Math 366 - Winter 2009

Exam 2A

27 February 2009

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Answer the following questions. The answers must be clear, intelligible, and you must show your work. Provide explanation for all your steps. Your grade will be determined by adherence to these criteria. Use of books, notes and calculators is strictly forbidden. The number of points each problem is worth is given in the above table.
1. Prove that there exist two real numbers which are not equal to each other and whose product is equal to their sum.

Solution:
Let $a$ and $b$ be two distinct real numbers. It is given that $ab = a + b$. Thus $b = \frac{a}{a-1}$. It is easy to see now that $a = 3$ and $b = 3/2$ satisfy the required condition.

2. Show that if $a, b$ and $c$ are integers such that $ac | bc$, then $a | b$.

Solution:
Let $a, b, c$ be integers. Since $ac | bc$ we have $bc = ack$ for some integer $k$. Observe that $ac \neq 0$ and this implies that $c \neq 0$. After cancelling $c$ in the previous equality we get $b = ak$ which proves that $a | b$. 
3. Prove or disprove that if $a$ and $b$ are rational numbers, then $a^b$ is also rational.

Solution:
The statement is false. There exist two numbers $a = 2$ and $b = \frac{1}{2}$, both rational, with $a^b = \sqrt{2}$ irrational.

4. Prove or disprove $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

Solution:
True; $\lfloor x \rfloor = n$ is an integer and $\lceil n \rceil = n$. 
5. For any integer \( n \geq 1 \), \( n^2 + 1 \) has the form \( 4k + 1 \) or \( 4k + 2 \) for some integer \( k \).

Solution 1:
Suppose \( n \) is an integer with \( n \geq 1 \).

Case 1: First assume that \( n \) is even. In this case \( n = 2q \) and \( n^2 + 1 = 4q^2 + 1 = 4k + 1 \) with \( k = q^2 \), an integer.

Case 2: Now consider the case \( n \) is odd. In this case \( n = 2q + 1 \) and \( n^2 + 1 = (2q + 1)^2 + 1 = 4q^2 + 4q + 1 + 1 = 4(q^2 + q) + 2 = 4k + 2 \) with \( k = q^2 + q \) an integer.

Solution 2:
Suppose \( n \) is an integer with \( n \geq 1 \). By the quotient-remainder theorem with \( d = 4 \) we know that \( n = 4q \), or \( n = 4q + 1 \), or \( n = 4q + 2 \), or \( n = 4q + 3 \) for some integer \( q \).

Case 1: In this case \( n = 4q \) and \( n^2 + 1 = 16q^2 + 1 = 4k + 1 \) with \( k = 4q^2 \), an integer.

Case 2: In this case \( n = 4q + 1 \) and \( n^2 + 1 = (4q + 1)^2 + 1 = 16q^2 + 8q + 1 + 1 = 4(4q^2 + 2q) + 2 = 4k + 2 \) with \( k = 4q^2 + 2q \) an integer.

Case 3: In this case \( n = 4q + 2 \) and \( n^2 + 1 = (4q + 2)^2 + 1 = 16q^2 + 16q + 4 + 1 = 4(4q^2 + 4q + 1) + 1 = 4k + 1 \) with \( k = 4q^2 + 4q + 1 \) an integer.

Case 4: In this case \( n = 4q + 3 \) and \( n^2 + 1 = (4q + 3)^2 + 1 = 16q^2 + 24q + 9 + 1 = 4(4q^2 + 6q + 2) + 2 = 4k + 2 \) with \( k = 4q^2 + 6q + 2 \) an integer.

6. Use a proof by contraposition to show that if \( n \) is an integer and \( n^3 + 5 \) is odd, then \( n \) is even. For full credit you must use the definition of even/odd numbers in your argument. You also have to clearly state the statement to be proved and its contrapositive.

Solution:
The statement to be proved: \( \forall n \in \mathbb{Z} \), if \( n^3 + 5 \) is odd then \( n \) is even.

The contrapositive of the above statement: \( \forall n \in \mathbb{Z} \), if \( n \) is odd then \( n^3 + 5 \) is even.

Let \( n \) be an odd integer. Thus \( n = 2k + 1 \) for some integer \( k \).

Then \( n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 12k^2 + 6k + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3) = 2l \) with \( l = 4k^3 + 6k^2 + 3k + 3 \) an integer. This proves that \( n^3 + 5 \) is even.
7. For the following list of integers, provide a simple formula that generates the terms of an integer sequence that begins with the given list 2, −6, 18, −54, 162, −486, . . .. Assume that \( a_0 = 2 \). Note that you will not receive full credit for a recursive formula or a formula which has more than one algebraic expression.

Solution:
The \( n \)th term of this sequence is \( a_n = 2 \cdot (-1)^n 3^n \).

8. Rewrite the following sum using summation notation:

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}
\]

Solution:

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = \sum_{k=1}^{n} \frac{k}{(k+1)!}
\]
9. Use Mathematical Induction to prove that
\[
1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}
\]
whenever \(n\) is a positive integer. For full credit you have to clearly state all the steps required in a proof by mathematical induction.

Solution:
Let \(P(n)\) be the property: \(1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}\).

Basis Step: Show that the property is true for \(n = 1\).
The property is true for \(n = 1\) because for \(n = 1\) the left hand side is 1 and the right-hand side is \(\frac{1^2(1+1)^2}{4} = 1\) also.

Inductive Step: Show that for all integers \(k \geq 1\), if the property is true for \(n = k\) then it is true for \(n = k + 1\).
Suppose \(1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}\) for some integer \(k \geq 1\).
We must show that \(1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = \frac{(k+1)^2(k+2)^2}{4}\).
But the left-hand side of this equation is:
\[
1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = \frac{k^2(k+1)^2}{4} + (k + 1)^3 =
\]
\[
(k + 1)^2 \left[\frac{k^2}{4} + (k + 1)\right] = (k + 1)^2 \frac{k^2 + 4k + 4}{4} = \frac{(k + 1)^2(k + 2)^2}{4}
\]
which equals the right-hand side of the equation, as was to be shown.
Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.