Answer the following questions. The answers must be clear, intelligible, and you must show your work. Provide explanation for all your steps. Your grade will be determined by adherence to these criteria. Use of books, notes and calculators is strictly forbidden. Problems 5 and 13 are worth 15 points each. All other problems are worth 10 points each.
1. Suppose \( g : A \rightarrow B \) and \( f : B \rightarrow C \) are two functions. Show that if both \( f \) and \( g \) are one-to-one functions, then \( f \circ g \) is also one-to-one.

Note \( f \circ g : A \rightarrow C \).

Assume that \( a_1, a_2 \in A \) are such that \( (f \circ g)(a_1) = (f \circ g)(a_2) \).

Thus \( f(g(a_1)) = f(g(a_2)) \).

Since \( f \) is one-to-one it follows that \( g(a_1) = g(a_2) \).

Also \( g \) is one-to-one, thus \( a_1 = a_2 \). This proves that \( f \circ g \) is one-to-one.
2. Suppose \( f : \mathbb{N} \rightarrow \mathbb{N} \) has the rule \( f(n) = 4n^2 + 1 \). Determine whether \( f \) is onto \( \mathbb{N} \).

\[ f \] is not onto \( \mathbb{N} \).

To see this let \( m = 2 \), which is not in the range of \( f \) since \( 4n^2 + 1 = 2 \) has no solutions in \( \mathbb{N} \); that is, there is no \( n \in \mathbb{N} \) with \( f(n) = 2 \).
3. Use an indirect proof to show that if $x^3$ is irrational then $x$ is irrational.

We have to show:

$$x^3 \text{ irrational } \rightarrow x \text{ irrational}$$

Use indirect proof. We show:

$$x \text{ rational } \rightarrow x^3 \text{ rational}$$

Since $x$ is rational, there are two integers $a$ and $b$, with $b \neq 0$, such that:

$$x = \frac{a}{b}$$

Then:

$$x^3 = \frac{a^3}{b^3}$$

which is rational as cubic powers of integers are integers.
4. Let $a$ and $b$ be integers.

(a). Prove or disprove: If $a \equiv b \pmod{5}$ then $a^2 \equiv b^2 \pmod{5}$.

(b). Prove or disprove: If $a^2 \equiv b^2 \pmod{5}$ then $a \equiv b \pmod{5}$.

(a). True.

Suppose that $a \equiv b \pmod{5}$. Then $5 \mid (a - b)$ so there is an integer $k$ with $a - b = 5k$ and:

$$a^2 - b^2 = (a - b)(a + b) = 5k(a + b) = 5l$$

with $l = k(a + b)$, an integer.

It follows that: $5 \mid (a^2 - b^2)$ and $a^2 \equiv b^2 \pmod{5}$.

(b). False.

We see that $1^2 \equiv 4^2 \pmod{5}$ but $1 \not\equiv 4 \pmod{5}$. 

5. Use the Principle of Mathematical Induction to prove that:

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{n}{2^n} = \frac{2^{n+1} - 2 - n}{2^n}
\]

for all \( n \geq 1 \).

Basis Step:

\[ P(1) : \quad \frac{1}{2} = \frac{2^2 - 2 - 1}{2} \]

Inductive Step: prove the implication \( P(k) \rightarrow P(k + 1) \) is true.

Assume \( P(k) \) is true for the positive integer \( k \). Prove \( P(k + 1) \) is true.

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{k}{2^k} + \frac{k + 1}{2^{k+1}} = \frac{2^{k+1} - 2 - k}{2^k} + \frac{k + 1}{2^{k+1}} = \frac{2^{k+2} - 4 - 2k + k + 1}{2^{k+1}} = \frac{2^{k+2} - 3 - k}{2^{k+1}} = \frac{2^{k+2} - 2 - (k + 1)}{2^{k+1}}
\]

which proves \( P(k+1) \) is true. The induction step is complete. Therefore by the Principle of Mathematical Induction the result follows.
6. Show that the set of odd positive integers greater that 3 is countable.

The function \( f(n) = 2n + 3 \) is a one-to-one correspondence from the set of positive integers to the set of odd integers greater than 3. Hence the set is countable.
7. How many people are needed to guarantee that at least two were born on the same day of the week and in the same month (perhaps in different years).

There are $7 \times 12 = 84$ day-month combinations. Therefore we need 85 people to ensure that two of them were born on the same day of the week and in the same month.
8. Find the number of subsets of \( S = \{1, 2, \ldots, 10\} \) that contain exactly five elements, the sum of which is even.

The only ways to get an even number from a sum of five numbers are: all five numbers are even, three numbers are even and two are odd, one number is even and four are odd. Note that \( S \) has five even elements and five odd elements.

The number of subsets of \( S \) that contain five elements, the sum of which is even is:

\[
1 + \binom{5}{3} \cdot \binom{5}{2} + \binom{5}{1} \cdot \binom{5}{4} = 1 + 100 + 25 = 126
\]
9. Use the Binomial Theorem to prove the following:

\[
\binom{50}{0} + \binom{50}{2} + \cdots + \binom{50}{50} = \binom{50}{1} + \binom{50}{3} + \cdots + \binom{50}{49}
\]

In

\[(a + b)^n = \sum_{k=1}^{n} \binom{n}{k} a^{n-k} b^k\]

use \(n = 50\), \(a = 1\) and \(b = -1\).

Then we get:

\[ (1 + (-1))^{50} = 0 = \sum_{k=1}^{50} \binom{50}{k} 1^{50-k} (-1)^k \]

For \(k\) even the terms will have positive sign and for \(k\) odd they will be negative:

\[ 0 = \binom{50}{0} - \binom{50}{1} + \binom{50}{2} + \cdots + \binom{50}{49} \]

Rewrite the above equality by moving all the negative terms on one side of the equation.
10. Nine people: Ann, Ben, Cal, Dot, Ed, Fran, Gail, Hail and Ida are in a room. Five of them stand in a row for a picture. In how many ways can this be done if Ann and Ben are in the picture, but not standing next to each other? Assume that the order people stand in a row for the picture matters.

Ann and Ben are always in the picture.
Choose the other three people for the picture. This can be done in \( \binom{7}{3} \) ways.
Arrange the three people in a row for the picture. This can be done in 3! ways.
Place Ann and Ben in two of the four available spots between the other three people. This can be done in \( 2 \cdot \binom{4}{2} = 12 \) ways.

Finally, the answer is:
\[
\binom{7}{3} \cdot 3! \cdot 2 \cdot \binom{4}{2} \quad \text{ways}
\]
11. Consider all the bit strings of length 12.
   How many begin with 11 or end with 10?

The answer is:

\[ 2^{10} + 2^{10} - 2^8 = 2^{11} - 2^8 = 7 \cdot 2^8 \]
12. Let $A = \{a, b, c, d\}$. Draw the directed graph for the relation on $A$ defined by the matrix:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{pmatrix}
\]
13. Suppose $A$ is the set composed of all ordered pairs of positive integers. Let $R$ be the relation defined on $A$ where $(a, b)R(c, d)$ means that $a + d = b + c$. Prove that $R$ is an equivalence relation.

Reflexivity:
let $(a, b) \in A$, then $a + b = b + a$ so $(a, b)R(b, a)$.

Symmetry:
Let $(a, b), (c, d) \in A$ with $(a, b)R(c, d)$, then $a + d = b + c$ and $c + b = d + a$ so $(c, d)R(a, b)$.

Transitivity:
Let $(a, b), (c, d), (e, f) \in A$ be such that $(a, b)R(c, d)$ so $a + d = b + c$. Also assume $(c, d)R(e, f)$ so $c + f = d + e$. Add the above two equalities to get: $a + d + c + f = b + c + d + e$ and therefore: $a + f = b + e$ which shows $(a, b)R(e, f)$. 
14. Assume $A$ is the set of all ordered pairs of positive integers and $R$ is the equivalence relation defined in problem 14. Find $[(2, 4)]$, the equivalence class of $(2, 4)$.

$$[(2, 4)] = \{(a, b) \in A \mid 2 + b = a + 4\} = \{(a, a + 2) \mid a \in A\}$$