Math 366 - Winter 2009

Exam 1B

30 January 2009

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<th>Problem</th>
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Answer the following questions. The answers must be clear, intelligible, and you must show your work. Provide explanation for all your steps. Your grade will be determined by adherence to these criteria. Use of books, notes and calculators is strictly forbidden. The number of points each problem is worth is given in the above table.
## Rules of Inference

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| 1. | Modus Ponens | $p \rightarrow q$
|   |   | $p$
|   |   | $\therefore q$
| 2. | Modus Tollens | $p \rightarrow q$
|   |   | $\sim q$
|   |   | $\therefore \sim p$
| 3. | Generalization | $p$
|   |   | $\therefore p \lor q$
| 4. | Specialization | $p \land q$
|   |   | $\therefore p$
| 5. | Conjunction | $p$
|   |   | $q$
|   |   | $\therefore p \land q$
| 6. | Elimination | $p \lor q$
|   |   | $\sim q$
|   |   | $\therefore p$
| 7. | Transitivity | $p \rightarrow q$
|   |   | $q \rightarrow r$
|   |   | $\therefore p \rightarrow r$
| 8. | Division by cases | $p \lor q$
|   |   | $p \rightarrow r$
|   |   | $q \rightarrow r$
|   |   | $\therefore r$
| 9. | Contraction Rule | $\sim p \rightarrow c$
|   |   | $\therefore p$
| 10. | Disjunctive syllogism | $p \lor q$
|   |   | $\sim p$
|   |   | $\therefore q$
| 11. | Resolution | $p \lor q$
|   |   | $\sim p \lor r$
|   |   | $\therefore q \lor r$
1. Write the following statement in the form “if \( p \), then \( q \)” in English. Clearly state which statement you labelled as \( p \) and which statement you labelled as \( q \). Then formulate the contrapositive of this implication.

   \( \text{It is necessary to have a valid password to log on to the server.} \)

Solution:
Let \( p \) be the statement: \( \text{You log on to the server.} \)
Let \( q \) be the statement: \( \text{You have a valid password.} \)

The given statement is \( p \rightarrow q \).

\( \text{If you log on to the server then you have a valid password.} \)

The contrapositive is \( \sim q \rightarrow \sim p \) and reads:

\( \text{If you don’t have a valid password then you cannot log on to the server.} \)
2. Determine whether the following argument is valid.

*If* $x$ *is rational, then* $x^2$ *is rational.*

$x^2$ *is rational.*

*Therefore* $x$ *is rational.*

The domain for the variable $x$ is $\mathbb{R}$, the set of real numbers.

*Solution:*

Let $P(x)$ denote the statement: *$x$ is rational.*

Let $Q(x)$ denote the statement: *$x^2$ is rational.*

With these notation, the argument given above reads:

$$
\forall x (P(x) \rightarrow Q(x))
$$

$$
\forall x Q(x)
$$

$$
\therefore \forall x P(x)
$$

From the hypothesis $Q(x)$ and the statement $\forall x (P(x) \rightarrow Q(x))$ we cannot conclude $P(x)$, because we are not using a valid rule of inference. Instead, this is an example of converse error, also known as the fallacy of affirming the conclusion. A counterexample is supplied by $x = \sqrt{2}$ for which $Q(\sqrt{2})$ is true and $P(\sqrt{2})$ is false.
3. Use symbols to write the logical form of the following argument. Indicate how to interpret each notation you defined, or any symbol used as a name. Use a truth table to determine whether the argument form is valid. Use the table given below for your truth table.

If I am dreaming and hallucinating then I see elephants running down the road.
Either I am dreaming or I am not hallucinating.
If I am not hallucinating then I am dreaming.
Therefore, I see elephants running down the road.

Solution:
Let \( p \) denote the statement: I am hallucinating.
Let \( q \) denote the statement: I am dreaming.
Let \( r \) denote the statement: I see elephants running down the road.
The argument given above can be written as:

\[
(p \land q) \rightarrow r \\
q \lor \sim p \\
\sim p \rightarrow q \\
\therefore r
\]

and it corresponds to the statement

\[
((p \land q) \rightarrow r) \land (q \lor \sim p) \land (\sim p \rightarrow q) \rightarrow r
\]

Let \( S \) be the statement \((p \land q) \rightarrow r) \land (q \lor \sim p) \land (\sim p \rightarrow q)\). Construct a truth table to determine if the statement \( S \) is a tautology or not.

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<th>( q )</th>
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<th>( p \land q )</th>
<th>( (p \land q) \rightarrow r )</th>
<th>( q \lor \sim p )</th>
<th>( \sim p \rightarrow q )</th>
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<th>( S \rightarrow r )</th>
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Since the statement \( P \) is not a tautology, the argument is invalid.
4. Let $P(x, y)$ be the statement “$y$ is greater or equal to $x$” where the domain for both $x$ and $y$ is the set of nonnegative integers. Determine the truth values of the following statements:
   a). $P(2, 4)$,
   b). $\forall x \exists y P(x, y)$.
For full credit you need to justify your answer for part b).

**Solution:**

a). The statement $P(2, 4)$ is the statement “$4 \geq 2$”, which is true.

b). The statement says that for every integer $x$ there exists an integer $y$ with the property $y \geq x$. The statement is true, just take $y = x + 1$. 
5. Use predicates, quantifiers, logical connectives, and mathematical operators to express the following statement:

*There is an integer that is the sum of the squares of three integers.*

The domain for all the variables you use should be $\mathbb{Z}$, the set of integers.

Solution:

$$\exists x \exists a \exists b \exists c \text{ such that } x = a^2 + b^2 + c^2$$
6. Express the negation of the following statement:

$$\exists x \exists y (x \neq 0 \lor ((xy \neq 4) \land (x + y = 0)))$$

For full credit, indicate all the steps you took in order to obtain the final form for the negated statement.

**Solution:**

$$\sim (\exists x \exists y (x \neq 0 \lor ((xy \neq 4) \land (x + y = 0))))$$

$$\equiv \forall x \forall y \sim (x \neq 0 \lor ((xy \neq 4) \land (x + y = 0)))$$

$$\equiv \forall x \forall y ((x = 0) \land \sim ((xy \neq 4) \land (x + y = 0)))$$

$$\equiv \forall x \forall y ((x = 0) \land (xy = 4) \lor (x + y \neq 0))$$