1. Find the number of subsets of $S = \{1, 2, \ldots, 10\}$ that contain exactly 5 elements, the sum of which is even.

The sum of 5 numbers is even if 5, 3, 1 of the numbers are even. In $S$ there are 5 even and 5 odd numbers. Therefore, the number of subsets of $S$ with the required properties is:

$$1 + \binom{5}{3} \cdot \binom{5}{2} + \binom{5}{1} \cdot \binom{5}{4} = 1 + 100 + 25 = 126$$

2. Nine people: Ann, Ben, Cal, Dot, Ed, Fran, Gail, Hail and Ida are in a room. Five of them stand in a row for a picture. In how many ways can this be done if Ann and Ben are in the picture, but not standing next to each other? Assume that the order people stand in a row for the picture matters.

Ann and Ben are always in the picture. Choose the other 3 people for the picture, this can be done in $\binom{7}{3}$ ways. Arrange these 3 people in a row for the picture in $3!$ ways. Place Ann and Ben in two of the four available spots between the three people, this can be done in $2 \cdot \binom{4}{2}$ ways. The answer is:

$$\binom{7}{3} \cdot 3! \cdot 2 \cdot \binom{4}{2} = 2520$$

3. How many ways are there to travel on a path in 3-dimensional space from the origin $(0,0,0)$ to the point $(4,3,5)$ by taking steps which always increase by one unit either the $x$-coordinate, or the $y$-coordinate, or the $z$-coordinate. (We are always stepping parallel to one of the axes, in the positive direction).

The number of ways to travel between the two points is the number of permutations of the multiset $\{4 \cdot x, 3 \cdot y, 5 \cdot z\}$, which is:

$$\frac{(4+3+5)!}{4! \cdot 3! \cdot 5!} = \frac{12!}{4! \cdot 3! \cdot 5!}$$

4. Let $d$ be a positive integer. Show that among any group of $d + 1$ integers there are two with exactly the same remainder when they are divided by $d$.

The possible remainders when dividing an integer by $d$ are $0, 1, \ldots, d-1$. Thus there are $d$ possible remainders and by the Pigeonhole Principle, any set of $d + 1$ integers will have at least two integers with the same remainders when divided by $d$. 
5. How many terms are there in the expansion of \((a + b)^{100}\)?

According to the binomial theorem we have:

\[(a + b)^{100} = \sum_{k=0}^{100} \binom{100}{k} a^{k} b^{100-k}\]

The above sum contains 101 terms.

6. Let \(n\) be a nonnegative integer. Give an algebraic proof to the following identity:

\[\binom{n}{2} + 6 \binom{n+1}{2} + \binom{n+2}{2} = (2n + 1)^2\]

Use the algebraic definition of the binomial coefficients to get the following:

\[
\frac{n!}{2!(n-2)!} + 6 \cdot \frac{(n+1)!}{2!(n-1)!} + \frac{(n+2)!}{2!n!} = \\
\frac{1}{2}n(n-1) + 3n(n+1) + \frac{1}{2}(n+1)(n+2) = 4n^2 + 4n + 1 = (2n + 1)^2
\]

7. Determine the number of permutations of \(\{1, 2, \ldots, 10\}\) in which exactly three integers are in their natural position.

There are \(\binom{10}{3}\) ways to choose the three integers left in their natural positions. Then the rest of the 7 integers have to be permuted in such a way that none of them remains in its natural position, thus we get:

\[\binom{10}{3} \cdot D_7 = \frac{10!}{3!7!} \cdot 1854 = 120 \cdot 1854 = 222480\]

8. A door lock is opened by pushing a series of buttons. Each of the three terms in the combination is entered by pushing either one button or two buttons simultaneously. If there are 5 buttons how many combinations are there? (Example: 2 \(-\) 3, 2, 5 \(-\) 4 is a valid combination.)

A push of buttons in the combinations is either the push of one of the 5 buttons or the simultaneously push of one of \(\binom{5}{2} = 10\) combinations of two of the 5 buttons. Hence there are: \((5 + 10) \cdot (5 + 10) \cdot (5 + 10) = 3375\) possible combinations for the door lock.

9. Consider all the bit strings of length 12. How many strings begin with 11 or end with 00?

The number of strings which begin with 11 or end with 00 is: \(2^{10} + 2^{10} - 2^{8} = 7 \cdot 2^{8}\).

10. How many different combinations of pennies, nickels, dimes and quarters can a piggy bank contain if it has 20 coins in it?

Consider the combinations with repetitions of a multiset of size 20 which contain four types of objects, we get:

\[\binom{20 + 4 - 1}{20} = \frac{23!}{3!20!} = \frac{21 \cdot 22 \cdot 23}{6} = 1771\]
11. What is the generating function for $a_k$, where $a_k$ is the number of solutions of $x_1 + x_2 + x_3 + x_4 = k$ where $x_1, x_2, x_3, x_4$ are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$ and $x_4 \geq 1$?

The generating function is:

$$G(x) = (x^3 + x^4 + \ldots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \ldots)$$

which in closed form is:

$$G(x) = \frac{x^5(1 + x + x^2 + x^3 + x^4)^2}{(1 - x)^2} = \frac{x^5(1 - x^5)^2}{(1 - x)^4}$$

12. Use the generating function you found in problem 11 to determine $a_7$.

We note that $a_7$ is the coefficient of $x^7$ in $G(x)$ which is the same as the coefficient of $x^2$ in

$$\frac{1 - 2x^5 + x^{10}}{(1 - x)^4}$$

Therefore:

$$a_7 = \left( \begin{array}{c} 4 + 2 - 1 \\ 2 \end{array} \right) = 10$$

13. Give a recurrence relation with initial conditions for $a_n$, the number of edges of the complete graph $K_n$.

Let $a_{n-1}$ denote the number of edges in $K_{n-1}$. To obtain $K_n$ from $K_{n-1}$ we add one vertex and $n - 1$ edges to it. Thus we get:

$$a_n = a_{n-1} + n - 1, \quad a_0 = 0$$

14. Use the recurrence relation you found in problem 13 to find a formula for $a_n$, the number of edges of $K_n$.

We have a nonhomogeneous first order recurrence relation. The homogeneous part of the solution is of the form:

$$a_n^h = C$$

with $C$ a constant to be determined from the initial conditions.

We make a guess for the particular solution (note that a linear polynomial won’t work):

$$a_n^p = an^2 + bn + c$$

We substitute the above in the recurrence relation:

$$an^2 + bn + c = a(n - 1)^2 + b(n - 1) + c + n - 1$$

which gives:

$$(1 - 2a)n + (a - b - 1) = 0$$

Therefore $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

The general solution of the recurrence relation is:

$$a_n = C + \frac{1}{2}n^2 - \frac{1}{2}n$$
From the initial condition we get $C = 0$.

Thus the answer is:

$$a_n = \frac{1}{2} n^2 - \frac{1}{2} n = \frac{n(n - 1)}{2} = \binom{n}{2}$$

which is equal to $\binom{n}{2}$ since $K_n$ contains all possible edges between its $n$ vertices.

15. For the bipartite graph and matching $M$:

$$\begin{align*}
&x_1 \bullet y_1 \\
&x_2 \bullet y_2 \\
&x_3 \bullet y_3 \\
&x_4 \bullet y_4 \\
&x_5 \bullet y_5 \\
&x_6 \bullet y_6
\end{align*}$$

Edges: $(x_1, y_2, y_3, y_4, y_6), (x_2, y_3, y_4, y_5), (x_3, y_1, y_2, y_3, y_4, y_6), (x_4, y_4, y_5), (x_5, y_4, y_5), (x_6, y_4, y_5)$.

Matching: $(x_1 y_2, x_2 y_3, x_3 y_4, x_5 y_6)$.

Find an $M$-alternating chain and hence a new matching $M'$ with 5 edges.

The $M$-alternating chain obtained by applying the matching algorithm is $\gamma = (y_1, x_3, y_4, x_4)$ and the corresponding matching with 5 edges is $M' = \{x_1 y_2, x_2 y_3, x_5 y_5, x_3 y_1, x_4 y_4\}$. 
16. Use data from the previous problem 15. For the max-matching $M'$ you obtained in problem 15 find a min-cover $S$ with 5 vertices.

\[
\begin{array}{c}
  x_1 \bullet & \bullet \ y_1 \\
  x_2 \bullet & \bullet \ y_2 \\
  x_3 \bullet & \bullet \ y_3 \\
  x_4 \bullet & \bullet \ y_4 \\
  x_5 \bullet & \bullet \ y_5 \\
  x_6 \bullet & \bullet \ y_6 \\
\end{array}
\]

Start with the matching $M' = \{x_1y_2, x_2y_3, x_5y_5, x_3y_1, x_4y_4\}$ obtained in the previous problem. Apply the algorithm again. A min-cover is obtained by taking the unlabelled vertices in $X$ and the labelled vertices in $Y$, thus $S = \{x_1, x_2, x_3, y_4, y_5\}$.

17. Use the deferred acceptance algorithm to find the women optimal stable complete marriage for the preferential ranking matrix

\[
\begin{bmatrix}
  2,1 & 3,4 & 1,3 & 4,1 \\
  4,2 & 2,3 & 1,4 & 3,4 \\
  3,3 & 1,1 & 2,1 & 4,2 \\
  4,4 & 1,2 & 2,2 & 3,3 \\
\end{bmatrix}
\]

The rows $A, B, C, D$ are the women and the column $a, b, c, d$ are the men.

1). Women choose first as follows: $A$ chooses $c$, $B$ chooses $c$, $C$ chooses $b$, $D$ chooses $b$.
2). $c$ rejects $B$ and $b$ rejects $D$.
3). $B$ chooses $b$ and $D$ chooses $c$.
4). $c$ rejects $A$ and $b$ rejects $B$.
5). $A$ chooses $a$ and $B$ chooses $d$.

We obtain the following complete marriage: $A - a, B - d, C - b, D - c$.

18. Does the graph $K_{2,5}$ have an Euler cycle? If not, does it have an Euler trail? Does the graph $K_{2,5}$ have a Hamilton trail? Justify all your answers.

$K_{2,5}$ has two vertices of degree 5, thus it has an Euler trail but no Euler cycle. The graph $K_{2,5}$ has no Hamilton trail since any path containing all 5 vertices of degree 2 must visit some of the vertices of degree 5 more than once.
19. How many nonisomorphic simple graphs are there with 6 vertices and 4 edges?

There are 9 such graphs: one with a 4-cycle, 2 with 3-cycles, 3 with no cycles (one with a 4-chain, one with a 3-chain, one with two 2-chains), 3 with stars (one with a 4-star, two with a 3-star).

20. A forest is a graph each of whose connected components is a tree. Prove that the removal of an edge from a tree leaves a forest of two trees.

First, note that removing an edge from a tree produces at least two trees. This follows from the fact that trees do not have cycles. If there were more than two trees, then putting the edge back could not result in a connected graph.
Review problems for the final exam

Sections included:

2.1
3.1, 3.2, 3.3, 3.4, 3.5
5.2, 5.3
6.1, 6.2, 6.3
7.2, 7.3, 7.4
9.2, 9.4
11.1, 11.2, 11.3, 11.5

1. Find the number of subsets of \( S = \{1, 2, \ldots, 12\} \) that contain exactly 4 elements, the sum of which is odd.

2. A group contains \( n \) men and \( n \) women. How many ways are there to arrange these people in a row if the men and women alternate?


4. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

5. Let \( n \) be a nonnegative integer. Give an algebraic proof to the following identity:

\[
\binom{n}{1} + 14 \binom{n}{2} + 36 \binom{n}{3} + 24 \binom{n}{4} = n^4
\]

6. Find the expansion of \((2x - 5y)^{13}\). What is the coefficient of \(x^5y^8\)? What is the coefficient of \(x^2y^{10}\)?

7. Problem 11, page 201.

8. A committee is formed containing either the governor or one of the two senators of each of the 50 states. How many ways are there to form the committee?

9. How many permutations of the ten digits either begin with the three digits 987, contain the digits 45 in the fifth and sixth positions, or end with the three digits 123?

10. How many different ways are there to choose a dozen donuts from the 21 varieties of a donut shop?


12. Problem 29, page 263. Find a closed formula for each of the generating functions in the problem. Also determine the following coefficients: \(e_6\) for part a; \(e_{15}\) for part b; \(e_7\) for part c; \(e_6\) for part d; \(e_9\) for part e.

13. Give a recurrence relation with initial conditions for \( a_n \), the number of edges of the complete bipartite graph \( K_{2,n} \). Then use this relation to find a formula for \( a_n \).

14. Give a recurrence relation with initial conditions for \( a_n \), the number of edges of the graph \( W_n \). Then use this relation to find a formula for \( a_n \). The wheel \( W_n \) is a graph with \( n + 1 \) vertices, \( n \) vertices form a cycle and the \((n + 1)\)th vertex is adjacent to the first \( n \) vertices.


19. Problem 2, page 482.

20. Problems 58 and 60, page 488.
Useful formulas

1. \( \frac{1}{1 - ax} = \sum_{n=0}^{\infty} a^n x^n \),

2. \( \frac{1}{1 + ax} = \sum_{n=0}^{\infty} (-1)^n a^n x^n \),

3. \( \frac{1}{(1 - x)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{n} x^n \),

4. \( \frac{1}{(1 + x)^k} = \sum_{n=0}^{\infty} (-1)^n \binom{n + k - 1}{n} x^n \),

5. \( \frac{1}{1 - x^k} = \sum_{n=0}^{\infty} x^{kn} \),

6. \( \frac{1}{1 + x^k} = \sum_{n=0}^{\infty} (-1)^n x^{kn} \),

7. \( \frac{1 - x^{n+1}}{1 - x} = \sum_{k=0}^{n} x^k \),

8. \( (1 + ax)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^k \),

9. \( \frac{1}{(1 - rx)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{n} r^n x^n \),

\[ D_n = n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - \binom{n}{3} (n-3)! + \ldots + (-1)^n \binom{n}{n} 0! \]