Saturated fusion systems with parabolic families

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Basics on Fusion Systems

A fusion system $\mathcal{F}$ over a finite $p$-group $S$ is a category whose:

- objects are the subgroups of $S$,
- morphisms are such that $\text{Hom}_S(P, Q) \subseteq \text{Hom}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q)$,
  every $\mathcal{F}$-morphism factors as an $\mathcal{F}$-isomorphism followed by an inclusion.

Let $\mathcal{F}$ be a fusion system over a finite $p$-group $S$. A subgroup $P$ of $S$ is

- fully $\mathcal{F}$-normalized if $|N_S(P)| \geq |N_S(\varphi(P))|$, for all $\varphi \in \text{Hom}_\mathcal{F}(P, S)$;
- $\mathcal{F}$-centric if $C_S(\varphi(P)) = Z(\varphi(P))$ for all $\varphi \in \text{Hom}_\mathcal{F}(P, S)$;
- $\mathcal{F}$-essential if $Q$ is $\mathcal{F}$-centric and $S_p(\text{Out}_\mathcal{F}(P)) = S_p(\text{Aut}_\mathcal{F}(P)/\text{Aut}_P(P))$ is disconnected.

The fusion system $\mathcal{F}$ over a finite $p$-group $S$ is saturated if the following hold:

- Sylow Axiom
- Extension Axiom

The normalizer of $P$ in $\mathcal{F}$ is the fusion system $N_\mathcal{F}(P)$ on $N_S(P)$

$$\varphi \in \text{Hom}_{N_\mathcal{F}}(Q, R) \text{ if } \exists \hat{\varphi} \in \text{Hom}_\mathcal{F}(PQ, PR) \text{ with } \hat{\varphi}(P) = P \text{ and } \hat{\varphi}|_Q = \varphi.$$ 

The fusion system $\mathcal{F}$ is constrained if $\mathcal{F} = N_\mathcal{F}(Q)$ for some $\mathcal{F}$-centric subgroup $Q \neq 1$ of $S$.

The group $G$ has (finite) Sylow $p$-subgroup $S$ if $S$ is a finite $p$-subgroup of $G$

and if every finite $p$-subgroup of $G$ is conjugate to a subgroup of $S$.

$\mathcal{F}_S(G)$ is the fusion system on $S$ with $\text{Hom}_\mathcal{F}(P, Q) = \text{Hom}_G(P, Q)$, for $P, Q \leq S$. 

...
A **chamber system** over a set $I$ is a nonempty set $C$ whose elements are called **chambers** together with a family of equivalence relations $(\sim_i; i \in I)$ on $C$ indexed by $I$.

The $i$-**panels** are the equivalence classes with respect to $\sim_i$.

Two distinct chambers $c$ and $d$ are called **$i$-adjacent** if they are contained in the same $i$-panel:

$$c \sim_i d$$

A **gallery** of length $n$ connecting two chambers $c_0$ and $c_n$ is a sequence of chambers

$$c_0 \sim_{i_1} c_1 \sim_{i_2} \ldots \sim_{i_{n-1}} c_{n-1} \sim_{i_n} c_n$$

The chamber system $C$ is **connected** if any two chambers can be joined by a gallery.

The **rank** of the chamber system is the cardinality of the set $I$.

A **morphism** $\varphi : C \to D$ between two chamber systems over $I$ is a map on chambers that preserves $i$-adjacency: if $c, d \in C$ and $c \sim_i d$ then $\varphi(c) \sim_i \varphi(d)$ in $D$.

$\text{Aut}(C)$ is the group of all automorphisms of $C$ (automorphism has the obvious meaning).

If $G$ is a group of automorphisms of $C$ then **orbit chamber system** $C/G$ is a chamber system over $I$. 
Fusion Systems with Parabolic Families

A fusion system $\mathcal{F}$ over a finite $p$-group $S$ has a family $\{\mathcal{F}_i; i \in I\}$ of parabolic subsystems if:

1. (F0) $\forall i \in I$, $\mathcal{F}_i$ is saturated, constrained, of $\mathcal{F}_i$-essential rank one;
2. (F1) $B := N_{\mathcal{F}}(S)$ is a proper subsystem of $\mathcal{F}_i$ for all $i \in I$;
3. (F2) $\mathcal{F} = \langle \mathcal{F}_i; i \in I \rangle$ and no proper subset $\{\mathcal{F}_j; j \in J \subset I\}$ generates $\mathcal{F}$;
4. (F3) $\mathcal{F}_i \cap \mathcal{F}_j = B$ for any pair of distinct elements $\mathcal{F}_i$ and $\mathcal{F}_j$;
5. (F4) $\mathcal{F}_{ij} := \langle \mathcal{F}_i, \mathcal{F}_j \rangle$ is saturated constrained subsystem of $\mathcal{F}$ for all $i, j \in I$.

**Proposition (Onofrei, 2011)**

If $\mathcal{F}$ contains a family of parabolic subsystems then there are:

- $p'$-reduced $p$-constrained finite groups $B, G_i, G_{ij}$ with $B = F_S(B)$, $\mathcal{F}_i = F_S(G_i)$, $\mathcal{F}_{ij} = F_S(G_{ij})$, $\forall i, j \in I$;
- injective homomorphisms $\psi_i : B \to G_i$, $\psi_{ij} : G_i \to G_{ij}$ such that $\psi_{ij} \circ \psi_j = \psi_{ij} \circ \psi_i$, $\forall i, j \in I$.

In other words, $\mathcal{A} = \{(B, G_i, G_{ij}), (\psi_i, \psi_{ij}); i, j \in I\}$ is a diagram of groups.

The proof is based on:

- [BCGLO, 2005]: Every saturated constrained fusion system $\mathcal{F}$ over $S$ is the fusion system $F_S(G)$ of a finite group $G$ that is $p'$-reduced $O_{p'}(G) = 1$ and $p$-constrained $C_G(O_p(G)) \leq O_p(G)$, and if we set $U := O_p(\mathcal{F})$ then $1 \longrightarrow Z(U) \longrightarrow G \longrightarrow \text{Aut}_\mathcal{F}(U) \longrightarrow 1$.
- [Aschbacher, 2008]: If $G_1$ and $G_2$ are such that $\mathcal{F} = F_S(G_1) = F_S(G_2)$ then there is an isomorphism $\varphi : G_1 \to G_2$ with $\varphi|_S = \text{Id}_S$. 
Lemma (Onofrei, 2011)

If $G$ is a faithful completion of the diagram of groups $\mathcal{A}$ then:

(P1) $G := \langle G_i, i \in I \rangle \neq \langle G_j, j \in J \not\subseteq I \rangle$

(P2) $G_i \cap G_j = B$ for all $i \neq j$ in $I$;

(P3) $B \neq G_i$ for all $i \in I$;

(P4) $\cap_{g \in G} B^g = 1$.

Hence $(G; B, G_i, i \in I)$ is a parabolic system of rank $n = |I|$.

The chamber system $C = C(G; B, G_i, i \in I)$ is defined as follows:

- the chambers are cosets $gB$ for $g \in G$;
- two chambers $gB$ and $hB$ are $i$-adjacent if $gG_i = hG_i$ where $g, h \in G$.

$G$ acts chamber transitively, faithfully on $C$ by left multiplication.

Definition (Onofrei, 2011)

A fusion - chamber system pair $(\mathcal{F}, C)$ consists of:

- a fusion system $\mathcal{F}$ with a family of parabolic subsystems $\{\mathcal{F}_i; i \in I\}$;
- a chamber system $C = C(G; B, G_i, i \in I)$ with $G$ a faithful completion of $\mathcal{A}$. 
Main Theorem on Fusion - Chamber System Pairs

Theorem (Onofrei, 2011)

Let $(\mathcal{F}, C)$ be a fusion-chamber system pair. Assume the following hold.

(i) $C^P$ is connected for all $p$-subgroups $P$ of $G$.

(ii) If $P$ is $\mathcal{F}$-centric and if $R$ is a $p$-subgroup of $\text{Aut}_G(P)$, then $(C^P/C_G(P))^R$ is connected.

Then $\mathcal{F} = \mathcal{F}_S(G)$ is a saturated fusion system over $S$.

Sketch of the Argument

**Step 1:** $S$ is a Sylow $p$-subgroup of $G$.

- Since $C^P$ is connected, $C^P \neq \emptyset$ and $\exists g \in G$ such that $gB \in C^P$, thus $P \leq gBg^{-1}$ and since $gSg^{-1} \in \text{Syl}_p(gBg^{-1})$, $\exists h \in G$ such that $hPh^{-1} \leq S$.

**Step 2:** $\mathcal{F}$ is the fusion system given by conjugation in $G$, this means $\mathcal{F} = \mathcal{F}_S(G)$.

- Clearly $\mathcal{F} \subseteq \mathcal{F}_S(G)$.

- $\mathcal{F}_S(G) \subseteq \mathcal{F}$ follows from the fact that every morphism in $\mathcal{F}_S(G)$ is a composite of morphisms $\varphi_1, \ldots, \varphi_n$ with $\varphi_i \in \mathcal{F}_S(G_{j_i})$, $j_i \in I$. 
Main Theorem: Sketch of the Argument

**Step 3:** Every morphism in $\mathcal{F}$ is a composition of restrictions of morphisms between $\mathcal{F}$-centric subgroups.

- Recall $\mathcal{F}_i$ is saturated, constrained and of essential rank one.
- If $E_i$ is $\mathcal{F}_i$-essential then $E_i$ is $\mathcal{F}$-centric.
- **Alperin-Goldschmidt Theorem:** Each morphism $\varphi_i \in \mathcal{F}_i$ can be written as a composition of restrictions of $\mathcal{F}_i$-automorphisms of $S$ and of automorphisms of fully $\mathcal{F}_i$-normalized $\mathcal{F}_i$-essential subgroups of $S$.

Hence we may use:

- **[BCGLO, 2005]:** It suffices to verify the saturation axioms for the collection of $\mathcal{F}$-centric subgroups only.

**Step 4:** **The Sylow Axiom:** For all $\mathcal{F}$-centric $P$ that are fully $\mathcal{F}$-normalized, $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(P))$.

- **Proposition [Stancu, 2004]:** Assume that
  - $\text{Aut}_S(S)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(S)$;
  - The Extension Axiom holds for all $\mathcal{F}$-centric subgroups $P$.
  Then if $Q$ is $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized then $\text{Aut}_S(Q)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(Q)$.
Main Theorem: Sketch of the Argument

Step 5: *The Extension Axiom*: Let $P$ be $\mathcal{F}$-centric.

For any $\varphi \in \text{Hom}_\mathcal{F}(P, S)$ there is a morphism $\hat{\varphi} \in \text{Hom}_\mathcal{F}(N_\varphi, S)$ such that $\hat{\varphi}|_P = \varphi$.

$$PC_S(P) \leq N_\varphi \leq N_S(P)$$

For $P \leq S$ we introduce a new chamber system $\text{Rep}(P, C)$ as follows:

- The chambers are the elements of $\text{Rep}(P, B) := \text{Inn}(B) \setminus \text{Inj}(P, B)$;
  
  $[\alpha] \in \text{Rep}(P, B)$ denotes the class of $\alpha \in \text{Inj}(P, B)$

- The $i$-panels are represented by the elements of
  
  $\text{Rep}(P, B, G_i) := \{ [\gamma] \in \text{Rep}(P, G_i) : \gamma \in \text{Inj}(P, G_i) \text{ with } \gamma(P) \leq B \}$.

- Let $\tau^K_H$ denote the inclusion map of the group $H$ into the group $K$.

- Two chambers $[\alpha]$ and $[\beta]$ are $i$-adjacent if $\left[ \tau^G_B \circ \alpha \right] = \left[ \tau^G_B \circ \beta \right]$ in $\text{Rep}(P, G_i)$.

- $N_G(P)$ acts on $\text{Rep}(P, C)$ via $g \cdot [\alpha] = [\alpha \circ g^{-1}]$ for $g \in N_G(P)$. 
Main Theorem: Sketch of the Argument

There is an $N_G(P)$-equivariant chamber system isomorphism

$$f_P : C^P \longrightarrow \text{Rep}(P, C)_0$$

given by

$$f_P(gB) = [c_{g^{-1}}]$$

that induces an isomorphism on the orbit chamber systems

$$C^P / C_G(P) \longrightarrow \text{Rep}(P, C)_0$$

where $\text{Rep}(P, C)_0$ is the connected component of $\text{Rep}(P, C)$ that contains $[\tau^B_P]$, affords the action of $\text{Aut}_G(P) = N_G(P)/C_G(P)$.

For $\varphi \in \text{Hom}_F(P, S)$ let $K = N_\varphi / Z(P) = \text{Aut}_{N_\varphi}(P)$.

The map

$$\Gamma : \text{Rep}(N_\varphi, C)_0 \longrightarrow \text{Rep}(P, C)_0^K \simeq (C^P / C_G(P))^K$$

is onto.

- that is induced by the restriction $N_\varphi \rightarrow P$
- between the connected component of $\text{Rep}(N_\varphi, C)$ that contains $[\tau^B_{N_\varphi}]$ and the fixed point set of $K$ acting on $\text{Rep}(P, C)_0$
Main Theorem: Sketch of the Argument

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is onto.

- that is induced by the restriction $N_\varphi \rightarrow P$

- between the connected component of $\text{Rep}(N_\varphi, C)$ that contains $[\tau^B_{N_\varphi}]$ and the fixed point set of $K$ acting on $\text{Rep}(P, C)_0$
Assume $\mathcal{F}$ contains a family of parabolic systems. Set $U_i = O_p(\mathcal{F}_i)$ and $U_{ij} = O_p(\mathcal{F}_{ij})$.

We say $\mathcal{F}$ contains a classical family of parabolic systems with diagram $\mathcal{M}$ if:

- For each $i \in I$, $\text{Out}_{\mathcal{F}_i}(U_i)$ is a rank one finite group of Lie type in characteristic $p$.
- For each pair $i, j \in I$, $\text{Out}_{\mathcal{F}_{ij}}(U_{ij})$ is either a rank two finite group of Lie type in characteristic $p$ or it is a (central) product of two rank one finite groups of Lie type in characteristic $p$.

The diagram $\mathcal{M}$ is a graph whose vertices are labeled by the elements of $I$,

- $\text{Out}_{\mathcal{F}_{ij}}(U_{ij})$ is a product of two rank one Lie groups then the nodes $i$ and $j$ are not connected,
- $\text{Out}_{\mathcal{F}_{ij}}(U_{ij})$ is a rank two Lie group the nodes $i$ and $j$ are connected by a bond of strength $m_{ij} - 2$, where $m_{ij}$ denotes the integer that defines the Weyl group.

**Proposition**

Let $(\mathcal{F}, C)$ be a fusion-chamber system pair with $|I| \geq 3$. Assume that:

(i). $\mathcal{F}$ contains a classical family of parabolic systems with diagram $\mathcal{M}$;

(ii). $\mathcal{M}$ is a spherical diagram.

Then $\mathcal{F}$ is the fusion system of a finite simple group of Lie type in characteristic $p$ extended by diagonal and field automorphisms.
Thank You

The End