1. Write each of these statements in the form “if \( p \), then \( q \)” in English.
   (i). It snows whenever the wind blows from the northeast.
   (ii). The apple trees will bloom if it stays warm for a week.
   (iii). It is necessary to walk 8 miles to get to the top of Long’s Peak.
   (iv). A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

2. Determine whether each of these implications is true or false.
   (i). If \( 1 + 1 = 2 \), then \( 2 + 2 = 5 \).
   (ii). If \( 1 + 1 = 3 \), then \( 2 + 2 = 5 \).
   (iii). If \( 2 + 2 = 4 \), then \( 1 + 2 = 3 \).

3. You are on an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if:
   (i). A says “At least one of us is a knave” and B says nothing.
   (ii). A says “I am a knave or B is a knight” and B says nothing.

4. Use truth tables to determine which of the following argument forms are valid and which are not.
   
   \[
   \begin{array}{ccc}
   \text{i).} & \sim p & \text{ii).} & p \lor q \\
   p \lor q & p \rightarrow r & q \rightarrow r \\
   \therefore q & \therefore r
   \end{array}
   \]

5. Express the negations of each of these statements:
   (i). \( \forall x \exists y(((x \geq 0) \land (y < 0)) \rightarrow (x - y > 0)) \).
   (ii). \( \exists x \exists y(((x \leq 0) \land (y \leq 0)) \lor (x - y > 0)) \).
   (iii). \( \forall x((\exists y(x = 2y)) \lor (\exists z(x = 2z + 1))) \).

6. Rewrite the following statements in formal language. Then write a formal negation and an informal negation for each statement. Clearly state the domain for each variable.
   (i). If a real number is less or equal to 3, then the square of the number is less than or equal to 9.
   (ii). Being divisible by 8 is a sufficient condition for being divisible by 4.
   (iii). Being divisible by 8 is not a necessary condition for being divisible by 4.

7. Determine the truth value of each of these statements if the domain for all variables consists of all integers. Justify your answers.
   (i). \( \forall n \exists m(n^2 < m) \).
   (ii). \( \exists n \forall m(n < m^2) \).
   (iii). \( \exists n \exists m(n^2 + m^2 = 5) \).
   (iv). \( \exists n \exists m(n^2 + m^2 = 6) \).
   (i). Prove that there exists an integer $n$ such that $2n^2 - 5n + 2$ is prime.
   (ii). For every positive integer $n$ there is an integer divisible by more than $n$ primes.

   (i). Prove that the sum of two odd numbers is even.
   (ii). For all integers $a, b$ and $c$, if $a|b$ and $a|c$ then $a|(b + c)$.
   (iii). The product of two rational numbers is a rational number.
   (iv). Prove that if $n$ is an integer and $1 - n$ is even then $n^2 + 1$ is even.

10. Proving or Disproving an Universal Statement or an Existential Statement.
   (i). Prove or disprove: \( \lceil xy \rceil = \lceil x \rceil \lceil y \rceil \) for all real numbers $x$ and $y$.
   (ii). Prove or disprove that $2^n + 1$ is prime for all nonnegative integers.
   (iii). Prove or disprove that $2n^2 - 5n + 2$ is composite whenever $n$ is a positive integer greater than 1.

11. Proof by Cases.
   (i). Prove that for any integer $n$, $n^2 + n$ is even.
   (ii). Use the quotient-remainder theorem with $d = 3$ to prove that the product of any three consecutive integers is divisible by 3.
   (iii). For all integers $n$, $n^2$ has the form $5k$, $5k + 1$ or $5k + 4$ for some integer $k$.

12. Proofs by Contradiction and Contraposition.
   (i). Prove the following statement in two ways: (a) by contradiction and (b) by contraposition. For any integer $n$, if $n^2$ is not divisible by 3 then $n$ is not divisible by 3.
   (ii). Prove the following statement in two ways: (a) by contradiction and (b) by contraposition. Prove that if $n$ is an integer and $n^2$ is even, then $n$ is even.
   (iii). Prove that the sum of an irrational number and a rational number is irrational using a proof by contradiction.

   (i). Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers.
   (ii). Use mathematical induction to prove to show that $1+2+2^2+\ldots+2^n = 2^{n+1} - 1$, for all nonnegative integers $n$.
   (iii). Prove that $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 = \frac{(-1)^{n-1}n(n+1)}{2}$ whenever $n$ is a positive integer.
   (iv). Show that $2^n > n^2$ whenever $n$ is an integer greater than 4.

14. Strong Mathematical Induction
   (i). Suppose $a_0, a_1, a_2, \ldots$ is a sequence defined as follows: $a_0 = 1, a_1 = 1, a_2 = 3, a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for all integers $k \geq 3$. Use a proof by strong mathematical induction to show that $a_n \leq 3^n$ for all integers $n \geq 0$.
   (ii). Suppose $a_0, a_1, a_2, \ldots$ is a sequence defined as follows: $a_0 = 12, a_1 = 29, a_k = 5a_{k-1} - 6a_{k-2}$ for all integers $k \geq 2$. Use a proof by strong mathematical induction to show that $c_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$. 
15. Let $A = \{1, 3, 4\}$ and $B = \{2, 3, 5\}$. Find $A \cup B$, $A \cap B$, $A - B$, $B - A$ and $A \times B$. Write your answers in "bracket" notation.

16. Determine whether these statements are true or false:

   (1) $0 \in \emptyset$
   (2) $\{0\} \in \{0\}$
   (3) $\emptyset \in \{\emptyset\}$
   (4) $\{0\} \subset \{0\}$
   (5) $x \in \{x\}$
   (6) $\{x\} \in \{x\}$
   (7) $\emptyset \subseteq \{x\}$
   (8) $\{0\} \subseteq \emptyset$
   (9) $\emptyset \subseteq \{\emptyset\}$
   (10) $\{0\} \in \{\emptyset\}$
   (11) $\{0\} \in \{\{\emptyset\}\}$
   (12) $\{x\} \subseteq \{x\}$
   (13) $\{x\} \in \{\{x\}\}$
   (14) $\emptyset \in \{x\}$

17. Let $A, B$ and $C$ be sets. Show that:

   (i). $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   (ii). $(B - A) \cup (C - A) = (B \cup C) - A$.
   (iii). If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.
   (iv). If $A \subseteq C$ then $A \cup (B \cap C) \subseteq C$.

18. Determine whether each of the following functions is one-to-one or onto. Provide a proof of your answer.

   (i). $f : \{0, 1, 2\} \to \{a, b, c, d\}$ where $f(0) = d$, $f(1) = b$ and $f(2) = c$.
   (ii). The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = -2x + 1$.
   (iii). The function $h : \mathbb{R} \to \mathbb{R}^{\text{non-negative}}$ given by $h(x) = x^2$.
   (iv). The function $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = n^3$.
   (v). The function $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

19. Let $f : \mathbb{R} \to \mathbb{R}$. Determine whether $f$ is a bijection if:

   (i). $f = (x + 2)/(x + 5)$.
   (ii). $f = (x^2 + 1)/(x^2 + 2)$.

Justify your answer.