INTRODUCTION TO PREDICATES AND QUANTIFIED STATEMENTS

Let $P(x)$ be a predicate with domain $D$.

- The universal statement $\forall x \in D, P(x)$ is the statement which is true iff $P(x)$ is true for every element $x$ in the domain $D$.
- The existential statement $\exists x \in D, P(x)$ is the statement that is true iff $P(x)$ is true for at least one element $x$ in the domain $D$.

Example 28/87
Rewrite each statement without quantifiers or variables. Indicate which are true and which are false. Justify. The following are given:

- The domain of $x$ is the set of integers $\mathbb{Z}$;
- $O(x)$ denotes the predicate “$x$ is odd”;
- $P(x)$ denotes the predicate “$x$ is prime”;
- $S(x)$ denotes the predicate “$x$ is perfect square”.

a. $\exists x$ such that $P(x) \land \sim O(x)$.
There exists an integer which is prime and not odd.
There exists an even prime integer.
This is a true statement. Take $x = 2$.

b. $\exists x$ such that $O(x) \land S(x)$.
There exists an odd integer which is a perfect square.
This is a true statement. Let $x = 25 = 5^2$.

c. $\forall x, P(x) \rightarrow \sim S(x)$.
If an integer is prime then it is not a perfect square.
Sketch of proof: If $x$ is a perfect square then $x = a^2$ and it has the following factors 1, $a$, $a^2$, thus it cannot be a prime.

Part c. provides an example of an universal conditional statement, that is a statement of the form:

$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$

If we change the domain for $x$ to $D_x = \{x \in D | P(x)\}$, we can rewrite the universal conditional statement as:

$\forall x \in D_x, Q(x)$
1. Negations of quantified statements

Consider the case when $D = \{x_1, x_2, \ldots x_n\}$. Then we can write:

\[ \forall x P(x) \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n); \]
\[ \exists x P(x) \equiv P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n). \]

Thus DeMorgan’s laws give:

\[ \sim \forall x P(x) \equiv \sim P(x_1) \lor \sim P(x_2) \lor \ldots \lor \sim P(x_n); \]
\[ \sim \exists x P(x) \equiv \sim P(x_1) \land \sim P(x_2) \land \ldots \land \sim P(x_n). \]

The above examples suggest that in general we will have:

- The negation of the universal statement $\sim \forall x P(x)$ is the statement $\exists x \sim P(x)$, which says that there is an element $x$ in the domain $D$ for which $P(x)$ is false. Equivalently, we can say “some $x$ are not $P(x)$”.
- The negation of the existential statement $\sim \exists x P(x)$ is the statement $\forall x \sim P(x)$, which says that for all elements $x$ in the domain $D$, $P(x)$ is false. Equivalently, we can say “none is $P(x)$”.

Example

Formulate the negations of the following statements.

a. Every bird can fly.
   Negations: There is a bird who cannot fly. Some birds cannot fly.

b. There is a horse that can add.
   Negations: Every horse cannot add. No horse can add.

Example

Translate each of the following statements into logical expressions using predicates, quantifiers and logical connectives. Let $D$ be the domain of all people. Denote by $P(x)$ the predicate “$x$ is perfect” and by $Q(x)$ the predicate “$x$ is your friend”.

a. No one is perfect.
   Answer: $\forall x \sim P(x)$.

b. Not everyone is perfect.
   Answer: $\sim \forall P(x)$.

c. Everyone is your friend and is perfect.
   Answer: $\forall x Q(x) \land \forall x P(x)$. 
INTRODUCTION TO PREDICATES AND QUANTIFIED STATEMENTS

Note that $\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$.
Both statements assert that both $P(x)$ and $Q(x)$ are true for all values of $x$ in the domain.

2. THE NEGATION OF THE UNIVERSAL CONDITIONAL STATEMENT

The negation of the universal conditional statement is the statement:

$$\exists x \in D \text{ such that } \sim (P(x) \rightarrow Q(x))$$

Recall from Chapter 1 that $p \rightarrow q \equiv \sim p \lor q$. Thus we have the following logical equivalences:

$$\forall x (P(x) \rightarrow Q(x)) \equiv \forall x (\sim P(x) \lor Q(x))$$
$$\sim \forall x (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \land \sim Q(x))$$

Example
Consider the statement: $\forall$ real numbers, if $x^2 \geq 1$ then $x > 0$.
The negation reads: There is a real number with $x^2 \geq 1$ and $x \leq 0$.

Example
The statement “If an integer is divisible by 2 then it is even”
can be rephrased as “An integer is either not divisible by 2 or is even”.
The negation is: “Some integers are divisible by 2 and are odd”.

Two more logically equivalent forms for the universal conditional statement:

$\forall x, P(x)$ is sufficient condition for $Q(x) \equiv \forall P(x) \rightarrow Q(x)$;

$\forall x, Q(x)$ is necessary condition for $P(x) \equiv \forall P(x) \rightarrow Q(x)$

Example 42/97
Rewrite the following statement without using the word necessary. Let $D = \mathbb{Z}$.

“Being divisible by 8 is not a necessary condition for being divisible by 4”.

If we let $P(x)$ to be the predicate “$x$ is divisible by 4” and $Q(x)$ to be the predicate “$x$ is divisible by 8”, then the given statement can be written as:

$$\sim \forall (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \lor \sim Q(x))$$

Some integers are divisible by 4 and are not divisible by 8.