\mathrm{HC}_2(\mathbb{Z}/p(\varepsilon),(\varepsilon))\cong(\mathbb{Z}/p\oplus\mathbb{Z}/2)\otimes\mathbb{Z}/p\cong\mathbb{Z}/p$ for p odd. Thus for $p\neq 2,3$, $\mathrm{HC}_2(\mathbb{Z}/p(\varepsilon),(\varepsilon))\otimes\mathbb{Z}[\frac{1}{3}]\neq K_3(\mathbb{Z}/p(\varepsilon),\ (\varepsilon))\otimes\mathbb{Z}[\frac{1}{3}]$, as HC_2 will only account for a single factor of $\mathbb{Z}/p$. On the other hand, Kassel has shown that the torsion in $K_3(\mathbb{Z}(\varepsilon),(\varepsilon))$ consists of 2-torsion and 3-torsion. Thus $K_3(\mathbb{Z}(\varepsilon),(\varepsilon))\otimes\mathbb{Z}[\frac{1}{3}]$ and $\mathrm{HC}_2(\mathbb{Z}(\varepsilon),(\varepsilon))\otimes\mathbb{Z}[\frac{1}{3}]\cong\mathbb{Z}[\frac{1}{3}]\oplus\hat{H}^0(\mathbb{Z}/2;\ (\varepsilon)\otimes(\varepsilon))\otimes\mathbb{Z}[\frac{1}{3}]\cong(\mathbb{Z}\oplus\mathbb{Z}/2)\otimes\mathbb{Z}[\frac{1}{3}]\cong\mathbb{Z}[\frac{1}{3}]\oplus\mathbb{Z}/2$ are both of the form $\mathbb{Z}[\frac{1}{3}]\oplus$ 2-torsion.

Finally, we give an application to the stable K-groups $K_*^n(R,A)$ of Hatcher–Igusa–Waldhausen (see, for example, [5] or [16]). $K_*^n(R,A)$ is defined as follows ($n\geq 1$): for a ring R and a bimodule A, one forms the simplicial ring $R(A,n)_\bullet$ which is the semi-direct product of R in degree 0 and A in degree $n-1$. Thus for $n=1$, $R(A,1)_\bullet=R\oplus A$. In any case, one can form the relative Waldhausen K-groups, $K_*(R(A,n)_\bullet,R)$ as in [16] or [17] and one has $K_*^n(R,A)=K_{*+n-1}(R(A,n)_\bullet,R)$. The techniques of the last two sections carry over to this simplicial context (by using cyclic and Hochschild 'hyper-homology', as in [2] or [7]). The result is

Corollary 4.7. *There are maps $\otimes_R^n A/(1-t)\xrightarrow{\phi_n}K_{nm}(R(A,m)_\bullet)=K_{nm-m+1}^m(R,A)\xrightarrow{c_n}\otimes_R^n A/(1-t)$ such that $c_n\circ\phi_n$ is multiplication by n.*

There are stabilization maps $K_*^m(R,A)\to K_*^{m+1}(R,A)$ [5, 6], and it is easy to show that the image of the elements constructed in the above corollary cannot be detected by the classes c_n after even a single stabilization. However, this does not imply that the contribution of these elements to the stable groups $K_*^s(R,A) = \lim_{n\to\infty} K_*^n(R,A)$ is zero; in fact, exactly how these elements contribute to the stable groups $K_*^s(R,A)$ is unclear.

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