43. The points of intersection of the cardioid \( r = 1 + \sin \theta \) and the spiral loop \( r = 2\theta, -\pi/2 \leq \theta \leq \pi/2 \), can’t be found exactly. Use a graphing device to find the approximate values of \( \theta \) at which they intersect. Then use these values to estimate the area that lies inside both curves.

44. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid \( r = 8 + 8 \sin \theta \), where \( r \) is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.

45–48 Find the exact length of the polar curve.

45. \( r = 3 \sin \theta, \ 0 \leq \theta \leq \pi/3 \)  
46. \( r = e^{2\theta}, \ 0 \leq \theta \leq 2\pi \)  
47. \( r = \theta^2, \ 0 \leq \theta \leq 2\pi \)  
48. \( r = \theta, \ 0 \leq \theta \leq 2\pi \)

49–52 Use a calculator to find the length of the curve correct to four decimal places.

49. \( r = 3 \sin 2\theta \)  
50. \( r = 4 \sin 3\theta \)  
51. \( r = \sin(\theta/2) \)  
52. \( r = 1 + \cos(\theta/3) \)

53–54 Graph the curve and find its length.

53. \( r = \cos^4(\theta/4) \)  
54. \( r = \cos^2(\theta/2) \)

55. (a) Use Formula 10.2.7 to show that the area of the surface generated by rotating the polar curve 
\[
    r = f(\theta), \quad a \leq \theta \leq b
\]
where \( f' \) is continuous and \( 0 \leq a < b \leq \pi \) about the polar axis is

\[
    S = \int_a^b 2\pi r \sin \theta \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta
\]

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate \( r^2 = \cos 2\theta \) about the polar axis.

56. (a) Find a formula for the area of the surface generated by rotating the polar curve \( r = f(\theta), a \leq \theta \leq b \) (where \( f' \) is continuous and \( 0 \leq a < b \leq \pi \)), about the line \( \theta = \pi/2 \).

(b) Find the surface area generated by rotating the lemniscate \( r^2 = \cos 2\theta \) about the line \( \theta = \pi/2 \).

#### 10.5 CONIC SECTIONS

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in Figure 1.
A parabola is the set of points in a plane that are equidistant from a fixed point $F$ (called the focus) and a fixed line (called the directrix). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 18 on page 268 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin $O$ and its directrix parallel to the $x$-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from $P$ to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from $P$ to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = y + p$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If we write $a = 1/(4p)$, then the standard equation of a parabola (1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the $y$-axis because (1) is unchanged when $x$ is replaced by $-x$. 

<table>
<thead>
<tr>
<th>Part</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$x^2 = 4py$, $p &gt; 0$</td>
</tr>
<tr>
<td>b</td>
<td>$x^2 = 4py$, $p &lt; 0$</td>
</tr>
<tr>
<td>c</td>
<td>$y^2 = 4px$, $p &gt; 0$</td>
</tr>
<tr>
<td>d</td>
<td>$y^2 = 4px$, $p &lt; 0$</td>
</tr>
</tbody>
</table>
If we interchange \( x \) and \( y \) in (1), we obtain

\[
y^2 = 4px
\]

which is an equation of the parabola with focus \((p, 0)\) and directrix \(x = -p\). (Interchanging \( x \) and \( y \) amounts to reflecting about the diagonal line \( y = x \).) The parabola opens to the right if \( p > 0 \) and to the left if \( p < 0 \) [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the \( x \)-axis, which is the axis of the parabola.

**EXAMPLE 1** Find the focus and directrix of the parabola \( y^2 + 10x = 0 \) and sketch the graph.

**SOLUTION** If we write the equation as \( y^2 = -10x \) and compare it with Equation 2, we see that \( 4p = -10 \), so \( p = -\frac{5}{2} \). Thus the focus is \((p, 0) = \left(-\frac{5}{2}, 0\right)\) and the directrix is \(x = \frac{5}{2}\).

The sketch is shown in Figure 5.

**ELLIPSES**

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points \( F_1 \) and \( F_2 \) is a constant (see Figure 6). These two fixed points are called the **foci** (plural of focus). One of Kepler’s laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

In order to obtain the simplest equation for an ellipse, we place the foci on the \( x \)-axis at the points \((-c, 0)\) and \((c, 0)\) as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be \(2a > 0\). Then \(P(x, y)\) is a point on the ellipse when

\[
|PF_1| + |PF_2| = 2a
\]

that is,

\[
\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a
\]

or

\[
\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}
\]

Squaring both sides, we have

\[
x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2
\]

which simplifies to

\[
a\sqrt{(x + c)^2 + y^2} = a^2 + cx
\]

We square again:

\[
a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2
\]

which becomes

\[
(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)
\]
From triangle $F_1F_2P$ in Figure 7 we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by $a^2b^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The $x$-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the vertices of the ellipse and the line segment joining the vertices is called the major axis. To find the $y$-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation 3 is unchanged if $x$ is replaced by $-x$ or $y$ is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

4. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

If the foci of an ellipse are located on the $y$-axis at $(0, \pm c)$, then we can find its equation by interchanging $x$ and $y$ in (4). (See Figure 9.)

5. The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

**Example 2** Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

**Solution** Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The $x$-intercepts are $\pm 4$ and the $y$-intercepts are $\pm 3$. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $\left( \pm \sqrt{7}, 0 \right)$. The graph is sketched in Figure 10.

**Example 3** Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

**Solution** Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$. 
Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 63). This principle is used in *lithotripsy*, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

**HYPERBOLAS**

A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points $F_1$ and $F_2$ (the foci) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle’s Law, Ohm’s Law, supply and demand curves). A particularly significant application of hyperbolas is found in the navigation systems developed in World Wars I and II (see Exercise 51).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 52 to show that when the foci are on the $x$-axis at $(c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the $x$-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the vertices of the hyperbola. But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no $y$-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geq a$. Therefore we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its branches.

When we draw a hyperbola it is useful to first draw its asymptotes, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. [See Exercise 69 in Section 4.5, where these lines are shown to be slant asymptotes.]
If the foci of a hyperbola are on the y-axis, then by reversing the roles of \( x \) and \( y \) we obtain the following information, which is illustrated in Figure 13.

The hyperbola

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

has foci \((0, \pm c)\), where \( c^2 = a^2 + b^2 \), vertices \((0, \pm a)\), and asymptotes \( y = \pm (a/b)x \).

**EXAMPLE 4** Find the foci and asymptotes of the hyperbola \( 9x^2 - 16y^2 = 144 \) and sketch its graph.

**SOLUTION** If we divide both sides of the equation by 144, it becomes

\[
\frac{x^2}{16} - \frac{y^2}{9} = 1
\]

which is of the form given in (7) with \( a = 4 \) and \( b = 3 \). Since \( c^2 = 16 + 9 = 25 \), the foci are \((\pm 5, 0)\). The asymptotes are the lines \( y = \frac{3}{4}x \) and \( y = -\frac{3}{4}x \). The graph is shown in Figure 14.

**EXAMPLE 5** Find the foci and equation of the hyperbola with vertices \((0, \pm 1)\) and asymptote \( y = 2x \).

**SOLUTION** From (8) and the given information, we see that \( a = 1 \) and \( a/b = 2 \). Thus \( b = a/2 = \frac{1}{2} \) and \( c^2 = a^2 + b^2 = \frac{5}{4} \). The foci are \((0, \pm \sqrt{5}/2)\) and the equation of the hyperbola is

\[
y^2 - 4x^2 = 1
\]

**SHIFTED CONICS**

As discussed in Appendix C, we shift conics by taking the standard equations (1), (2), (4), (5), (7), and (8) and replacing \( x \) and \( y \) by \( x - h \) and \( y - k \).

**EXAMPLE 6** Find an equation of the ellipse with foci \((2, -2)\), \((4, -2)\) and vertices \((1, -2)\), \((5, -2)\).

**SOLUTION** The major axis is the line segment that joins the vertices \((1, -2)\), \((5, -2)\) and has length 4, so \( a = 2 \). The distance between the foci is 2, so \( c = 1 \). Thus \( b^2 = a^2 - c^2 = 3 \). Since the center of the ellipse is \((3, -2)\), we replace \( x \) and \( y \) in (4) by \( x - 3 \) and \( y + 2 \) to obtain

\[
\frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{3} = 1
\]

as the equation of the ellipse.
to \( \mathbf{r} - \mathbf{r}_0 \) and so we have

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0
\]

which can be rewritten as

\[
\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0
\]

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write \( \mathbf{n} = \langle a, b, c \rangle \), \( \mathbf{r} = \langle x, y, z \rangle \), and \( \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \). Then the vector equation (5) becomes

\[
\langle a, b, c \rangle \cdot (x - x_0, y - y_0, z - z_0) = 0
\]

or

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

Equation 7 is the **scalar equation of the plane through** \( P(x_0, y_0, z_0) \) **with normal vector** \( \mathbf{n} = \langle a, b, c \rangle \).

**EXAMPLE 4** Find an equation of the plane through the point \( (2, 4, -1) \) with normal vector \( \mathbf{n} = \langle 2, 3, 4 \rangle \). Find the intercepts and sketch the plane.

**SOLUTION** Putting \( a = 2 \), \( b = 3 \), \( c = 4 \), \( x_0 = 2 \), \( y_0 = 4 \), and \( z_0 = -1 \) in Equation 7, we see that an equation of the plane is

\[
2(x - 2) + 3(y - 4) + 4(z + 1) = 0
\]

or

\[
2x + 3y + 4z = 12
\]

To find the \( x \)-intercept we set \( y = z = 0 \) in this equation and obtain \( x = 6 \). Similarly, the \( y \)-intercept is 4 and the \( z \)-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

![Figure 7](image)

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

\[
ax + by + cz + d = 0
\]

where \( d = -(ax_0 + by_0 + cz_0) \). Equation 8 is called a **linear equation** in \( x, y, \) and \( z \).

Conversely, it can be shown that if \( a, b, \) and \( c \) are not all 0, then the linear equation (8) represents a plane with normal vector \( \langle a, b, c \rangle \). (See Exercise 77.)

**EXAMPLE 5** Find an equation of the plane that passes through the points \( P(1, 3, 2) \), \( Q(3, -1, 6) \), and \( R(5, 2, 0) \).

**SOLUTION** The vectors \( \mathbf{a} \) and \( \mathbf{b} \) corresponding to \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are

\[
\mathbf{a} = \langle 2, -4, 4 \rangle \quad \mathbf{b} = \langle 4, -1, -2 \rangle
\]
CYLINDERS

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

**EXAMPLE 1** Sketch the graph of the surface \( z = x^2 \).

**SOLUTION** Notice that the equation of the graph, \( z = x^2 \), doesn’t involve \( y \). This means that any vertical plane with equation \( y = k \) (parallel to the \( xz \)-plane) intersects the graph in a curve with equation \( z = x^2 \). So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola \( z = x^2 \) in the \( xz \)-plane and moving it in the direction of the \( y \)-axis. The graph is a surface, called a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the \( y \)-axis.

We noticed that the variable \( y \) is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables \( x, y, \) or \( z \) is missing from the equation of a surface, then the surface is a cylinder.

**EXAMPLE 2** Identify and sketch the surfaces.

(a) \( x^2 + y^2 = 1 \)  
(b) \( y^2 + z^2 = 1 \)

**SOLUTION**

(a) Since \( z \) is missing and the equations \( x^2 + y^2 = 1, \ z = k \) represent a circle with radius 1 in the plane \( z = k \), the surface \( x^2 + y^2 = 1 \) is a circular cylinder whose axis is the \( z \)-axis. (See Figure 2.) Here the rulings are vertical lines.

(b) In this case \( x \) is missing and the surface is a circular cylinder whose axis is the \( x \)-axis. (See Figure 3.) It is obtained by taking the circle \( y^2 + z^2 = 1, \ x = 0 \) in the \( yz \)-plane and moving it parallel to the \( x \)-axis.

**NOTE** When you are dealing with surfaces, it is important to recognize that an equation like \( x^2 + y^2 = 1 \) represents a cylinder and not a circle. The trace of the cylinder \( x^2 + y^2 = 1 \) in the \( xy \)-plane is the circle with equations \( x^2 + y^2 = 1, \ z = 0 \).

QUADRIC SURFACES

A quadric surface is the graph of a second-degree equation in three variables \( x, y, \) and \( z \). The most general such equation is

\[
Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0
\]
but the traces in the $xz$- and $yz$-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \quad \text{and} \quad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a hyperboloid of one sheet and is sketched in Figure 9.

The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of $k$, and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the $z$-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

**Table 1** Graphs of quadric surfaces

<table>
<thead>
<tr>
<th>Surface</th>
<th>Equation</th>
<th>Surface</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipsoid</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>All traces are ellipses.</td>
<td>Cone</td>
<td>$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$</td>
</tr>
<tr>
<td></td>
<td>If $a = b = c$, the ellipsoid is a sphere.</td>
<td></td>
<td>Horiztonal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.</td>
</tr>
<tr>
<td>Elliptic Paraboloid</td>
<td>$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.</td>
<td>Hyperboloid of One Sheet</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Horiztonal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.</td>
</tr>
<tr>
<td>Hyperbolic Paraboloid</td>
<td>$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c &lt; 0$ is illustrated.</td>
<td>Hyperboloid of Two Sheets</td>
<td>$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Horiztonal traces in $z = k$ are ellipses if $k &gt; c$ or $k &lt; -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.</td>
</tr>
</tbody>
</table>